Absolutely \((r, p, q)\)-summing inclusions

by

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Abstract. As a continuation of the work of Bennett and Carl for the case \(q = \infty\), we consider absolutely \((r, p, q)\)-summing inclusion maps between Minkowski sequence spaces, \(1 \leq p, q \leq 2\). Using these results we deduce parts of the limit orders of the corresponding operator ideals and an inclusion theorem between the ideals of \((u, s, t)\)-nuclear and of absolutely \((r, p, q)\)-summing operators, which gives a new proof of a result of Carl on Schatten class operators. Furthermore, we also consider inclusions between arbitrary Banach sequence spaces and inclusions between finite-dimensional Schatten classes. Finally, applications to Hilbert numbers of inclusions are given.

1. Introduction and basic tools. Let \(1 \leq r, p, q \leq \infty\) be such that \(1/p + 1/q \geq 1/r\). According to Pietsch [31, 17.1.1], an operator \(T : X \to Y\) between Banach spaces \(X\) and \(Y\) is called absolutely \((r, p, q)\)-summing if there exists a constant \(C > 0\) such that for any choice of \(x_1, \ldots, x_n \in X\) and \(y'_1, \ldots, y'_n \in Y'\), the inequality
\[
\left( \sum_{k=1}^{n} |y'_k(Tx_k)|^r \right)^{1/r} \leq C \sup_{x' \in B_{X'}} \left( \sum_{k=1}^{n} |x'(x_k)|^p \right)^{1/p} \sup_{y \in B_Y} \left( \sum_{k=1}^{n} |y'_k(y)|^q \right)^{1/q}
\]
holds. We put \(\pi_{r,p,q}(T) := \inf C\) with \(C\) as above. In this way, we obtain the maximal Banach operator ideal \((\Pi_{r,p,q}, \pi_{r,p,q})\). Let us list the most prominent special cases which have been dealt with in the literature so far:

- \(\Pi_{r,p} := \Pi_{r,p,\infty}\), the ideal of all absolutely \((r, p)\)-summing operators;
- \(\Pi_p := \Pi_{p,p} = \Pi_{p,p,\infty}\), the ideal of all absolutely \(p\)-summing operators;
- \(D_{p,q} := \Pi_{r,p,q}\) with \(r, p, q\) such that \(1/r = 1/p + 1/q\), the ideal of all \((p, q)\)-dominated operators;
- \(D_p := D_{p,p} = \Pi_{1,p,p'}\), the ideal of all \(p\)-dominated operators.

In this paper we deal with the ideal \(\Pi_{r,p,q}\) where \(1 \leq p, q \leq 2\). For the special case \(r = p = 1\) and \(q = 2\), this ideal has become of interest recently in an article of Bu [4], where the author has shown that a Banach
space $X$ of cotype 2 is a G.T. space (i.e., $\mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2)$) if and only if $X \overset{\varepsilon}{\otimes} \ell_1 \subset X \overset{\pi}{\otimes} \ell_2$, i.e., the identity map on $X$ is absolutely $(1, 1, 2)$-summing.

Bennett [2] and Carl [6] independently successfully investigated under which assumptions on the indices involved the inclusion mapping $\text{id} : \ell_u \hookrightarrow \ell_v$ is in $\Pi_{q,p}$. In this article, we present analogs of their results for the Banach operator ideal $\Pi_{r,p,q}$ and then derive in part the limit order of these ideals. This can also be used to give an alternative proof of a result due to Carl [5] on Schatten class operators. Furthermore, in the spirit of [13] and [16], we also consider inclusions $E \hookrightarrow F$, $E$ and $F$ arbitrary Banach sequence spaces, and $S^n_u \hookrightarrow S^n_v$. Finally, we give applications to Hilbert numbers of inclusions.

We start with some basic notations. For $1 \leq p \leq \infty$, its conjugate number $p'$ is defined by $1/p + 1/p' = 1$. For two real sequences $(a_n)$ and $(b_n)$ we mean by $a_n \prec b_n$ that there exists $C > 0$ such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$, and by $a_n \succeq b_n$ that $b_n \leq a_n$. If $a_n \prec b_n$ and $a_n \succ b_n$ simultaneously, then we write $a_n \asymp b_n$.

We shall use standard notation and notions from Banach space theory, as presented e.g. in [10, 19]. If $X$ is a Banach space, then $B_X$ is its (closed) unit ball and $X'$ its dual. As usual $\mathcal{L}(X,Y)$ denotes the Banach space of all (bounded linear) operators from $X$ into $Y$ endowed with the operator norm $\|\cdot\|$, where $X$ and $Y$ are Banach spaces, and $\mathcal{N}(X,Y)$ the Banach space of all nuclear operators endowed with the nuclear norm $N(\cdot)$. By $X \overset{\varepsilon}{\otimes} Y$ and $X \overset{\pi}{\otimes} Y$ we denote their injective and projective tensor products, respectively, and by $X \overset{\varepsilon}{\otimes} Y$ and $X \overset{\pi}{\otimes} Y$ the respective completions. If one of the spaces involved is finite-dimensional, we can identify $\mathcal{L}(X,Y) = X' \overset{\varepsilon}{\otimes} Y$ and $\mathcal{N}(X,Y) = X' \overset{\pi}{\otimes} Y$ isometrically. Furthermore, if $X$ is finite-dimensional, the tensor norm $\Delta_2$ on $\ell_2 \otimes X$ is given by the identification $\ell_2 \otimes_{\Delta_2} X = \ell_2(X)$, where the latter is as usual the corresponding Köthe–Bochner space. For this and more information on tensor products of Banach spaces we refer to [10].

For $1 \leq r < \infty$ we denote by $S_r$ the Banach space of all compact operators $T : \ell_2 \to \ell_2$ for which the sequence of singular numbers is in $\ell_r$. We put $S_\infty := \mathcal{L}(\ell_2)$.

Standard techniques (see, e.g., [19] or [10]) allow us to formulate the following useful characterization.

**Proposition 1.1.** Let $1 \leq p, q, r \leq \infty$ such that $1/r \leq 1/p + 1/q$. Then for an operator $T : X \to Y$ between Banach spaces $X$ and $Y$, the following are equivalent:

(i) $T$ is absolutely $(r,p,q)$-summing;
(ii) the bilinear mappings \( \varphi_n : X \otimes_{\varepsilon} \ell^n_p \times Y' \otimes_{\varepsilon} \ell^n_q \to \ell^n_r \), defined by 
\[ \varphi_n((x_1, \ldots, x_n), (y'_1, \ldots, y'_n)) := (y'_1(Tx_1), \ldots, y'_n(Tx_n)), \]
are uniformly bounded.

In this case, \( \pi_{r,p,q}(T) = \sup_n \| \varphi_n \| \). If \( r = 1 \), then the above is equivalent to

(iii) the mappings \( T \otimes \text{id}_{\ell^n_p} : X \otimes_{\varepsilon} \ell^n_p \to Y \otimes_{\pi} \ell^n_q \) are uniformly bounded.

In this case, \( \pi_{1,p,q}(T) = \sup_n \| T \otimes \text{id}_{\ell^n_p} \| \).

The following inclusion can be found in [31, 17.1.4].

**Proposition 1.2.** Let \( r_0 \leq r_1, p_0 \leq p_1 \) and \( q_0 \leq q_1 \). Suppose that \( 0 \leq 1/p_0 + 1/q_0 < 1/r_0 \leq 1/p_1 + 1/q_1 < 1/r_1 \). Then \( \Pi_{r_0,p_0,q_0} \subseteq \Pi_{r_1,p_1,q_1} \).

### 2. Interpolation of summing norms.

Basics on interpolation theory of Banach spaces can be found in [3]. Let us just introduce our notation for the complex interpolation method. For a given compatible Banach couple \((X_0, X_1)\) and \( 0 < \theta < 1 \), we denote by \([X_0, X_1]_\theta\) the resulting complex interpolation space. We will frequently use the fact that for \( 1 \leq p_0, p_1 \leq \infty \) and \( 0 < \theta < 1 \) one has

\[
[l_{p_0}, l_{p_1}]_\theta = l_p,
\]
where \( p \) is determined by \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \). The following crucial interpolation tool is due to Kouba [25].

**Proposition 2.1.** Let \( 1 \leq p_0, p_1, q_0, q_1 \leq 2, 0 < \theta < 1 \) and \( 1 \leq p, q \leq 2 \) with \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) and \( 1/q = (1 - \theta)/q_0 + \theta/q_1 \). Then \([l_{p_0} \otimes_{\varepsilon} l_{q_0}, l_{p_1} \otimes_{\varepsilon} l_{q_1}]_\theta = l_p \otimes_{\varepsilon} l_q \). In particular,

\[
\sup_{n,m} \| \ell^n_{p_0} \otimes_{\varepsilon} \ell^n_{q_0} \to [\ell^n_{p_0} \otimes_{\varepsilon} \ell^n_{q_0}, \ell^n_{p_1} \otimes_{\varepsilon} \ell^n_{q_1}]_\theta \| < \infty.
\]

The following one-sided interpolation formula can be found in [29].

**Proposition 2.2.** Let \( 1 \leq p_0 < p_1 < 2, 1 \leq r < p'_1, 0 < \theta < 1 \) and \( 1 \leq p < p_1 \) with \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \). Then \([l_r \otimes_{\varepsilon} l_{p_0}, l_r \otimes_{\varepsilon} l_{p_1}]_\theta = l_r \otimes_{\varepsilon} l_p \). In particular,

\[
\sup_{n,m} \| \ell^n_{r_0} \otimes_{\varepsilon} \ell^n_{r_1} \to [\ell^n_{r_0} \otimes_{\varepsilon} \ell^n_{r_0}, \ell^n_{r_0} \otimes_{\varepsilon} \ell^n_{r_1}]_\theta \| < \infty.
\]

To simplify our statements, we denote for \( 1 \leq u \leq v \leq \infty \) the inclusion map \( \text{id} : l_u \hookrightarrow l_v \) by \( \text{id}_{uv} \), and by \( \text{id}_{uv}^n \) the finite-dimensional inclusion map \( : l_u^n \hookrightarrow l_v^n \) (for \( u \) and \( v \) not necessarily ordered as above).

**Lemma 2.3.** Let \( 1 \leq u_0, u_1 \leq 2, u_0 \leq v_0 \leq \infty, u_1 \leq v_1 \leq \infty, 1 \leq r_0, r_1 \leq \infty \) and \( 1 \leq s_0, s_1, t_0, t_1 \leq 2 \). Then for all \( 0 < \theta < 1 \), there exists \( C > 0 \) such that

\[
\pi_{r,s,t}(\text{id}_{uv}^n) \leq C \pi_{r_0,s_0,t_0}(\text{id}_{u_0v_0}^n)^{1-\theta} \pi_{r_1,s_1,t_1}(\text{id}_{u_1v_1}^n)^\theta,
\]
where $1/u = (1-\theta)/u_0 + \theta/u_1$, $1/v = (1-\theta)/v_0 + \theta/v_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$, $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/t = (1-\theta)/t_0 + \theta/t_1$, in each of the following cases:

(i) $2 \leq v_0, v_1 \leq \infty$;
(ii) $v = v_0 = v_1$ and $t = t_0 = t_1$;
(iii) $\max(t_0, t_1) < v = v_0 = v_1$.

In particular, in all of these cases, $\text{id}_{u_0v_0} \in \Pi_{r_0,s_0,t_0}$ and $\text{id}_{u_1v_1} \in \Pi_{r_1,s_1,t_1}$ imply $\text{id}_{uv} \in \Pi_{r,s,t}$.

