On upper and lower bounds of the numerical radius
and an equality condition

by

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Abstract. We give an inequality relating the operator norm of $T$ and the numerical radii of $T$ and its Aluthge transform. It is a more precise estimate of the numerical radius than Kittaneh’s result [Studia Math. 158 (2003)]. Then we obtain an equivalent condition for the numerical radius to be equal to half the operator norm.

1. Introduction. For a bounded linear operator $T$ on a complex Hilbert space $H$, we denote the operator norm and the numerical radius of $T$ by $\|T\|$ and $w(T)$, respectively. It is well known that $w(T)$ is an equivalent norm of $T$, since (see [5, Theorem 1.3-1])

\begin{equation}
\frac{1}{2} \|T\| \leq w(T) \leq \|T\|.
\end{equation}

Concerning the second inequality, Kittaneh [8] has shown the following precise estimate of $w(T)$ by using several norm inequalities and ingenious techniques:

\begin{equation}
w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} \|T^2\|^{1/2}.
\end{equation}

Obviously, (1.2) is sharper than the right inequality of (1.1). We remark that we cannot compare $w(T)$ with $\|T^2\|^{1/2}$, generally. In fact, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then $0 = \|T^2\|^{1/2} < w(T) = 1/2$; but if

\[ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]

then $1/\sqrt{2} = w(T) < \|T^2\|^{1/2} = 1$.

We obtain a sufficient condition for $w(T) = \frac{1}{2} \|T\|$ to hold from (1.1), (1.2) and [8]: if $T^2 = 0$, then $w(T) = \frac{1}{2} \|T\|$. But this condition is not

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necessary: if \( T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \), then \( w(T) = \frac{1}{2} \|T\| = 1 \), but \( T^2 \neq 0 \). We remark that some (necessary or sufficient) conditions for \( w(T) = \frac{1}{2} \|T\| \) to hold are given in [5, Theorems 1.3-4 and 1.3-5], but no equivalent condition has been known yet.

Let \( T = U|T| \) be the polar decomposition of \( T \). The Aluthge transform \( \tilde{T} \) of \( T \) is defined by \( \tilde{T} = |T|^{1/2}U|T|^{1/2} \) (see [1]). The following properties of \( \tilde{T} \) are well known:

(i) \( \|\tilde{T}\| \leq \|T\| \),
(ii) \( w(\tilde{T}) \leq w(T) \),
(iii) \( r(\tilde{T}) = r(T) \).

The first and last properties are easy by the definition of \( \tilde{T} \), and the second one is shown in [7], [9] and [11]. Moreover for a non-negative integer \( n \), we denote the \( n \)th Aluthge transform by \( \tilde{T}_n \), i.e.,

\[
\tilde{T}_n = \tilde{T}_{n-1} \quad \text{and} \quad \tilde{T}_0 = T.
\]

This was first considered in [7] and [10], independently.

In this paper, first, we obtain a more precise estimate than (1.2). In the inequality, we use a bigger term, \( \|T\| \), and a smaller one, \( w(\tilde{T}) \), than \( w(T) \). Moreover the proof is very simple and needs only a generalized polarization identity. Next, we give a condition equivalent to \( w(T) = \frac{1}{2} \|T\| \).

2. An inequality sharper than Kittaneh’s. In this section, we prove a sharper estimate of \( w(T) \) than Kittaneh’s [8], as follows:

**THEOREM 2.1.** For any \( T \in B(\mathcal{H}) \), \( w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} w(\tilde{T}) \).

We remark that by the Heinz inequality [6], \( \|A^rXB^r\| \leq \|AXB\|^r \|X\|^{1-r} \) for \( A, B \geq 0 \) and \( r \in [0, 1] \), we have

\[
w(\tilde{T}) \leq \|\tilde{T}\| = \|T\|^{1/2}U|T|^{1/2} \|U\|^{1/2} = \|T^2\|^{1/2},
\]

i.e., Theorem 2.1 is sharper than (1.2).