Proof. We define the bilinear mappings

$$\psi_{n,m} : \mathbb{K}^n \otimes \mathbb{K}^m \times \mathbb{K}^n \otimes \mathbb{K}^m \to \mathbb{K}^m, \quad ((x_k), (y_k)) \mapsto (y_k(x_k)).$$

Then the mappings

$$\psi_{n,m} : \ell^m_{u_0} \otimes \ell^m_{v_1} \times \ell^m_{v_0} \otimes \ell^m_{v_1} \to \ell^m_{r_0}$$

are bounded from above by $\pi_{r_1,s_1,t_1}(\text{id}_{u_0v_0}, i = 0, 1$. Thus, by bilinear interpolation (see, e.g., [3, 4.4.1]), the mappings

$$\psi_{n,m} : [\ell^m_{u_0} \otimes \ell^m_{v_1}, \ell^m_{v_0} \otimes \ell^m_{v_1}]_\theta \times [\ell^m_{v_0} \otimes \ell^m_{t_0}, \ell^m_{v_1} \otimes \ell^m_{t_1}]_\theta \to [\ell^m_{r_0}, \ell^m_{r_1}]_\theta$$

are bounded from above by $\pi_{r_0,s_0,t_0}(\text{id}_{u_0v_0})^{1-\theta} \pi_{r_1,s_1,t_1}(\text{id}_{u_1v_1})^\theta$. Clearly, by (2.1) we have $[\ell^m_{r_0}, \ell^m_{r_1}]_\theta = \ell^m_{r_0}$ isometrically. By Proposition 2.1,

$$\sup_{n,m} \|\text{id} : \ell^m_{u_0} \otimes \ell^m_{v_1} \to [\ell^m_{u_0} \otimes \ell^m_{v_0}, \ell^m_{v_0} \otimes \ell^m_{v_1}]_\theta \| < \infty$$

and by [3, 4.5.2] together with Proposition 2.1 in case (i), [3, 4.2.1(c)] in case (ii), and Proposition 2.2 in case (iii),

$$\sup_{n,m} \|\text{id} : \ell^m_{u_0} \otimes \ell^m_{v_0} \to [\ell^m_{u_0} \otimes \ell^m_{t_0}, \ell^m_{v_0} \otimes \ell^m_{t_1}]_\theta \| < \infty.$$ 

Thus, the mappings

$$\psi_{n,m} : \ell^m_{u_0} \otimes \ell^m_{v_0} \times \ell^m_{v_0} \otimes \ell^m_{t_0} \to \ell^m_{r_0}$$

have norm less than or equal to $C \pi_{r_0,s_0,t_0}(\text{id}_{u_0v_0})^{1-\theta} \pi_{r_1,s_1,t_1}(\text{id}_{u_1v_1})^\theta$ for some $C > 0$ not depending on $m$ and $n$. The final assertion then follows by the maximality of the operator ideal $\Pi_{r,s,t}$ and by density. \hfill \blacksquare

3. Absolutely summing inclusion maps. To apply the above lemma, we need some extreme cases.

**Lemma 3.1.** The following hold true:

(i) $\text{id}_{2^2} \in \Pi_{1,1,1}$;
(ii) $\text{id}_{1^\infty} \in \Pi_{1,2,2}$;
(iii) $\text{id}_{12} \in \Pi_{1,1,2} \cap \Pi_{1,2,1}$;
(iv) $\text{id}_{2^\infty} \in \Pi_{1,1,2} \cap \Pi_{1,2,1}$.
Proof. (i) is clear as $\Pi_{1,1,1} = \mathcal{L}$. This means that
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_2^m \otimes_{\mathcal{E}} \ell_1^m \rightarrow \ell_2^m \otimes_{\pi} \ell_1^m \| = 1,
$$
which also gives $\text{id}_{1,\infty} \in \Pi_{1,2,2}$. Furthermore, by Grothendieck, $\text{id}_{1,2} \in \Pi_2$ implies $\text{id}_{1,2} \in \Pi_1$, i.e.,
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_1^m \otimes_{\mathcal{E}} \ell_1^m \rightarrow \ell_2^m \otimes_{\pi} \ell_1^m \| < \infty.
$$
Thus, by factorization,
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_1^m \otimes_{\mathcal{E}} \ell_1^m \rightarrow \ell_2^m \otimes_{\pi} \ell_1^m \| < \infty,
$$
which yields $\text{id}_{1,2} \in \Pi_{1,1,2}$ and $\text{id}_{2,\infty} \in \Pi_{1,2,1}$ (by duality).

By [31, 22.4.8] it is known that $\pi_2(\text{id}_{1,2}^n) = \pi_{2,\infty}(\text{id}_{1,2}^n) = 1$, i.e.,
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_2^m \otimes_{\mathcal{E}} \ell_1^m \rightarrow \ell_2^m \otimes_{\Delta_2} \ell_2^m \| = 1.
$$
Since $\varepsilon^t = \varepsilon$, and $\Delta_2^t = \Delta_2$ on the tensor product of two Hilbert spaces, it also follows that
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_1^m \otimes_{\mathcal{E}} \ell_2^m \rightarrow \ell_2^m \otimes_{\Delta_2} \ell_2^m \| = 1.
$$
Furthermore, by duality,
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_2^m \otimes_{\Delta_2} \ell_2^m \rightarrow \ell_2^m \otimes_{\pi} \ell_{\infty}^m \| = 1.
$$
Thus, by factorization,
$$
\sup_{n,m} \| \text{id} \otimes \text{id} : \ell_1^m \otimes_{\mathcal{E}} \ell_2^m \rightarrow \ell_2^m \otimes_{\pi} \ell_{\infty}^m \| = 1,
$$
which gives $\text{id}_{1,2} \in \Pi_{1,2,1}$ and $\text{id}_{2,\infty} \in \Pi_{1,1,2}$. ■

Theorem 3.2. Let $1 \leq p, q \leq 2$ and $1 \leq u \leq 2 \leq v \leq \infty$ with $1/u - 1/v \geq 2 - 1/p - 1/q$. Then $\text{id}_{uv} \in \Pi_{1,p,q}$.

Proof. We start by applying Lemma 2.3(i) for $r = r_0 = r_1$ and using Lemma 3.1 to obtain more extreme cases. Fix $1 \leq q \leq 2$ and define $1 \leq \tilde{q} \leq 2 \leq \overline{q} \leq \infty$ by $1/\overline{q} = 3/2 - 1/q$ and $1/\overline{q} = 1/q - 1/2$. Then we obtain the following (taking $\theta := 2/\overline{q}$ in Lemma 2.3(i) whenever $1 < q < 2$):

<table>
<thead>
<tr>
<th>$u_0$</th>
<th>$v_0$</th>
<th>$u_1$</th>
<th>$v_1$</th>
<th>$s_0$</th>
<th>$t_0$</th>
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<tr>
<td>1</td>
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<td>$\text{id}<em>{1,2} \in \Pi</em>{1,\tilde{q},q}$ (i)</td>
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<td>$\text{id}<em>{2,\infty} \in \Pi</em>{1,\tilde{q},q}$ (ii)</td>
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<td>$\text{id}<em>{1,2} \in \Pi</em>{1,1,q}$ (iii)</td>
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<tr>
<td>2</td>
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<td>$\text{id}<em>{1,2} \in \Pi</em>{1,1,q}$ (iv)</td>
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<td>$\text{id}<em>{1,2} \in \Pi</em>{1,2,q}$ (v)</td>
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<td>1</td>
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<td>$\text{id}<em>{1,2} \in \Pi</em>{1,2,q}$ (vi)</td>
</tr>
</tbody>
</table>
Coming to the parameter $p$, we have to consider two cases, where we use again Lemma 2.3(i) (taking $\theta = q'/u_0'$, $\theta = q/v_0$, $\theta = q'(1/u_1 - 1/2)$ and $\theta = q'(1/2 - 1/v_1)$, respectively) together with the results from the above table:

- $1 \leq p < \tilde{q}$: (i) and (iii) give
  \[ id_{u_2} \in II_{1,p,q}, \text{ where } 1/u_0 = 5/2 - 1/p - 1/q, \]
  and (ii) and (iv) give
  \[ id_{v_0} \in II_{1,p,q}, \text{ where } 1/v_0 = 1/p + 1/q - 3/2. \]
- $\tilde{q} \leq p \leq 2$: (ii) and (v) give
  \[ id_{u_1} \in II_{1,p,q}, \text{ where } 1/u_1 = 2 - 1/p - 1/q, \]
  and (i) and (vi) give
  \[ id_{v_1} \in II_{1,p,q}, \text{ where } 1/v_1 = 1/p + 1/q - 1. \]

If we take $s_0 = s_1 = p$ and $t_0 = t_1 = q$, Lemma 2.3(i) applied to these cases for all $0 < \theta < 1$ gives $id_{uv} \in II_{1,p,q}$ for all those $1 \leq u \leq 2 \leq v \leq \infty$ such that $1/u - 1/v = 2 - 1/p - 1/q$. The rest is clear by factorization. 

The above can be extended to the general case of absolutely $(r,p,q)$-summing operators. When considering limit orders in the next section, we will see that the assumption $1/u - 1/v \geq 1 + 1/r - 1/p - 1/q$ in the corollary below cannot be weakened.

**Corollary 3.3.** Let $1 \leq p, q \leq 2$, $1 \leq r \leq \infty$ and $1 \leq u \leq 2 \leq v \leq \infty$ with $1/u - 1/v \geq 1 + 1/r - 1/p - 1/q \geq 0$. Then $id_{uv} \in II_{r,p,q}$.

**Proof.** The idea is to find $p_0 \leq p$ and $q_0 \leq q$ such that $1/p_0 + 1/q_0 - 1 = 1/p + 1/q - 1/r$. Then $2 - 1/p_0 - 1/q_0 = 1/u$, so that the above theorem together with Proposition 1.2 gives

\[ id_{uv} \in II_{1,p_0,q_0} \subseteq II_{r,p,q}. \]

We have to consider several cases, for which we simply list our choices of $p_0, q_0$ and leave the verification to the reader:

- $r \geq \max(p,q)$: $q_0 = 1$ and $p_0$ such that $1/p_0 = 1/p + 1/q - 1/r$;
- $r \leq q$: $p_0 = p$ and $q_0$ such that $1/q_0 = 1 - 1/r + 1/q$;
- $r \leq p$: $q_0 = q$ and $p_0$ such that $1/p_0 = 1 - 1/r + 1/p$. 

The situation in the cases other than $1 \leq u \leq 2 \leq v \leq \infty$ seems to be more complicated. We can give the following partial result for $1 \leq u \leq v \leq 2$. It will turn out later on that (i) is optimal in the case $q = 2$ and that (ii) is almost optimal in the case $p = 1$ (that is, the only improvement possible in this case is to replace “$>$” by “$\geq$”). However, (iii) shows that (i) is not optimal in general.
Proposition 3.4. Let $1 \leq p, q \leq 2$, $1 \leq r \leq \infty$ and $1 \leq u < v \leq 2$.

(i) $\text{id}_{uv} \in \Pi_{r,p,q}$ whenever $1/r \leq 1/p - v'/u'q'$;
(ii) $\text{id}_{uv} \in \Pi_{p,p,q}$ whenever $1/u - 1/v > 1/q$;
(iii) $\text{id}_{uv} \in \Pi_{r,p,q}$ whenever $v > q$ and $1/r < 1/p - (v'/q')(1/q' + 1/v - 1/u)$.

Proof. (i) Since $\text{id}_{11} \in \Pi_{1,1,2}$, it follows that $\text{id}_{1v} \in \Pi_{1,1,2} \subseteq \Pi_{1,1,q} \subseteq \Pi_{p,p,q}$. Furthermore, $\text{id}_{1v} \in \Pi_{\tilde{r},p,q}$, where $1/\tilde{r} = 1/p + 1/q - 1$. Set $\theta := v'/u'$. Then $1/u = (1 - \theta)/1 + \theta/v$. Now Lemma 2.3(ii) implies $\text{id}_{uv} \in \Pi_{r,p,q}$, where

$$\frac{1}{r} = \frac{1}{p} + \frac{\theta}{p} - \frac{\theta}{q} = \frac{1}{p} - \frac{\theta}{q'} = \frac{1}{p} - \frac{v'}{q'}.
$$

(ii) Let $t < v$ be arbitrary. Then as above, $\text{id}_{1v} \in \Pi_{1,1,t}$ and $\text{id}_{tv} \in \Pi_{1,1,1}$. Set $\theta := t'/u'$; then $1/u = (1 - \theta)/1 + \theta/t$. Now Lemma 2.3(iii) implies $\text{id}_{uv} \in \Pi_{1,1,q}$, where

$$\frac{1}{q} = \frac{1}{t} + \frac{\theta}{t} = \frac{1}{1} + \frac{1}{t} + \frac{1}{1} + \frac{1}{t}.
$$

The claim for $p$ arbitrary follows by Proposition 1.2.

(iii) Let $\tilde{u} < u$ be such that $1/\tilde{u} > 1/q' + 1/v$. Then (ii) implies $\tilde{u}v \in \Pi_{p,p,q}$, and once again $\text{id}_{uv} \in \Pi_{\tilde{r},p,q}$ with $1/\tilde{r} = 1/p + 1/q - 1$. Set $\theta = v'(1/\tilde{u} - 1/u)$; then $1/u = (1 - \theta)/\tilde{u} + \theta/v$. Now Lemma 2.3(ii) implies $\text{id}_{uv} \in \Pi_{r,p,q}$, where

$$\frac{1}{r} = \frac{1}{p} + \frac{\theta}{p} - \frac{\theta}{q} = \frac{1}{p} - \frac{v'}{q'} \left(\frac{1}{\tilde{u}} - \frac{1}{u} \right) < \frac{1}{p} - \frac{v'}{q'} \left(\frac{1}{q'} + \frac{1}{v - u} \right),
$$

which gives the claim. ■

4. Limit orders. We continue with a result on limit orders. For the definition and basic facts mentioned subsequently, we refer to [31, 14.4].

Let $1 \leq u, v \leq \infty$ and $\sigma = (\sigma_n) \in \ell_\infty$ be such that the diagonal operator $D_\sigma : \ell_u \to \ell_v$, $(x_n) \mapsto (\sigma_n x_n)$, is defined (and continuous). Then for a Banach operator ideal $(\mathcal{A}, \Lambda)$, its limit order $\lambda(\mathcal{A}, u, v)$ is defined by

$$\lambda(\mathcal{A}, u, v) := \inf\{1/r \geq 0; D_\sigma \in \mathcal{A}(\ell_u, \ell_v) \text{ for all } \sigma \in \ell_r\}.
$$

Very useful in computing special limit orders is the following formula:

$$\lambda(\mathcal{A}, u, v) = \inf\{\lambda \geq 0; \exists q \geq 0 \text{ such that } \Lambda(\text{id}_{uv}^n) \leq qn^\lambda\}.
$$

König [24] in a famous paper proved an important connection to embedding maps of Sobolev spaces and weakly singular integral operators (see also [31, 22.7]).

Before stating our partial result for the limit order of $\Pi_{r,p,q}$, we prove a minor lemma first.

Lemma 4.1. Let $1 \leq r \leq p \leq 2$. Then $\pi_{r,p,2}(\text{id}_{11}^n) \sim n^{1/r - 1/p}$. 

Proof. We start with the case \( r = 1 \). Since \( \mathcal{L}(\ell_1, \ell_2) = \Pi_1(\ell_1, \ell_2) \) by Grothendieck (see, e.g., [31, 22.4.4]), we have \( \text{id}_{12} \in \Pi_1 \), hence

\[
\| \text{id} \otimes \text{id} : \ell_1 \otimes_{\varepsilon} \ell_1^\pi \to \ell_2 \otimes_{\pi} \ell_2^\pi \| \leq \pi_1(\text{id}_{12}) < 1,
\]

which gives \( \text{id}_{11} \in \Pi_{1,1,2} \). Furthermore, \( \pi_2(\text{id}_{12}^n) = 1 \), and \( \pi_2(\text{id}_{11}^n) = n^{1/2} \) (see, e.g., [31, 22.4.9]), hence,

\[
\| \text{id} \otimes \text{id} : \ell_2 \otimes_{\varepsilon} \ell_2^\pi \to \ell_2 \otimes_{\Delta_2} \ell_2^\pi \| = 1
\]

and (by duality)

\[
\| \text{id} \otimes \text{id} : \ell_2 \otimes_{\Delta_2} \ell_2^\pi \to \ell_2 \otimes_{\pi} \ell_2^\pi \| \leq n^{1/2}.
\]

Thus, by factorization,

\[
\| \text{id} \otimes \text{id} : \ell_2 \otimes_{\varepsilon} \ell_1^\pi \to \ell_2 \otimes_{\pi} \ell_1^\pi \| \leq n^{1/2},
\]

which gives \( \pi_{1,2,2}(\text{id}_{11}^n) \leq n^{1/2} \). The claim for \( 1 < p < 2 \) now follows by Lemma 2.3(ii) with \( r_0 = r_1 = 1 \), \( s_0 = 1 \), \( s_1 = 2 \), \( s = p \) and \( \theta = 2/p' \).

Coming to the general case \( r \leq p \), we observe that \( \Pi_{1,1,2} \subseteq \Pi_{p,p,2} \) by Proposition 1.2. Hence, \( \text{id}_{11} \in \Pi_{p,p,2} \), which implies \( \pi_{p,p,2}(\text{id}_{11}^n) \leq 1 \). By the above, \( \pi_{1,p,2}(\text{id}_{11}^n) \leq n^{1-1/p} \). Thus, Lemma 2.3(ii) with \( r_0 = 1, r_1 = p, s_0 = s_1 = p \) and \( \theta = p'/r' \) gives the claim. \( \blacksquare \)

We now get the following partial result for the limit order of \( \Pi_{r,p,q} \):

**Theorem 4.2.** Let \( 1 \leq r \leq p, q \leq 2 \) with \( 1/p + 1/q \leq 1/2 + 1/r \). Then \( \Pi_{r,p,q}(\text{id}_{uv}^n) \asymp n^{\alpha_{r,p,q}(u,v)} \) where \( \alpha_{r,p,q}(u,v) \) is given by the following (incomplete) diagram:

![](https://via.placeholder.com/150)

In particular, \( \chi(\Pi_{r,p,q}, u, v) = \alpha_{r,p,q}(u, v) \). In the case \( r = 1 \), the diagram can be completed by substituting “1” for “?” in the upper left quadrant. In particular,

\[
\pi_{1,p,q}(\text{id}_{uv}^n) \asymp \| \text{id} \otimes \text{id} : \ell_u \otimes_{\varepsilon} \ell_v^p \to \ell_u \otimes_{\pi} \ell_v \otimes_{\pi} \ell_q \| \asymp \max \left( 1, \frac{N(\text{id}_{uv}^q)}{\| \text{id}_{uv}^q \|} \right)
\]

for all \( 1 \leq u, v \leq \infty \) whenever \( 1 \leq p, q \leq 2 \) are such that \( 1/p + 1/q \leq 3/2 \).
Proof. Consider the standard unit vectors $e_1, \ldots, e_n$ in $\ell^u_n$ and $\ell^v_n$, respectively. Then
\[
\sup_{x' \in B_{\ell_u}} \left( \sum_{k=1}^n |x'(e_k)|^p \right)^{1/p} = \|\text{id}^{n}_{u'p}\|, \quad \sup_{y \in B_{\ell_v}} \left( \sum_{k=1}^n |e_k(y)|^q \right)^{1/q} = \|\text{id}^{n}_{q'v'}\|
\]
and
\[
\left( \sum_{k=1}^n |e_k(\text{id}_{uv}(e_k))|^r \right)^{1/r} = n^{1/r}.
\]
Thus,
\[
\pi_{r,p,q}(\text{id}^{n}_{uv}) \geq \max \left( 1, \frac{n^{1/r}}{\|\text{id}^{n}_{u'p}\| \|\text{id}^{n}_{q'v'}\|} \right),
\]
which gives the lower estimates.

Concerning the upper ones, Corollary 3.3 gives the part of the diagram where we have a “0”, so we are left with the remaining four parts. Since $(\mathcal{N}, N)$ is the smallest operator ideal, we have $\Pi_{r,p,q}(\text{id}^{n}_{uv}) \leq N(\text{id}^{n}_{uv}) \leq n$, which gives the exact estimate in the case $r = 1$ for the “?”-part. The upper estimates for the part above the “0” follow by factorization from Corollary 3.3. For the last two parts, observe that by Lemma 4.1,
\[
\pi_{r,p,q}(\text{id}^{n}_{11}) \leq \pi_{r,p,2}(\text{id}^{n}_{11}) \sim n^{1/r-1/p},
\]
and by duality,
\[
\pi_{r,p,q}(\text{id}^{n}_{\infty\infty}) = \pi_{r,q,p}(\text{id}^{n}_{11}) \sim n^{1/r-1/q}.
\]
The remaining estimates then follow by factorization from these corner cases.

Remark 4.3.

(a) For the case $1 \leq p, q \leq 2$ and $1/p + 1/q > 3/2$, the same techniques give the following incomplete picture for the limit order of $\Pi_{1,p,q}$:

1/\(v\)
(b) Apart from the gaps in the above (which, as we conjecture, may be filled according to the first diagram), there is not much hope that the “easy” formula

$$\pi_{1,p,q}(\ell^m_u \rightarrow \ell^m_v) \asymp \max(1, N(id^u_{v',q'})/\|id^u_{u}/p\|)$$

or the (weaker) formula for the limit order hold whenever one of the indices $p$ and $q$ is strictly greater than 2; e.g., they do not hold for the ideals $\mathcal{D}_p = \Pi_{1,p,p'}$ whenever $p \neq 2$ (see, e.g., [31, 22.5]).

An immediate consequence of the above considerations (the case $r = 1$) for the norms of tensor product identities is the following:

**Corollary 4.4.** Let $1 \leq p, q \leq 2 \leq r, s \leq \infty$. Then

$$\|id \otimes id : \ell^m_p \otimes \ell^m_q \rightarrow \ell^m_p \otimes \ell^m_q\| \asymp \min(n, m)^{\max(0,1-1/p-1/q+1/r+1/s)}.$$ 

Let $\mathcal{S}_r$ and $\mathcal{S}_{r,\infty}$ denote the operator ideals of all operators $T$ with $(h_n(T)) \in \ell_r$ and $(h_n(T)) \in \ell_{r,\infty}$, respectively (for the definition of Hilbert numbers and the facts mentioned here we refer to the very last section). By the fact that $\Pi_{r,2,2}$ is the largest extension of $\mathcal{S}_r$ and by (8.1) we know that $\mathcal{S}_r \subseteq \Pi_{r,2,2} \subseteq \mathcal{S}_{r,\infty}$, hence for all $1 \leq u, v, r \leq \infty$ it follows that $\lambda(\mathcal{S}_r, u, v) = \lambda(\Pi_{r,2,2}, u, v)$. Thus, the diagrams for the limit order of $\mathcal{S}_r$ given in [20] give the following ones for $\Pi_{r,2,2}$:

$$1 \leq r \leq 2 \quad \Pi_{r,2,2} \quad 2 < r < \infty$$

Note that the special case $id_{u,v} \in \Pi_{r,2,2}$ whenever $1 \leq u < v \leq 2$ and $1/r \leq (v'/2)(1/u - 1/v)$ has also been proved in Proposition 3.4(i).