To prove Theorem 2.1, we use the following famous formula which is called the generalized polarization identity:

**THEOREM A (Generalized Polarization Identity).** For each \( T \in B(\mathcal{H}) \) and \( x, y \in \mathcal{H} \),

\[
\langle Tx, y \rangle = \frac{1}{4}(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle) + \frac{1}{4}i(\langle T(x+iy), x+iy \rangle - \langle T(x-iy), x-iy \rangle).
\]

**Proof of Theorem 2.1.** First of all, we note that

\[
w(T) = \sup_{\theta \in \mathbb{R}} \|\text{Re}(e^{i\theta}T)\|,
\]

\[
\|\|T\|^{1/2}U|T|^{1/2} \|U\|^{1/2} = \|T^2\|^{1/2},
\]

\[
w(\tilde{T}) \leq \|\tilde{T}\| = \|T\|^{1/2}U|T|^{1/2} \|U\|^{1/2} = \|T^2\|^{1/2},
\]

i.e., Theorem 2.1 is sharper than (1.2).
Bounds of the numerical radius

\[ \sup_{\theta \in \mathbb{R}} \text{Re}\{e^{i\theta} \langle Tx, x \rangle\} = |\langle Tx, x \rangle| \]

and

\[ \sup_{\theta \in \mathbb{R}} \|\text{Re}(e^{i\theta}T)\| = \sup_{\theta \in \mathbb{R}} w(\text{Re}(e^{i\theta}T)) = w(T). \]

Let \( T = U|T| \) be the polar decomposition. Then by (2.2), we have

\[ \langle e^{i\theta}Tx, x \rangle = \langle e^{i\theta}|T|x, U^*x \rangle \]

\[ = \frac{1}{4}(\langle |T|(e^{i\theta} + U^*)x, (e^{i\theta} + U^*)x \rangle - \langle |T|(e^{i\theta} - U^*)x, (e^{i\theta} - U^*)x \rangle) \]

\[ + \frac{1}{4}i(\langle |T|(e^{i\theta} + iU^*)x, (e^{i\theta} + iU^*)x \rangle - \langle |T|(e^{i\theta} - iU^*)x, (e^{i\theta} - iU^*)x \rangle). \]

Note that the inner products on the right hand side are all positive since \( |T| \) is positive. Hence we have

\[ \text{Re}\langle e^{i\theta}Tx, x \rangle \]

\[ = \frac{1}{4}(\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*)x, x \rangle - \langle (e^{-i\theta} - U)|T|(e^{i\theta} - U^*)x, x \rangle) \]

\[ \leq \frac{1}{4}\|\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*)x, x \rangle \| \]

\[ \leq \frac{1}{4}\|\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*) \| \]

\[ = \frac{1}{4}\| |T|^{1/2}(e^{i\theta} + U^*)(e^{-i\theta} + U)|T|^{1/2} \| \quad (\text{since } \|X^*X\| = \|XX^*\|) \]

\[ = \frac{1}{4}\|2|T| + e^{i\theta}\tilde{T} + e^{-i\theta}(\tilde{T})^* \| \]

\[ = \frac{1}{2}\| |T| + \text{Re}(e^{i\theta}\tilde{T}) \| \]

\[ \leq \frac{1}{2}\| |T| \| + \frac{1}{2}\|\text{Re}(e^{i\theta}\tilde{T}) \| \]

\[ \leq \frac{1}{2}\| |T| \| + \frac{1}{2}w(\tilde{T}) \quad (\text{by (2.3)).} \]

Hence we have the desired inequality. ■

**Corollary 2.2.** If \( \tilde{T} = 0 \), then \( w(T) = \frac{1}{2}\| |T| \|. \)

**Proof.** The proof is easy by Theorem 2.1 and (1.1). ■

**Remark.**

(i) In Corollary 2.2, the conditions \( \tilde{T} = 0 \) and \( w(T) = \frac{1}{2}\| |T| \| \) are not equivalent: if \( T = 1 \oplus \left( \begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array} \right) \), then \( w(T) = \frac{1}{2}\| |T| \| = 1 \), but \( \tilde{T} = 1 \oplus 0 \neq 0 \).

(ii) The conditions \( \tilde{T} = 0 \) and \( T^2 = 0 \) are equivalent. Indeed, let \( T = U|T| \) be the polar decomposition. If \( \tilde{T} = 0 \), then

\[ T^2 = U|T|U|T| = U|T|^{1/2}\tilde{T}|T|^{1/2} = 0. \]