Proposition 3.4 can be used to give more results for the limit order of $\Pi_{r,p,q}$; as an example we will validate the following two diagrams, which may give some impression how diverse the limit orders of $\Pi_{r,p,q}$ may be:
\[ \Pi_{q,q,q} \quad 1 < q < 2 \quad \Pi_{2,2,q} \]

\[
\begin{array}{c|c|c}
? & 1 - \frac{1}{u} & ? \\
\hline
\frac{1}{v} & \frac{1}{q'} - \frac{1}{u} + \frac{1}{v} & 0 \\
\end{array}
\]

By what has been done before, we can exclude the case \( 1 \leq u \leq 2 \leq v \leq \infty \). Now let \( 1 \leq u < v \leq 2 \). Then by Proposition 3.4(ii) we know that \( \text{id}_{uv} \in \Pi_{p,p,q} \) whenever \( 1/u - 1/v > 1/q' \). This gives, by symmetry, the diagram for \( \Pi_{q,q,q} \), and the right-hand side of the diagram for \( \Pi_{2,2,q} \). For the left-hand side, consider the ideal \( \Pi_{2,q,2} \). By Proposition 3.4(i) we have \( \text{id}_{uv} \in \Pi_{2,q,2} \) whenever \( 1 \leq u < v \leq 2 \) and \( 1/q' \leq (v'/2)(1/u - 1/v) \). By duality, this gives \( \text{id}_{uv} \in \Pi_{2,2,q} \) whenever \( 2 \leq u < v \leq \infty \) and \( 1/q' \leq (u/2)(1/u - 1/v) \). Factorization now gives the upper estimates for the left-hand side of the diagram for \( \Pi_{2,2,q} \), and the lower ones follow from the diagram for \( \Pi_{q',2,2} \), since \( \Pi_{2,2,q} \subseteq \Pi_{q',2,2} \).

5. Connections to nuclear operators and Schatten classes. Going back to the definition, it is not clear (and very often false) whether for a given Banach operator ideal \( (A, A) \) its limit order is attained, i.e., whether \( D_\sigma \in \mathcal{A}(\ell_u, \ell_v) \) for all \( \sigma \in \ell_1/\lambda(A,u,v) \). For special choices of the indices involved, we can confirm this. The proof goes along similar lines to the one of [11, Lemma 3], but we give the details for the convenience of the reader. Let us first recall a result of [7, 1.4.3], for which we introduce the following temporary notation: Let \( x_1, \ldots, x_n \in \ell_u^m \). Then for \( 1 \leq p \leq \infty \) we set

\[ w_p(x_i) := \sup_{x' \in B_{\ell_u^m}} \left( \sum_{i=1}^n |x'(x_i)|^p \right)^{1/p}. \]

**Lemma 5.1.** Let \( 1 \leq p, u \leq \infty \). Then there exists a constant \( C > 0 \) such that for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \ell_u^m \),

\[ w_p(x_i \otimes y_j; \ell_u^{m^2}) \leq C w_p(x_i; \ell_u^m) w_p(y_j; \ell_u^m) \]

whenever either \( 1 \leq p' \leq u \leq \infty \), \( 1 \leq u \leq p' = 2 \), or \( 1 \leq u \leq 2 < p' \leq \infty \). In particular, such a constant exists for all \( u \) whenever \( p \in \{2, \infty\} \).

**Proposition 5.2.** Let \( 1 \leq p, q, r \leq \infty \) with \( 1/p + 1/q = 1/r \). Then for \( 1 \leq u, v \leq \infty \), the limit order \( \lambda(\Pi_{r,p,q}, u, v) \) is attained whenever \( 1 \leq u, p \leq 2 \).
or \(1 \leq p' \leq u \leq \infty\), and \(1 \leq q \leq 2 \leq v \leq \infty\) or \(1 \leq v \leq q \leq \infty\). In particular, it is attained for all \(u,v\) whenever \(p,q \in \{2, \infty\}\).

Proof. First we show that under the given assumptions, there exists a constant \(C > 0\) such that for all \(\sigma_1, \ldots, \sigma_m \in \mathbb{K}\),

\[
    \pi_{r,p,q}(D_\sigma : \ell^m_u \to \ell^m_v)^2 \leq C \pi_{r,p,q}(D_\sigma \otimes D_\sigma : \ell^m_u \to \ell^m_v^2).
\]

Let \(x_1, \ldots, x_n \in \ell^m_u\) and \(y_1, \ldots, y_n \in \ell^m_v\). Then

\[
    \left( \left( \sum_{k=1}^{n} |y_k(D_\sigma x_k)\|^r \right)^{1/r} \right)^2 = \left( \left( \sum_{k=1}^{n} |y_k(D_\sigma x_k)\|^r_1 \right)^{1/r} \right)^2 \leq \pi_{r,p,q}(D_\sigma \otimes D_\sigma) w_p(x_k \otimes x_l; \ell^m_u) w_q(y_k \otimes y_l; \ell^m_v).\]

Thus, the assumptions together with the lemma above give \((5.1)\).

Now set \(\lambda := \lambda(\Pi_{r,p,q}, u, v)\). Then for all \(\varepsilon > 0\) sufficiently small and all \(\sigma \in \ell(\lambda+\varepsilon)^{-1}\) we have \(D_\sigma \in \Pi_{r,p,q}(\ell_u, \ell_v)\), i.e.,

\[
    \pi_{r,p,q}(D_\sigma : \ell_u \to \ell_v) \leq c(\varepsilon)\|\sigma\|_{(\lambda+\varepsilon)^{-1}}.
\]

Denote by \(D\) the set of all finite-dimensional diagonal operators \(D_\sigma : \mathbb{K}^m \to \mathbb{K}^m\), \(m\) arbitrary. Obviously, \(D_\sigma \otimes D_\sigma \in D\) for all \(\sigma \in \mathbb{K}^m\). Define on \(D\) two functions \(A\) and \(B\) by

\[
    A(D_\sigma : \mathbb{K}^m \to \mathbb{K}^m) := \pi_{r,p,q}(D_\sigma : \ell^m_u \to \ell^m_v),
\]

\[
    B(D_\sigma : \mathbb{K}^m \to \mathbb{K}^m) := \|\sigma\|_{\ell^{-1}}.
\]

Then it follows from the above that for all \(\sigma \in \mathbb{K}^m\) and \(\varepsilon > 0\) sufficiently small,

\[
    A(D_\sigma) \leq c(\varepsilon)\|\sigma\|_{(\lambda+\varepsilon)^{-1}} \leq \tilde{c}(\varepsilon)m^\varepsilon\|\sigma\|_{\lambda^{-1}}.
\]

Clearly, \(B(D_\sigma \otimes D_\sigma) = B(D_\sigma)^2\) and, by \((5.1)\), \(A(D_\sigma)^2 \leq CA(D_\sigma \otimes D_\sigma)\). Hence, an application of [7, 1.3.1] yields, for all \(\sigma \in \mathbb{K}^m\),

\[
    \pi_{r,p,q}(D_\sigma) = A(D_\sigma) \leq CB(D_\sigma) = C\|\sigma\|_{\lambda^{-1}},
\]

which by an obvious continuity argument gives the claim. \(\blacksquare\)

**Corollary 5.3.** Let \(1 \leq r \leq \infty\) and \(1 \leq p, q, u \leq 2 \leq v \leq \infty\), and define \(1 \leq s \leq \infty\) by \(1/s := \max(0, 1 + 1/r - 1/q - 1/p + 1/v - 1/u)\). Then

\[
    D_\sigma \in \Pi_{r,p,q}(\ell_u, \ell_v)\quad \text{for all } \sigma \in \ell_s.
\]
Proof. This is now a direct consequence of Proposition 5.2 and our
diagrams in the previous section. ■

The above result for diagonal operators has deep consequences for the
connection to nuclear operators. Let $0 < u \leq \infty$ and $1 \leq s, t \leq \infty$ with
$1 + 1/u \geq 1/s + 1/t$. Then an operator $T : X \to Y$ between Banach spaces $X$
and $Y$ is called $(u, s, t)$-nuclear (shorthand: $T \in \mathcal{N}_{u, s, t}(X, Y)$) if $T$ factorizes
through a diagonal operator $D_{\sigma} : \ell_t \to \ell_s$ with $\sigma \in \ell_u$ if $u < \infty$, and $\sigma \in c_0$
if $u = \infty$ (see, e.g., [31, 18.1]). We start by recalling a useful inclusion result
related to Proposition 1.2 (see, e.g., [31, 18.1.5]).

\textbf{Proposition 5.4.} Let $0 < u_0 \leq u_1 \leq \infty$, $1 \leq s_0 \leq s_1 \leq \infty$ and
$1 \leq t_0 \leq t_1 \leq \infty$ with $1/s_0 + 1/t_0 - 1/u_0 \leq 1/s_1 + 1/t_1 - 1/u_1 \leq 1$. Then
\[ \mathcal{N}_{u_0, s_0, t_0} \subseteq \mathcal{N}_{u_1, s_1, t_1}. \]

\textbf{Proposition 5.5.} Let $1 \leq u, r \leq \infty$, and either $1 \leq s, t \leq 2$ or $2 \leq
s, t \leq \infty$ or $1/\min(s, t) - 1/u \leq 1/2$, and either $1 \leq p, q \leq 2$ or $2 \leq p, q \leq \infty$
or $1/r - 1/\max(p, q) \leq 1/2$, and $0 \leq 1/p + 1/q - 1/r \leq 1$. Then
\[ \mathcal{N}_{u, s, t} \subseteq \Pi_{r, p, q} \]
whenever $1/s + 1/t - 1/u \leq 1/p + 1/q - 1/r$.

Proof. The case $1 \leq p, q \leq 2 \leq s, t \leq \infty$ follows from the corollary above
by definition.

Now let $1 \leq s, t \leq 2$. Choose $u \leq u_0 \leq \infty$ such that $1/u_0 = 1 + 1/u -
1/s - 1/t$, i.e., $1/s + 1/t - 1/u = 1/2 + 1/2 - 1/u_0$. Then by Proposition 5.4
we have $\mathcal{N}_{u, s, t} \subseteq \mathcal{N}_{u_0, 2, 2}$. If $1 \leq s \leq 2 \leq \infty$ and $1/s - 1/u \leq 1/2$, then define
$u \leq u_0$ by $1/u_0 = 1/u + 1/2 - 1/s$. Proposition 5.4 then gives $\mathcal{N}_{u, s, t} \subseteq \mathcal{N}_{u_0, 2, t}$.
The case $1 \leq t \leq 2 \leq s \leq \infty$ goes similarly.

For $2 \leq p \leq q \leq \infty$, we let $r \leq r_0 \leq \infty$ be defined by $1/r_0 = 1 + 1/r -
1/p - 1/q$, i.e., $1/2 + 1/2 - 1/r_0 = 1/p + 1/q - 1/r$. Thus, Proposition 1.2
gives $\Pi_{r_0, 2, 2} \subseteq \Pi_{r, p, q}$. Now, if $1 \leq p \leq 2 \leq q \leq \infty$ and $1/r - 1/q \leq 1/2$,
then we define $r_0 \leq r$ by $1/r_0 = 1/r + 1/2 - 1/q$. Proposition 1.2 then gives
$\Pi_{r_0, p, 2} \subseteq \Pi_{r, p, q}$. The case $1 \leq q \leq 2 \leq p \leq \infty$ and $1/r - 1/p \leq 1/2$ goes
similarly.

Combining all these observations with the case $1 \leq p, q \leq 2 \leq s, t \leq \infty$
gives the claim. ■

Our exposition now culminates in the confirmation of a result of [5] for
absolutely $(r, p, q)$-summing operators on $\ell_2$ as well as of a related result for
$(u, s, t)$-nuclear operators on $\ell_2$ (see also [22, 2.7]). It also shows that the
above inclusion result in the case $1 \leq p, q \leq 2 \leq s, t \leq \infty$ is optimal.

\textbf{Corollary 5.6.} Let $1 \leq u, r \leq \infty$ and $1 \leq p, q \leq 2 \leq s, t \leq \infty$ with
$1/s + 1/t - 1/u = 1/p + 1/q - 1/r < 1$. Then
\[ N_{u,s,t}(\ell_2) = \Pi_{r,p,q}(\ell_2) = S_v, \]

where \( 1/v = 1 + 1/u - 1/s - 1/t = 1 + 1/r - 1/p - 1/q. \)

**Proof.** By Propositions 5.4, 1.2 and 5.5,

\[ N_{v,2,2} \subseteq N_{u,s,t} \subseteq \Pi_{r,p,q} \subseteq \Pi_{v,2,2}. \]

Thus, by [31, 17.5.2, 18.5.4],

\[ S_v = N_{v,2,2}(\ell_2) \subseteq N_{u,s,t}(\ell_2) \subseteq \Pi_{r,p,q}(\ell_2) \subseteq \Pi_{v,2,2}(\ell_2) = S_v, \]

which gives the claim. \( \blacksquare \)

6. **Inclusions between arbitrary sequence spaces.** In this section we need to extend the definition of absolutely \((r, p, q)\)-summing operators. For technical reasons we will only consider the case \( p = q = 2 \), and the \( r \)-norm replaced by a sequence space norm.

We refer to [26] for all notation and information on symmetric Banach sequence spaces and recall only briefly the notions needed here. For a symmetric Banach sequence space \( E \), its fundamental sequence \((\lambda_E(n))\) is defined by \( \lambda_E(n) := \|\sum_{i=1}^{n} e_i\|_E \), where \( e_i \) is the \( i \)-th standard unit vector. The span of the first \( n \) standard unit vectors, equipped with the norm induced by \( E \), is denoted by \( E^n \). If \( E^\times \) denotes the Köthe dual of \( E \), then \( \lambda_{E^\times}(n) = n/\lambda_E(n) \). For two symmetric Banach sequence spaces \( E \) and \( F \), we define the space of multipliers \( M(E, F) \) by

\[ M(E, F) := \{ \lambda \in \ell_\infty; \lambda \mu \in F \text{ for all } \mu \in E \}, \]

equipped with the norm \( \|\lambda\|_{M(E,F)} := \sup_{\|\mu\|_E \leq 1} \|\lambda \mu\|_F \). If \( E \) is 2-concave and \( F \) is 2-convex (for these notions, we refer to [26]), then the following hold (see, e.g., [18, 2.1]):

(6.1) \[ \|\text{id}_E : \ell_2^n \hookrightarrow E^n\| \asymp \lambda_E(n)/\sqrt{n}; \]

(6.2) \[ \|\text{id} : E^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_F(n); \]

(6.3) \[ \lambda_{M(F,E)}(n) \asymp \lambda_E(n)/\lambda_F(n). \]

For a symmetric Banach sequence space \( E \) we denote by \( S_E \) the Banach space of all compact operators \( T : \ell_2 \to \ell_2 \) for which the sequence of singular numbers is contained in \( E \), equipped with the norm \( \|T\|_{S_E} := \|\sum_{i=1}^\infty s_i(T)e_i\|_E \). By \( S_E^n \) we denote the space \( \mathcal{L}(\ell_2^n) \) equipped with the norm \( \|T\|_{S_E^n} := \|\sum_{i=1}^n s_i(T)e_i\|_E \).

Let \( E \) be a maximal symmetric Banach sequence space. We call an operator \( T : X \to Y \) between Banach spaces \( X \) and \( Y \) absolutely \((E, 2, 2)\)-summing if there exists a constant \( C > 0 \) such that for any choice of \( x_1, \ldots, x_n \in X \)
and $y_1', \ldots, y_n' \in Y'$, the inequality
\[
\left\| \sum_{k=1}^{n} y_k'(Tx_k)e_k \right\|_E \leq C \sup_{x' \in B_{X'}} \left( \sum_{k=1}^{n} |x'(x_k)|^2 \right)^{1/2} \sup_{y' \in B_{Y'}} \left( \sum_{k=1}^{n} |y_k'(y)|^2 \right)^{1/2}
\]
holds. We put $\pi_{E,2,2}(T) := \inf C$ with $C$ as above. In this way, we obtain the maximal Banach operator ideal $(\Pi_{E,2,2}, \pi_{E,2,2})$.

**Lemma 6.1.** Let $E_0$ and $E_1$ be symmetric Banach sequence spaces and $\mathcal{F}$ an exact interpolation functor. Then
\[
\mathcal{F}(\Pi_{E_0,2,2}(X,Y), \Pi_{E_1,2,2}(X,Y)) \subseteq \Pi_{\mathcal{F}(E_0,E_1),2,2}(X,Y)
\]
for any fixed pair of Banach spaces $X$ and $Y$.

**Proof.** Fix $x_1, \ldots, x_n \in X$, $y_1', \ldots, y_n' \in Y'$. For $T \in \mathcal{L}(X,Y)$ consider the mapping $\psi_n(T) := (y_1'(Tx_1), \ldots, y_n'(Tx_n))$. Then by definition
\[
\|\psi_n : \Pi_{E_i,2,2}(X,Y) \rightarrow E_i\| \leq w_2(x_k)w_2(y_k), \quad i = 0, 1.
\]
Thus, interpolation and the definition give the claim. ■

The following result for operators on a Hilbert space is an extension of [31, 17.5.2].

**Proposition 6.2.** Let $E$ be a maximal symmetric Banach sequence space such that $E \neq \ell_\infty$. Then $\Pi_{E,2,2}(\ell_2) = S_E$. Moreover, $\Pi_{E,2,2}$ is the largest Banach operator ideal extending $S_E$ to the class of all Banach spaces.

**Proof.** By Mityagin [30] (see also [24, 1.b.10]) there exists an exact interpolation functor $\mathcal{F}$ such that $E = \mathcal{F}(\ell_1, \ell_\infty)$. Since $S_1 = \Pi_{1,2,2}(\ell_2)$ and $S_\infty \subseteq \Pi_{\infty,2,2}(\ell_2)$, the above lemma together with [1] yields
\[
S_E = \mathcal{F}(S_1, S_\infty) \subseteq \mathcal{F}(\Pi_{1,2,2}(\ell_2), \Pi_{\infty,2,2}) \subseteq \Pi_{E,2,2}(\ell_2).
\]
Conversely, we have $\pi_{E,2,2}(\text{id}_{\ell_2}^n) \geq \lambda_E(n)$. Thus, $\text{id}_{\ell_2} \notin \Pi_{E,2,2}$. Now proceed as in [31, 17.5.2] to obtain $\Pi_{E,2,2}(\ell_2) \subseteq S_E$. For the last part, note that by [31, 15.6] an operator $T : X \rightarrow Y$ belongs to the largest extension of $S_E$ whenever $RTS \in S_E$ for all $S \in \mathcal{L}(\ell_2, X)$ and $R \in \mathcal{L}(Y, \ell_2)$. By the definition of $\Pi_{E,2,2}$ it follows that such an operator $T$ also belongs to $\Pi_{E,2,2}$. Since $\Pi_{E,2,2}(\ell_2) = S_E$ by the above, the claim follows. ■

We now focus again on inclusion maps. As before, we denote for simplicity by $\text{id}_{E,F}$ the identity map $id : E \hookrightarrow F$ whenever $E$ and $F$ are symmetric Banach sequence spaces such that $E$ is continuously embedded in $F$. If $E = \ell_p$ ($F = \ell_p$, respectively), we write $\text{id}_{E,F}$ ($\text{id}_{E,p}$, respectively) instead of $\text{id}_{\ell_p,F}$ ($\text{id}_{\ell_p,p}$, respectively).

**Lemma 6.3.** Let $E$ and $F$ be symmetric Banach sequence spaces both not isomorphic to $\ell_2$ such that $E$ is $2$-concave, and $F$ is maximal and $2$-convex. Then $S_M(F,\ell_2) \circ S_M(\ell_2,E) \subseteq S_M(F,E)$.
Proof. Simply imitate the first part of the proof of [19, 6.3].

This now gives the following more general result.

**Proposition 6.4.** Let $E$ and $F$ be symmetric Banach sequence spaces such that $E$ is 2-concave, and $F'$ is maximal and 2-convex. Then $\text{id}_{EF} \in \Pi(M(F,E),2,2)$.

Proof. If $F = \ell_2$, then by [13] it is known that $\text{id}_{E2} \in \Pi(M(\ell_2,E),2,2)$. If $E = \ell_2$, then $\text{id}_{F^*2} \in \Pi(M(\ell_2,F^*),2) \subset \Pi(M(F,\ell_2),2,2)$, hence by duality also $\text{id}_{2F} \in \Pi(M(F,\ell_2),2,2)$. Thus assume that both spaces involved are not isomorphic to $\ell_2$. The proof is then only a slight modification of one in [15], but we give the details for the convenience of the reader. It is sufficient to show that $R \circ \text{id}_{EF} \circ S \in S(M(F,E))$ whenever $R \in L(F,\ell_2)$ and $S \in L(\ell_2,E)$. By [13] it is known that $\text{id}_{E2} \in \Pi(M(\ell_2,E),2)$, thus $\text{id}_{E2} \circ S \in \Pi(M(\ell_2,E),2)(\ell_2) = S(M(\ell_2,E))$. Similarly, $\text{id}_{F^*2} \circ R' \in S(M(\ell_2,F^*))$. Hence, $R \circ \text{id}_{EF} \circ S = R \circ \text{id}_{2F} \circ \text{id}_{E2} \circ S \in S(M(F,E))$, which gives the claim.

This result is best possible in the following sense: Let $G$ be a symmetric Banach sequence space such that $\text{id}_{EF} \in \Pi(G,2,2)$, where $E$ and $F$ are as above. Then $\lambda_G(n) \prec \lambda_{M(F,E)}(n)$. Indeed, as in the proof of Theorem 4.2 and with the help of (6.1)–(6.3), we deduce that

$$\pi_{G,2,2}(\text{id}_{EF}) \geq \frac{\lambda_G(n)}{||\text{id}_{E2}|| \ ||\text{id}_{F^2}||} \geq \frac{\lambda_E(n)}{\lambda_{F'}(n)} \geq \frac{\lambda_G(n)}{\lambda_{M(F,E)}(n)}.$$ 

Clearly, the above result includes the case $u = 1 \leq u \leq 2 \leq v \leq \infty$. The case $1 \leq u < v < 2$ or $2 < u < v \leq \infty$ turned out to be more complicated, which is also the case in this more general setting.

**Proposition 6.5.** Let $E$ and $F$ be 2-concave symmetric Banach sequence spaces and $F$ an exact interpolation functor such that

$$\sup_{n,m} ||L(\ell_2^m, E^n) \hookrightarrow F(L(\ell_2^m, \ell_1^n), L(\ell_2^m, F^n))|| < \infty.$$ 

Then $\text{id}_{EF} \in \Pi(F(\ell_2,\ell_\infty),2,2)$.

Proof. Fix $y'_1, \ldots, y'_m \in F^m$ and consider the mappings

$$\psi_{n,m} : K^m \otimes K^n \rightarrow K^m, \quad \psi_{n,m}(x_1, \ldots, x_m) := (y'_1(x_1), \ldots, y'_m(x_m)).$$

Since $\text{id}_{11} \in \Pi(2,2,2)$, it follows that $\text{id}_{1F} \in \Pi(2,2,2)$, thus

$$||\psi_{n,m} : L(\ell_2^m, \ell_1^n) \rightarrow \ell_2^m|| \leq C w_2(y'_i)$$

for some $C > 0$ independent of $n$ and $m$. Trivially, $\text{id}_{FF} \in \Pi(\ell_\infty,2,2)$ with norm equal to 1, that is,

$$||\psi_{n,m} : L(\ell_2^m, F^n) \rightarrow \ell_\infty^m|| \leq w_2(y'_i).$$
Then the assumption and interpolation give
\[ \|\psi_{n,m} : \mathcal{L}(\ell_2^n, E^n) \to F(\ell_2^n, \ell_\infty^n)\| \leq C w_2(y'_i), \]
where \( C > 0 \) is some other constant independent of \( n \) and \( m \). This finishes the proof. 