Conversely, if \( T^2 = 0 \), then by (2.1) we have \( \|\tilde{T}\| \leq \|T^2\|^{1/2} = 0. \)
**Corollary 2.3.** For $T \in B(\mathcal{H})$,

$$w(T) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|\tilde{T}_{n-1}\|.$$ 

**Proof.** By using Theorem 2.1 several times, we have

$$w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} w(\tilde{T}) \leq \frac{1}{2} \|T\| + \frac{1}{2} \left\{ \frac{1}{2} \|\tilde{T}\| + \frac{1}{2} w(\tilde{T}_2) \right\}$$

$$= \frac{1}{2} \|T\| + \frac{1}{4} \|	ilde{T}\| + \frac{1}{4} w(\tilde{T}_2)$$

$$\leq \frac{1}{2} \|T\| + \frac{1}{4} \|	ilde{T}\| + \frac{1}{8} \|	ilde{T}_2\| + \frac{1}{8} w(\tilde{T}_3) \leq \cdots \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|	ilde{T}_{n-1}\|.$$ 

Let

$$s(T) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|	ilde{T}_{n-1}\|.$$ 

By (2.1), $\|\tilde{A}\| \leq \|A^2\|^{1/2} \leq \|A\|$ for any $A \in B(\mathcal{H})$, and we obtain

(2.4) $r(T) \leq w(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} w(\tilde{T}) \leq s(T) \leq \frac{1}{2} \|T\| + \frac{1}{2} \|T^2\|^{1/2} \leq \|T\|$, 

where $r(T)$ means the spectral radius of $T$.

It is well known that $T$ is normaloid (i.e., $\|T\| = r(T)$) if and only if $\|T\| = w(T)$. Here we give other conditions of normaloidity of $T$:

**Corollary 2.4.** The following conditions are equivalent:

(i) $T$ is normaloid,

(ii) $\|T\| = s(T)$,

(iii) $r(T) = \frac{1}{2} \|T\| + \frac{1}{2} w(\tilde{T})$,

(iv) $s(T) = s(\tilde{T})$.

**Remark.**

(i) In Corollary 2.4, condition (ii) cannot be replaced by the weaker condition $\|T\| = \frac{1}{2} \|T\| + \frac{1}{2} \|T^2\|^{1/2}$. For example, let

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then $\|T\| = \frac{1}{2} \|T\| + \frac{1}{2} \|T^2\|^{1/2} = 1$ but $0 = r(T) < \|T\|$.

(ii) In Corollary 2.4, condition (iii) cannot be replaced by the weaker condition $r(T) = w(T)$ either. In fact, let $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $1 = r(T) = w(T) < \|T\| = 2$. (We call an operator satisfying $r(T) = w(T)$ spectraloid.)

To prove Corollary 2.4, the following formula will be used.

**Theorem B ([10]).** For any $T \in B(\mathcal{H})$, $\lim_{n \to \infty} \|\tilde{T}_n\| = r(T)$. 
**Proof of Corollary 2.4.** (i)⇒ (ii), (iii) and (iv) are obvious by (2.4) and $r(T) = r(T) \leq s(T) \leq s(T) \leq \|T\|$. 

(ii)⇒(i). By the definition of $s(T)$,

\begin{equation}
(2.5) \quad s(T) = \frac{1}{2} \|T\| + \frac{1}{2} s(T).
\end{equation}

Hence (ii) yields $s(T) = \|T\|$. Since $\|\tilde{T}\| \leq \|T\|$, this gives $s(T) \leq \|\tilde{T}\| \leq \|T\| = s(T)$, and so $s(T) = \|\tilde{T}\| = \|T\|$. By using the same technique, we have $\|T\| = \|\tilde{T}_n\|$ for all $n \in \mathbb{N}$. Hence by Theorem B, we have $\|T\| = \lim_{n \to \infty} \|\tilde{T}_n\| = r(T)$, that is, $T$ is normaloid.

(iii)⇒(i). Since $r(T) = r(T)$, by (iii) we have

$$r(T) = \frac{1}{2} \|T\| + \frac{1}{2} w(T) \geq \frac{1}{2} \|T\| + \frac{1}{2} r(T) = \frac{1}{2} \|T\| + \frac{1}{2} r(T),$$

that is, $r(T) \geq \|T\|$ and so $r(T) = \|T\|$.

(iv)⇒(ii). Evident by (2.5). ■

In [2], Ando shows that the equality $W(T) = W(T)$ of numerical ranges is equivalent to $co \sigma(T) = W(T)$ (i.e., $T$ is convexoid) for any matrix $T$, where $co \sigma(T)$ means the convex hull of the spectrum of $T$. We think that this result is parallel to the equivalence between (i) and (iv). So we expect that $s(T)$ has some interesting properties.