We refer the reader to [26] for the proper definition of Lorentz and Orlicz sequence spaces.

**Corollary 6.6.**

(i) Let \( 1 \leq u < v < 2 \). Then \( \text{id}_{uv} \in \Pi_{r,2,2} \), where \( 1/r = (v'/2)(1/u - 1/v) \).

(ii) Let \( 1 < p < r < 2 \) and \( 1 \leq q, s \leq 2 \). Then \( \text{id}_{l_{p,q}l_{r,s}} \in \Pi_{t,\tilde{q},2,2} \), where \( 1/t = (r'/2)(1/p - 1/r) \) and \( 1/\tilde{q} = 1/q - 1/2 \).

(iii) Let \( \varphi \) and \( \psi \) be Orlicz functions such that the functions \( t \mapsto \varphi(\sqrt{t}) \) and \( t \mapsto \psi(\sqrt{t}) \) are equivalent to concave functions. If \( \varphi^{-1}(t) = t \varrho(\psi^{-1}(t)/t) \) for some continuous and concave function \( \varrho : [0, \infty) \to [0, \infty) \) which is positive on \( (0, \infty) \), then \( \text{id}_{\varphi, \psi} \in \Pi_{\lambda,2,2} \), where \( \lambda^{-1}(t) = t^{1/2} \varrho(t^{-1/2}) \).

**Proof.** (i) This is already included in Proposition 3.4(i).

(ii) In [28, 2.1] it was shown that under the assumptions above,
\[ \sup_{n,m} \| \mathcal{L}(\ell_2^m, \ell_{p,q}^n) \to (\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_r^n))_\theta,\tilde{q} \| < \infty, \]
where \( \theta = r'/p' \). A quick inspection of the proof shows that \( \ell_p^n \) can be replaced by \( \ell_{r,s}^n \). Thus, the above proposition applies with the interpolation functor \( \mathcal{F} = (\cdot, \cdot)_\theta,\tilde{q} \). Furthermore, \( (\ell_2, \ell_\infty)_{\theta,\tilde{q}} = \ell_{t,\tilde{q}} \), which gives the claim.

(iii) The assumptions on \( \varphi \) and \( \psi \) ensure that \( \ell_\varphi \) and \( \ell_\psi \) are 2-concave (see, e.g., [23]). Let \( \varrho \) be the lower Orlicz-Minkov functor associated to \( \varrho \) (see, e.g., [14] for more details and references). Then (see, e.g., [27, p. 179]) we have \( \varrho_{\ell}(\ell_1, \ell_\psi) = \ell_\varphi \), and by [14, Proposition 3],
\[ \sup_{n,m} \| \mathcal{L}(\ell_2^m, \ell_{\varphi}^n) \to \varrho_{\ell}(\mathcal{L}(\ell_2^m, \ell_{\varphi}^n), \mathcal{L}(\ell_2^m, \ell_{\psi}^n)) \| < \infty. \]
Thus, the above proposition applies, and \( \varrho_{\ell}(\ell_2, \ell_\infty) = \ell_\lambda \) (see, e.g., [27, p. 178]), which gives the claim. 

7. Inclusions between finite-dimensional Schatten classes. We finally consider inclusions \( \text{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n \), where \( E \) and \( F \) are symmetric Banach sequence spaces. Since both unitary ideals involved contain \( \ell_2^n \), it follows that
\[ \pi_{r,p,q}(\text{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \geq \pi_{r,p,q}(\text{id}_{22}^n) = n^{\max(0,1+1/r-1/p-1/q)} \]
and
\[ \pi_{G,2,2}(\text{id} : \mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \geq \pi_{G,2,2}(\text{id}_{22}^n) = \lambda_G(n) \]
for all $1 \leq p, q \leq 2$, $1 \leq r \leq \infty$ and every symmetric Banach sequence space $G$. To give an analogue of Corollary 3.3, we need some more interpolation formulas.

**Proposition 7.1.** Let $1 \leq p_0, p_1, q_0, q_1 \leq 2$, $0 < \theta < 1$ and $1 \leq p, q \leq 2$ with $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $1/q = (1 - \theta)/q_0 + \theta/q_1$. Then

$$\sup_{n,m} \|\ell^m_p \otimes \varepsilon S^n_q \mapsto [\ell^m_{p_0} \otimes \varepsilon S^n_{q_0}, \ell^m_{p_1} \otimes \varepsilon S^n_{q_1}]\varepsilon\| < \infty$$

and

$$\sup_{n,m} \|S^n_p \otimes \varepsilon S^m_q \mapsto [S^n_{p_0} \otimes \varepsilon S^m_{q_0}, S^n_{p_1} \otimes \varepsilon S^m_{q_1}]\varepsilon\| < \infty.$$

**Proof.** This follows from the cases $q_0 = q_1 = q = 2$ (Proposition 2.1) and $p_0 = p_1 = p = 2$ ([16, 4.3]) by applying [17, Lemma 9] together with Pisier's factorization theorem as in [17, p. 450].

**Lemma 7.2.** Let $1 \leq u_0, u_1 \leq 2 \leq v_0, v_1 \leq \infty$, $1 \leq r_0, r_1 \leq \infty$ and $1 \leq s_0, s_1, t_0, t_1 \leq 2$. Then for all $0 < \theta < 1$,

$$\pi_{r,s,t}(id : S^n_{u_0} \hookrightarrow S^n_{v_0}) \leq \pi_{r_0,s_0,t_0}(id : S^n_{u_0} \hookrightarrow S^n_{v_0})^{1-\theta} \pi_{r_1,s_1,t_1}(id : S^n_{u_1} \hookrightarrow S^n_{v_1})^{\theta},$$

where $1/u = (1-\theta)/u_0 + \theta/u_1$, $1/v = (1-\theta)/v_0 + \theta/v_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$, $1/s = (1-\theta)/s_0 + \theta/s_1$ and $1/t = (1-\theta)/t_0 + \theta/t_1$.

**Proof.** The proof goes along similar lines to the one of Lemma 2.3(i), using the above proposition.

As before, we have to verify some extreme cases.

**Lemma 7.3.** The following hold true:

(i) $\pi_{1,1,1}(id : S^n_2 \hookrightarrow S^n_2) = 1$;

(ii) $\pi_{1,2,2}(id : S^n_1 \hookrightarrow S^n_\infty) = n$;

(iii) $\pi_{1,2,1}(id : S^n_1 \hookrightarrow S^n_2) = \pi_{1,1,2}(id : S^n_2 \hookrightarrow S^n_\infty) \times \sqrt{n}$;

(iv) $\pi_{1,1,2}(id : S^n_1 \hookrightarrow S^n_2) = \pi_{1,2,1}(id : S^n_2 \hookrightarrow S^n_\infty) \times \sqrt{n}$.

**Proof.** (i) is clear as $\Pi_{1,1,1} = L$. Since $\pi_2(id : S^n_1 \hookrightarrow S^n_2) = \sqrt{n}$ (see, e.g., [16, 5.2]), we have

$$\sup_m \|id \otimes id : S^n_1 \otimes \varepsilon \ell^m_2 \rightarrow \ell^m_2\| = \sqrt{n}.$$  

(7.1)

Thus, by duality,

$$\sup_m \|id \otimes id : \ell^m_2 \rightarrow S^n_\infty \otimes \pi \ell^m_2\| = \sqrt{n}.$$  

Hence, by factorization,

$$\sup_m \|id \otimes id : \ell^m_2 \rightarrow S^n_\infty \otimes \pi \ell^m_2\| = n,$$

which means $\pi_{1,2,2}(id : S^n_1 \hookrightarrow S^n_\infty) = n$. Next, the identity map $id_{S_2}$ is absolutely $(2,1)$-summing, that is,

$$\sup_m \|id \otimes id : S^n_2 \otimes \varepsilon \ell^m_1 \rightarrow \ell^m_2\| < \infty.$$  

(7.2)
By duality and factorization, this together with (7.1) yields
\[
\sup \| \text{id} \otimes \text{id} : \mathcal{S}_1^n \otimes \ell_2^n \to \mathcal{S}_2^n \otimes \ell_\infty^n \| \asymp \sqrt{n},
\]
which gives \( \pi_{1,2,1}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,1,2}(\text{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n} \). Finally, since \( \Pi_1(X,Y) = \Pi_2(X,Y) \) whenever \( X \) is of cotype 2, we have \( \pi_1(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_2(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \sqrt{n} \). Thus,
\[
\sup \| \text{id} \otimes \text{id} : \mathcal{S}_1^n \otimes \ell_1^n \to \mathcal{S}_2^n \otimes \ell_\infty^n \| \asymp \sqrt{n},
\]
which by factorization gives
\[
\sup \| \text{id} \otimes \text{id} : \mathcal{S}_1^n \otimes \ell_1^n \to \mathcal{S}_2^n \otimes \ell_2^n \| \asymp \sqrt{n}.
\]
Hence, \( \pi_{1,1,2}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,2,1}(\text{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n} \), which finishes the proof.

**Proposition 7.4.** Let \( 1 \leq p, q \leq 2, 1 \leq r \leq \infty \) and \( 1 \leq u \leq 2 \leq v \leq \infty \) with \( 1/u - 1/v = 1 + 1/r - 1/p - 1/q \). Then
\[
\pi_{r,p,q}(\text{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/u-1/v}.
\]

**Proof.** As in the proof of Theorem 3.2, we fix \( 1 \leq q \leq 2 \) and define \( \bar{q} \) and \( \bar{q} \) accordingly. Using Lemmas 7.2 and 7.3, we arrive at the following six cases (ordered according to the proof of Theorem 3.2):

- \( \pi_{1,\bar{q},q}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,\bar{q},q}(\text{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{n} \);
- \( \pi_{1,1,q}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_{1,1,q}(\text{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp n^{1/q} \);
- \( \pi_{1,2,q}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \pi_{1,2,q}(\text{id} : \mathcal{S}_1^n \hookrightarrow \mathcal{S}_\infty^n) \asymp n^{1/q} \).

Then proceed by interpolation as in the proof of Theorem 3.2 to obtain the statement in the case \( r = 1 \). The general case then follows as in the proof of Corollary 3.3.

**Corollary 7.5.** Let \( 1 \leq p, q \leq 2 \) and \( 1 \leq r \leq \infty \) with \( 1/p + 1/q - 1/r \leq 1 \). Then for all \( 1 \leq u \leq 2 \leq v \leq \infty \),
\[
\pi_{r,p,q}(\text{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1+1/r-1/p-1/q} \pi_{r,p,q}(\text{id} : \ell_u^n \hookrightarrow \ell_v^n).
\]
Moreover, for \( 2 \leq r, s \leq \infty \),
\[
\| \text{id} \otimes \text{id} : \mathcal{S}_p^n \otimes \ell_{q}^n \to \mathcal{S}_r^n \otimes \ell_s^n \| \asymp n^{2-1/p-1/q+\max(0,2-1/p-1/q+1/r+1/s)}.
\]

**Proof.** Appropriate factorizations give the upper estimates. For the lower estimates observe first that
\[
\pi_{r,p,q}(\text{id} : \mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \geq \pi_{r,p,q}(\text{id} : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_2^n) = n^{1/t},
\]
where \( 1/t = 1 + 1/r - 1/p - 1/q \). By (8.1) and the lower estimate from Corollary 8.8 below,
\[
\pi_{r,p,q}(\text{id}: S^n_u \hookrightarrow S^n_v) \geq \pi_{t,2,2}(\text{id}: S^n_u \hookrightarrow S^n_v) \geq \pi_{t,2,2}(n^2)(\text{id}: S^n_u \hookrightarrow S^n_v) \\
\geq n^{2/t}h_{n^2}(\text{id}: S^n_u \hookrightarrow S^n_v) \\
\geq n^{2/t}n^{1/v-1/u} = n^{2+2/r-2/p-2/q+1/v-1/u}.
\]

This calculation also gives the last part of the statement on taking \(r = 1\).

Mathematical routine lets us formulate and prove an analogue of Corollary 4.4:

**Proposition 7.6.** Let \(1 \leq p, q \leq 2 \leq r, s \leq \infty\). Then
\[
||\text{id} \otimes \text{id} : S^n_p \otimes \varepsilon S^n_q \rightarrow S^n_r \otimes \pi S^n_s|| \asymp n||\text{id} \otimes \text{id} : \ell^n_p \otimes \varepsilon \ell^n_q \rightarrow \ell^n_r \otimes \pi \ell^n_s||.
\]

**Proof.** We have to show that
\[
||\text{id} \otimes \text{id} : S^n_p \otimes \varepsilon S^n_q \rightarrow S^n_r \otimes \pi S^n_s|| \asymp n^{1+\max(0,1-1/p-1/q+1/r+1/s)}.
\]

Again, we first establish the cases where the norm is asymptotically equivalent to \(n\)—note that this behaviour is best possible, since all spaces involved contain \(\ell^2_2\).

By [18, 11.4],
\[
||\text{id} \otimes \text{id} : S^n_1 \otimes \varepsilon S^n_2 \rightarrow S^n_2^2|| \asymp \sqrt{n}.
\]

Thus, by duality and factorization the following identities have norms asymptotically equivalent to \(n\):
\[
\text{id} \otimes \text{id} : S^n_1 \otimes \varepsilon S^n_2 \rightarrow S^n_\infty \otimes \pi S^n_2, \quad \text{id} \otimes \text{id} : S^n_1 \otimes \varepsilon S^n_2 \rightarrow S^n_2 \otimes \pi S^n_2. \quad \text{Furthermore, by [18, 11.3] we have}
\]
\[
||\text{id} \otimes \text{id} : S^n_1 \otimes \varepsilon S^n_2 \rightarrow S^n_2 \otimes \pi S^n_2|| \asymp n
\]
and, by duality,
\[
||\text{id} \otimes \text{id} : S^n_2 \otimes \varepsilon S^n_2 \rightarrow S^n_\infty \otimes \pi S^n_2|| \asymp n.
\]

Now an interpolation strategy similar to the one in the proof of Theorem 3.2 together with Proposition 7.1 establishes
\[
||\text{id} \otimes \text{id} : S^n_p \otimes \varepsilon S^n_q \rightarrow S^n_r \otimes \pi S^n_s|| \asymp n
\]
whenever \(1/p + 1/q - 1/r - 1/s = 1\). The upper estimates now follow by appropriate factorizations. For the lower ones, recall that by [10, p. 35] we have \(N(\text{id}_E) = \dim E\) for all finite-dimensional Banach spaces \(E\). Thus,
\[
N(\text{id} : S^n_r \hookrightarrow S^n_s) \geq \frac{n^2}{n^{1/r'-1/s}} = n^{1+1/r+1/s}.
\]

Hence,
\[
||\text{id} \otimes \text{id} : S^n_p \otimes \varepsilon S^n_q \rightarrow S^n_r \otimes \pi S^n_s|| \geq \frac{N(\text{id} : S^n_r \hookrightarrow S^n_s)}{||\text{id} : S^n_p \hookrightarrow S^n_q||} \geq n^{2-1/p-1/q+1/r+1/s},
\]
which together with the general lower bound \(n\) gives the claim.
To formulate an analogue of Proposition 6.4 causes some problems. So far, we are only able to state the following; the proof is similar to the one of Proposition 6.4. We leave the details to the reader.

**Proposition 7.7.** Let $E_0, E_1$ be 2-concave symmetric Banach sequence spaces such that

$$\pi_M(\ell_2, E_i), \mathbf{2}(\lambda : S^n_{E_i} \hookrightarrow S^n_2) \succeq \frac{\lambda_{E_i}(n)}{\sqrt{n}}, \quad i = 0, 1.$$  

Then

$$\pi_M(\ell_2^x, E_0), \mathbf{2}(\lambda : S^n_{E_0} \hookrightarrow S^n_{E_1^x}) \succeq \frac{\lambda_{E_0}(n)\lambda_{E_1}(n)}{n}.$$  

In [16, 5.3] the following examples of spaces satisfying (7.3) were given:

- $\ell_p$, where $1 \leq p \leq 2$;
- $\ell_{p,q}$, where $1 < p < 2$ and $1 \leq q \leq 2$;
- $\ell_\varphi$, where $\varphi(t)$ is a submultiplicative Orlicz function not equivalent to $t^2$ in a neighbourhood of zero, such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function in a neighbourhood of zero.

However, they also gave examples of Lorentz and Orlicz sequence spaces that are 2-concave but do not satisfy (7.3), which makes it impossible to state a more general result in the spirit of Proposition 6.4.

**8. Applications to Hilbert numbers.** We refer to [24] and [32] for the general theory of s-numbers of operators. For an operator $T : X \to Y$ between Banach spaces $X$ and $Y$ recall the definition of its $k$th approximation number

$$a_k(T) := \inf \{\|T - S\|; S \in \mathcal{L}(X, Y) \text{ with } \|S\| \leq 1 \text{ and rank } S < k\},$$

and its $k$th Hilbert number

$$h_k(T) := \sup \{a_k(RTS); R \in \mathcal{L}(Y, \ell_2), S \in \mathcal{L}(\ell_2, X), \|S\|, \|R\| \leq 1\}.$$  

It is clear from the definition that $a_1(T) \geq a_2(T) \geq \cdots \geq 0$ and $h_1(T) \geq h_2(T) \geq \cdots \geq 0$. Furthermore, for a compact operator between Hilbert spaces, the sequences of approximation and Hilbert numbers coincide with the sequence of singular numbers.

An important inequality due to König (see, e.g., [24, 2.a.3]) states that $k^{1/r} x_k(T) \leq \pi_{r,2}(T)$ for all $T \in \Pi_{r,2}$, where $x_k(T)$ denotes the $k$th Weyl number of $T$ (see, e.g., [32]). We now provide an analogue for Hilbert numbers and $(E, 2, 2)$-summing operators.

For an operator $T$ denote by $\pi_{E,2,2}^{(k)}(T)$ the $(E, 2, 2)$-summing norm of $T$ computed with at most $k$ vectors $x_1, \ldots, x_k$ and $k$ vectors $y_1', \ldots, y_k'$.
Proposition 8.1. Let $E$ be a maximal symmetric sequence space. Then

\[(8.1)\quad \lambda_E(k)h_k(T) \leq \pi_{E,2,2}^{(k)}(T)\]

for all operators $T \in \mathcal{L}$.

Proof. Let $T \in \mathcal{L}(X, Y)$ where $X$ and $Y$ are Banach spaces. By Bauhardt’s characterization of Hilbert numbers (see, e.g., [31, 11.4.3]) there exist operators $S : \ell^k_2 \to X$ and $R : Y \to \ell^k_2$ such that $\|S\|, \|R\| \leq 1$ and

\[RTS = (1 + \varepsilon)^{-1}h_k(T)\text{id}_{\ell^k_2}^k.\]

Equivalently, this means that there exist $x_1, \ldots, x_k \in X$ and $y'_1, \ldots, y'_k \in Y'$ such that $w_2(x_i) \leq 1$, $w_2(y'_i) \leq 1$ and $y'_i(Tx_i) = (1 + \varepsilon)^{-1}h_k(T)$, $i = 1, \ldots, k$. Then by the definition of $\pi_{E,2,2}^{(k)}(T)$,

\[(1 + \varepsilon)^{-1}h_k(T)\lambda_E(k) = \left\| \sum_{i=1}^{k} |y'_i(Tx_i)|e_i \right\|_E \leq \pi_{E,2,2}^{(k)}(T),\]

which gives the claim. □

For a symmetric Banach sequence space $E$, denote by $\lambda(E)$ and $m(E)$ the Lorentz and Marcinkiewicz spaces associated to the fundamental function $\lambda_E$ of $E$, respectively, in the sense of [12, p. 59]. Furthermore, for a scale $s$ of $s$-numbers in the sense of [32] and a symmetric Banach sequence space $F$, we define $\mathcal{L}^s_F$ to be the class of all operators $T$ between Banach spaces such that $(s_n(T)) \in F$, equipped with the norm $s_F(T) := \|(s_n(T))\|_F$, $T \in \mathcal{L}^s_F$. In [12, 3.1] the authors proved the following:

For every symmetric Banach sequence space $E$ such that $\ell_2 \hookrightarrow E$, we have $\Pi_{E,2} \hookrightarrow \mathcal{L}^x_{m(E)}$. If in addition $E$ is an interpolation space with respect to the couple $(\ell_2, \ell_\infty)$, then $\mathcal{L}^x_{\lambda(E)} \hookrightarrow \Pi_{E,2}$.

Moreover, for $r > 2$ Pietsch [32, 27.5] showed that $\mathcal{L}_r \subseteq \Pi_{r,2} \subseteq \mathcal{L}_{r,\infty}$. The above proposition together with Proposition 6.2 now yields the following analogue for the scale of Hilbert numbers and $(E, 2, 2)$-summing operators:

Corollary 8.2. Let $E \neq \ell_\infty$ be a maximal symmetric Banach sequence space. Then

\[\mathcal{L}^h_E \hookrightarrow \Pi_{E,2,2} \hookrightarrow \mathcal{L}^h_{m(E)}.\]

A first application to inclusion maps is the following:

Proposition 8.3. Let $E$ and $F$ be symmetric Banach sequence spaces such that $E$ is $2$-concave, and $F$ is $2$-convex and maximal. Then

\[h_k(\text{id}_{EF}) \lesssim \frac{\lambda_F(k)}{\lambda_E(k)}.\]
Proof. We have \( h_k(id_{22}^k) = 1 \), hence by factorization,
\[
h_k(id_{EF}) \geq h_k(id_{EF}) \geq \frac{||id_{2E}^k||}{||id_{F2}^k||} \leq \frac{\lambda_F(k)}{\lambda_E(k)}.
\]
Conversely, by Proposition 6.4 we know that \( id_{EF} \in \Pi_{M(F,E),2,2} \). Thus, the proposition above gives \( h_k(id_{EF}) \leq 1/\lambda_M(F,E)(k) \). By [18], we have \( \lambda_M(F,E)(k) \approx \lambda_E(k)/\lambda_F(k) \), which gives the claim. \( \blacksquare \)

Now the above and the results from the previous section give the following examples. We guess that (i) is already known; however, we have not found a source where it is written up in this form.