### 3. Condition equivalent to $w(T) = \frac{1}{2} \|T\|$. In Corollary 2.2, we have obtained a sufficient condition for $w(T) = \frac{1}{2} \|T\|$ to hold. Some (necessary or sufficient) conditions for $w(T) = \frac{1}{2} \|T\|$ to hold are given in [5, Theorems 1.3-4 and 1.3-5]. But no condition equivalent to $w(T) = \frac{1}{2} \|T\|$ has been known. In this section, we give such a condition:

**Theorem 3.1.** Let $T \in B(H)$. The following conditions are equivalent:

(i) $w(T) = \frac{1}{2} \|T\|$,

(ii) $\|T\| = \|Re(e^{i\theta}T)\| + \|Im(e^{i\theta}T)\|$ for all $\theta \in \mathbb{R}$.

We remark that (ii) cannot be replaced by “$\|T\| = \|Re(e^{i\theta}T)\| + \|Im(e^{i\theta}T)\|$ for some $\theta \in \mathbb{R}$,” because if $T$ is a non-zero self-adjoint operator, then $\|T\| = \|Re T\| + \|Im T\| = \|Re T\|$, but $w(T) = \|T\| > \frac{1}{2} \|T\|$.

To prove Theorem 3.1, we need the following theorem:

**Theorem C ([3]).** Let $A, B \in B(H)$ be non-zero. Then the equality $\|A + B\| = \|A\| + \|B\|$ holds if and only if $\|A\| \|B\| \in W(A^*B)$.
Proof of Theorem 3.1. Let $e^{i\theta}T = H_\theta + iK_\theta$ be the Cartesian decomposition of $e^{i\theta}T$. We remark that

(3.1) \[ K_\theta = H_{\theta - \pi/2}, \]

because $e^{i(\theta - \pi/2)}T = -ie^{i\theta}T = K_\theta - iH_\theta$.

(i)$\Rightarrow$(ii). Since $w(T) = \sup_{\theta \in \mathbb{R}} \|H_\theta\| = \sup_{\theta \in \mathbb{R}} \|K_\theta\|$ by (2.3) and (3.1), we have

\[ \|T\| = \|e^{i\theta}T\| = \|H_\theta + iK_\theta\| \leq \|H_\theta\| + \|K_\theta\| \leq w(T) + w(T) = \|T\|, \]

proving (ii).

(ii)$\Rightarrow$(i). For any $\theta \in \mathbb{R}$, (ii) ensures $\|H_\theta\| \|K_\theta\| \in \overline{W(H_\theta^* iK_\theta)}$ by Theorem C, i.e., $-i\|H_\theta\| \|K_\theta\| \in \overline{W(H_\theta K_\theta)}$. Since $-i\|H_\theta\| \|K_\theta\|$ is a purely imaginary number and $\text{Im}(H_\theta K_\theta) = \text{Im}(H_0 K_0)$ for all $\theta \in \mathbb{R}$, we have

\[ \|H_\theta\| \|K_\theta\| = w(H_\theta K_\theta) = \|\text{Im}(H_\theta K_\theta)\| = \|\text{Im}(H_0 K_0)\|. \]

Thus for all $\theta \in \mathbb{R}$,

\[ \|H_\theta\| + \|K_\theta\| = \|T\|, \quad \|H_\theta\| \|K_\theta\| = \|\text{Im}(H_0 K_0)\|, \]

that is,

\[ \|H_\theta\| = \frac{\|T\| + \sqrt{\|T\|^2 - 4 \|\text{Im}(H_0 K_0)\|^2}}{2}, \]

\[ \|K_\theta\| = \frac{\|T\| - \sqrt{\|T\|^2 - 4 \|\text{Im}(H_0 K_0)\|^2}}{2}, \]

or the other way round. We remark that these values do not depend on $\theta \in \mathbb{R}$. So the function $\|H_\theta\|$ of $\theta \in \mathbb{R}$ takes only two values by (3.1). By an easy calculation, we have

\[ H_\theta = H_0 \cos \theta - K_0 \sin \theta. \]

Hence by the continuity of the operator norm, the function $\|H_\theta\|$ is continuous on $\theta \in \mathbb{R}$. Therefore it must be constant, i.e.,

\[ \|H_\theta\| = \|K_\theta\| = \frac{1}{2} \|T\|. \]

Hence we have (i). \phantom{.} ■

References


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