**Corollary 8.4.**

(i) **Let** \( 1 \leq u \leq v \leq \infty \). **Then**
\[
h_k(id_{uv}) \geq \begin{cases} k^{(v'/2)(1/v-1/u)}, & 1 \leq u < v < 2; \\ k^{(u/2)(1/v-1/u)}, & 2 \leq u < v \leq \infty; \\ k^{1/v-1/u}, & 1 \leq u \leq 2 \leq v \leq \infty; \\ k^{-1/2}, & u = v = 1 \text{ or } u = v = \infty; \\ 1, & 1 < u = v < \infty. \end{cases}
\]

(ii) **Let** \( 1 < p \leq r < \infty \) **and** \( 1 \leq q, s \leq \infty \). **Then**
\[
h_k(id_{p,q}) \geq \begin{cases} k^{(r'/2)(1/r-1/p)}, & 1 < p < r < 2, p \leq q \leq 2 \text{ and } 1 \leq s \leq r; \\ k^{(p/2)(1/r-1/p)}, & 2 < p < r < \infty, 2 \leq q \leq p \text{ and } r \leq s \leq \infty; \\ k^{1/r-1/p}, & 1 < p < 2 < r \leq \infty \text{ and } 1 \leq q \leq 2 \leq s \leq \infty; \\ 1, & p = r \text{ and } 1 < q = s < \infty. \end{cases}
\]

(iii) **Let** \( 1 < p < 2 \leq q < \infty \) **and** \( w \) **be a Lorentz sequence such that** \( n_{2'}(2-p) \leq \sum_{i=1}^n w_i^{2/(2-p)} \). **Then**
\[
h_k(id_{d(w,p)d(w,q)}) \geq (kw_k)^{1/q-1/p}.
\]

(iv) **Let** \( \varphi \) **and** \( \psi \) **be Orlicz functions such that** \( t \mapsto \varphi(\sqrt{t}) \) **and** \( t \mapsto \sqrt{\psi(t)} \) **are equivalent to concave and convex functions, respectively, and** \( \psi \) **satisfies the** \( \Delta_2 \)-**condition. **Then**
\[
h_k(id_{\varphi}) \geq \frac{\varphi^{-1}(1/k)}{\psi^{-1}(1/k)}.
\]

(v) **Let** \( \varphi \) **and** \( \psi \) **be Orlicz functions such that** \( t \mapsto \varphi(\sqrt{t}) \) **and** \( t \mapsto \psi(\sqrt{t}) \) **are equivalent to concave functions, respectively. If** \( \varphi^{-1}(t) = t\varphi(\psi^{-1}(t)/t) \) **for some continuous and concave function** \( \varphi : [0, \infty) \rightarrow [0, \infty) \text{ which is positive on } (0, \infty), \text{ then} \)
\[
h_k(id_{\varphi}) \geq \frac{\varphi(k^{1/2})}{k^{1/2}}.
\]
Proof. (i) The case $1 \leq u \leq 2 \leq v \leq \infty$ is contained in the above proposition. Now let $1 \leq u < v < 2$. Then the upper estimate follows from (8.1) together with Corollary 6.6(i). For the lower estimate, choose $m \in \mathbb{N}$ such that $m^{2/v'}/2 \leq k \leq m^{2/v'}$. Now [20, Proposition (2)] gives

$$h_k(\text{id}_{uv}) \geq h_k(\text{id}_{uv}^m) \geq m^{1/v-1/u} \geq 2^{-1}k(v'/2)(1/v-1/u),$$

which gives the lower estimate. The case $2 < u < v \leq \infty$ then follows by duality. Since $\text{id}_{11} \in \Pi_{2,2,2}$, we have $h_k(\text{id}_{11}) \sim k^{-1/2}$ by (8.1); the lower estimate follows by factorizing $\text{id}_{uv}^m$ through $\text{id}_{11}$ (see also [32, 2.9.19]). The claim for $\text{id}_{\infty\infty}$ then follows by duality. Finally, any $K$-convex infinite-dimensional Banach space (for this notion see, e.g., [19]) contains a complemented copy of $\ell_2^k$ (see, e.g., [19, 19.3]). Thus, $h_k(\text{id}_X) \sim 1$ for any $K$-convex infinite-dimensional Banach space $X$, in particular for $X = \ell_1$, $1 < u < \infty$.

(ii) This follows as in (i) together with Corollary 6.6(ii)—note that $\lambda_{\ell_{1,q}}(k) \asymp k^{1/2}$—and the lower estimate for $h_k(\text{id}_{pr})$ in (i).

(iii) The assumption on $w$ implies that $d(w,p)$ is 2-concave (see, e.g., [33]), and $d(w,q)$ for $q \geq 2$ is always 2-convex (and maximal). Thus, the above proposition gives the claim, if we take into account that $\lambda_{d(w,r)}(k) \asymp (kw_k)^{1/r}$ for any $1 < r < \infty$.

(iv) The assumptions ensure that $\ell_\varphi$ is 2-concave and that $\ell_\psi$ is 2-convex and maximal. Hence, the claim follows from the proposition above—note that $\lambda_{\ell_{\varphi}}(k) \asymp 1/\varphi^{-1}(1/k)$ for any Orlicz sequence space $\ell_\varphi$.

(v) This follows from (8.1) together with Corollary 6.6(iii).

We now show that one can even obtain all asymptotically exact upper estimates for the Hilbert numbers of the finite-dimensional inclusions $\text{id}_{uv}^n$ by using (8.1). The lower ones can be found in [20]. Note that the case $1 \leq v < u' \leq \infty$ follows from the one below by the duality of Hilbert numbers.

**Proposition 8.5.** Let $1 \leq u' \leq v \leq \infty$ and $1 \leq k \leq n$. Then

$$h_k(\text{id}_{uv}^n) \asymp \begin{cases} 
\min(n^{1/v-1/u}, n^{1/vk^{-1/2}}, nk^{-1}), & 1 \leq u' \leq v \leq 2, \\
\min(n^{1/v-1/u}, n^{1/vk^{-1/2}}), & 2 \leq v \leq u \leq \infty, \\
\min(ku/(1/v-1/u), n^{1/vk^{-1/2}}), & 2 \leq u < v \leq \infty, \\
k^{1/v-1/u}, & 2 \leq u' \leq \infty.
\end{cases}$$

**Proof.** Let $1 \leq u' \leq v \leq 2$. Then $\pi_{2,2,2}(\text{id}_{uv}^n) \asymp n^{1/v}$, hence $h_k(\text{id}_{uv}^n) \asymp n^{1/vk^{-1/2}}$. Moreover, $\pi_{1,2,2}(\text{id}_{uv}^n) \asymp n$, which gives $h_k(\text{id}_{uv}^n) \asymp nk^{-1}$. Finally, by the monotonicity of Hilbert numbers, $h_k(\text{id}_{uv}^n) \leq h_1(\text{id}_{uv}^n) = n^{1/v-1/u}$.

Let $2 \leq u, v \leq \infty$. Then $\pi_{2,2,2}(\text{id}_{uv}^n) \asymp n^{1/v}$, and therefore $h_k(\text{id}_{uv}^n) \asymp n^{1/vk^{-1/2}}$. If $v \leq u$, then as before $h_k(\text{id}_{uv}^n) \leq h_1(\text{id}_{uv}^n) = n^{1/v-1/u}$. If $u < v$, then $h_k(\text{id}_{uv}^n) \leq h_k(\text{id}_{uv}^n) \asymp k(u/2)^{1/v-1/u}$ as already seen in the above corollary. The lower estimate in this special case also follows similarly to the
above: if $k \leq n^{2/u}$, then choose $1 \leq \ell \leq m$ such that $\ell^{2/u}/2 \leq k \leq \ell^{2/u}$. Then again [20, Proposition (2)] gives

$$h_k(id_{uv}^n) \geq h_k(id_{uv}^\ell) \sim \ell^{1/v-1/u} \geq 2^{-1}k^{(u/2)(1/v-1/u)}.$$

The estimate $h_k(id_{uv}^n) \sim n^{1/v}k^{-1/2}$ for all $n^{2/u} \leq k \leq m$ from [20, Proposition (2)] gives the remaining lower estimate in this case.

Let $2 \leq u' \leq v \leq \infty$. Then $id_{uv} \in II_{r,2,2}$, where $1/r = 1/u - 1/v$. Hence, $h_k(id_{uv}^n) \leq k^{1/v-1/u}$. \[\square\]

We finish with inclusions between finite-dimensional Schatten classes. Since $\ell_2^n$ is contained in both spaces involved, $h_k(id : S_E^n \hookrightarrow S_F^n) = 1$ whenever $E$ is continuously embedded into $F$, and $1 \leq k \leq n$. Proposition 7.7 together with (8.1) gives the following upper estimate:

**Proposition 8.6.** Let $E_0, E_1$ be 2-concave symmetric Banach sequence spaces satisfying (7.3). Then for $n \leq k \leq n^2$,

$$h_k(id : S_{E_0}^n \hookrightarrow S_{E_1}^n) \sim \frac{k \lambda_{E_0}(n)\lambda_{E_1}(n)}{n \lambda_{E_0}(k)\lambda_{E_1}(k)}.$$

The situation for the lower estimate is more satisfactory. Here, for a symmetric Banach sequence space $G$ let $\lambda_G : [1, \infty) \to [1, \infty)$ be a monotone function extending $\lambda_G : \mathbb{N} \to [1, \infty)$.

**Proposition 8.7.** Let $E$ and $F$ be symmetric Banach sequence spaces such that $E$ is 2-concave and $F$ is 2-convex. Then for $n \leq k \leq n^2$,

$$h_k(id : S_E^n \hookrightarrow S_F^n) \geq \frac{\lambda_F(k/n)}{\lambda_E(k/n)}.$$

**Proof.** We proceed similarly to the proof of [21, 4.2]. Choose $1 \leq h \leq n$ such that $nh - 1 \leq k \leq nh$. Identify $L(\ell_2^n, \ell_2^n)$ and $L(\ell_2^n, \ell_2^n)$ with the sets of all $n \times h$-matrices and $n \times n$-matrices, respectively. Furthermore, denote the space $L(\ell_2^n, \ell_2^n)$ equipped with the Hilbert–Schmidt norm by $S_2(\ell_2^n, \ell_2^n)$, and define $S_F(\ell_2^n, \ell_2^n)$ likewise. Clearly, the natural injection $i_{2E} : S_2(\ell_2^n, \ell_2^n) \hookrightarrow S_E^n$ has norm asymptotically equivalent to $\lambda_E(h)/\sqrt{h}$. Now let $P_{F_2} : S_F^n \to S_2(\ell_3^n, \ell_3^n)$ be the natural projection which cuts off the last $n-h+1$ rows. Observe that any matrix in $L(\ell_2^n, \ell_2^n)$ has at most $h$ nonzero singular values. Since $s_k(P_{F_2}A) \leq s_k(A)$ for all $A \in L(\ell_2^n)$, we have $\|P_{F_2} : S_F^n \hookrightarrow S_2(\ell_2^n, \ell_2^n)\| \leq 1$. Hence, for $A \in L(\ell_2^n)$ and $\sigma = (\sigma_1, \ldots, \sigma_\ell)$ the nonzero singular values of $P_{F_2}A$, with $\ell \leq h$,

$$\|P_{F_2}A|S_2(\ell_2^n, \ell_2^n)\| = \|\sigma\|_2 \leq \|id_{F_2}\|\|\sigma\|_F \leq \frac{\lambda_F(h)}{\sqrt{h}} \|A|S_F^n\|.$$

Thus, since $id_{S_2(\ell_2^n, \ell_2^n)} = P_{F_2} \circ (id : S_E^n \hookrightarrow S_F^n) \circ i_{2E}$, it follows by the definition
of $h_k$ that

$$h_k(\text{id} : S^n_{\ell_E} \hookrightarrow S^n_{\ell_F}) \geq \frac{a_k(\text{id} S_2(\ell_2^\ast, \ell_2^\ast))}{\lambda_E(h)/\lambda_F(h)} = \frac{\lambda_F(h)}{\lambda_E(h)} \geq \frac{1}{2} \frac{\lambda_F(k/n)}{\lambda_E(k/n)}.$$

which gives the desired estimate. ■

All the above together now gives the following examples. As usual, we set $S^n_{\ell_{p,q}} := S^n_{\ell_{p,q}}$.

**Corollary 8.8.** Let $n \leq k \leq n^2$.

(i) Let $1 \leq u \leq 2 \leq v \leq \infty$. Then

$$h_k(\text{id} : S^n_u \hookrightarrow S^n_v) \preccurlyeq (n/k)^{1/u-1/v}.$$

(ii) Let $1 < u < 2 < v < \infty$ and $1 \leq r \leq 2 \leq s \leq \infty$. Then

$$h_k(\text{id} : S^n_{u,r} \hookrightarrow S^n_{v,s}) \preccurlyeq (n/k)^{1/u-1/v}.$$

**References**


Absolutely \((r, p, q)\)-summing inclusions


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Received February 9, 2006
Revised version September 25, 2006 (5858)