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On upper and lower bounds of the numerical radius and an equality condition

by

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Abstract. We give an inequality relating the operator norm of T and the numerical radii of T and its Aluthge transform. It is a more precise estimate of the numerical radius than Kittaneh's result [Studia Math. 158 (2003)]. Then we obtain an equivalent condition for the numerical radius to be equal to half the operator norm.

1. Introduction. For a bounded linear operator T on a complex Hilbert space $\mathcal{H}$, we denote the operator norm and the numerical radius of T by $\|T\|$ and $w(T)$, respectively. It is well known that $w(T)$ is an equivalent norm of T, since (see [5, Theorem 1.3-1])

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|. \tag{1.1}$$

Concerning the second inequality, Kittaneh [8] has shown the following precise estimate of $w(T)$ by using several norm inequalities and ingenious techniques:

$$w(T) \le \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{1/2}. \tag{1.2}$$

Obviously, (1.2) is sharper than the right inequality of (1.1). We remark that we cannot compare $w(T)$ with $\|T^2\|^{1/2}$, generally. In fact, if $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $0 = \|T^2\|^{1/2} < w(T) = 1/2$; but if

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

then $1/\sqrt{2} = w(T) < \|T^2\|^{1/2} = 1$.

We obtain a sufficient condition for $w(T) = \frac{1}{2}\|T\|$ to hold from (1.1), (1.2) and [8]: if $T^2 = 0$, then $w(T) = \frac{1}{2}\|T\|$. But this condition is not

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necessary: if $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $w(T) = \frac{1}{2}\|T\| = 1$, but $T^2 \neq 0$. We remark that some (necessary or sufficient) conditions for $w(T) = \frac{1}{2}\|T\|$ to hold are given in [5, Theorems 1.3-4 and 1.3-5], but no equivalent condition has been known yet.

Let $T = U|T|$ be the polar decomposition of T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ (see [1]). The following properties of $\widetilde{T}$ are well known:

(i) $\|\widetilde{T}\| \leq \|T\|$,
(ii) $w(\widetilde{T}) \leq w(T)$,
(iii) $r(\widetilde{T}) = r(T)$.

The first and last properties are easy by the definition of $\widetilde{T}$, and the second one is shown in [7], [9] and [11]. Moreover for a non-negative integer n, we denote the nth Aluthge transform by $\widetilde{T}_n$, i.e.,

$$\widetilde{T}_n = \widetilde{\widetilde{T}}_{n-1} \quad \text{and} \quad \widetilde{T}_0 = T.$$

This was first considered in [7] and [10], independently.

In this paper, first, we obtain a more precise estimate than (1.2). In the inequality, we use a bigger term, $\|T\|$, and a smaller one, $w(\widetilde{T})$, than $w(T)$. Moreover the proof is very simple and needs only a generalized polarization identity. Next, we give a condition equivalent to $w(T) = \frac{1}{2}\|T\|$.

2. An inequality sharper than Kittaneh's. In this section, we prove a sharper estimate of $w(T)$ than Kittaneh's [8], as follows:

THEOREM 2.1. *For any $T \in B(\mathcal{H})$, $w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})$.*

We remark that by the Heinz inequality [6], $\|A^r X B^r\| \leq \|AXB\|^r \|X\|^{1-r}$ for $A, B \geq 0$ and $r \in [0,1]$, we have

$$w(\widetilde{T}) \leq \|\widetilde{T}\| = \| \, |T|^{1/2}U|T|^{1/2}\| \leq \| \, |T|U|T|\|^{1/2}\|U\|^{1/2} = \|T^2\|^{1/2}, \tag{2.1}$$

i.e., Theorem 2.1 is sharper than (1.2).

To prove Theorem 2.1, we use the following famous formula which is called the generalized polarization identity:

THEOREM A (Generalized Polarization Identity). *For each $T \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$,*

$$\begin{aligned} \langle Tx, y\rangle = {} & \tfrac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle) \\ & + \tfrac{1}{4}i(\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle). \end{aligned} \tag{2.2}$$

Proof of Theorem 2.1. First of all, we note that

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)\|, \tag{2.3}$$

since

$$\sup_{\theta\in\mathbb{R}} \mathrm{Re}\{e^{i\theta}\langle Tx, x\rangle\} = |\langle Tx, x\rangle|$$

and

$$\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)\| = \sup_{\theta\in\mathbb{R}} w(\mathrm{Re}(e^{i\theta}T)) = w(T).$$

Let $T = U|T|$ be the polar decomposition. Then by (2.2), we have

$$\begin{aligned}
&\langle e^{i\theta}Tx, x\rangle = \langle e^{i\theta}|T|x, U^*x\rangle \\
&\quad = \tfrac{1}{4}(\langle |T|(e^{i\theta} + U^*)x, (e^{i\theta} + U^*)x\rangle - \langle |T|(e^{i\theta} - U^*)x, (e^{i\theta} - U^*)x\rangle) \\
&\qquad + \tfrac{1}{4}i(\langle |T|(e^{i\theta} + iU^*)x, (e^{i\theta} + iU^*)x\rangle - \langle |T|(e^{i\theta} - iU^*)x, (e^{i\theta} - iU^*)x\rangle).
\end{aligned}$$

Note that the inner products on the right hand side are all positive since $|T|$ is positive. Hence we have

$$\begin{aligned}
&\mathrm{Re}\langle e^{i\theta}Tx, x\rangle \\
&\qquad = \tfrac{1}{4}(\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*)x, x\rangle - \langle (e^{-i\theta} - U)|T|(e^{i\theta} - U^*)x, x\rangle) \\
&\qquad \le \tfrac{1}{4}\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*)x, x\rangle \\
&\qquad \le \tfrac{1}{4}\|(e^{-i\theta} + U)|T|(e^{i\theta} + U^*)\| \\
&\qquad = \tfrac{1}{4}\||T|^{1/2}(e^{i\theta} + U^*)(e^{-i\theta} + U)|T|^{1/2}\| \quad (\text{since } \|X^*X\| = \|XX^*\|) \\
&\qquad = \tfrac{1}{4}\|2|T| + e^{i\theta}\widetilde{T} + e^{-i\theta}(\widetilde{T})^*\| \\
&\qquad = \tfrac{1}{2}\|\,|T| + \mathrm{Re}(e^{i\theta}\widetilde{T})\| \\
&\qquad \le \tfrac{1}{2}\|T\| + \tfrac{1}{2}\|\mathrm{Re}(e^{i\theta}\widetilde{T})\| \\
&\qquad \le \tfrac{1}{2}\|T\| + \tfrac{1}{2}w(\widetilde{T}) \qquad (\text{by (2.3)}).
\end{aligned}$$

Hence we have the desired inequality. ∎

COROLLARY 2.2. *If* $\widetilde{T} = 0$, *then* $w(T) = \frac{1}{2}\|T\|$.

Proof. The proof is easy by Theorem 2.1 and (1.1). ∎

REMARK.

(i) In Corollary 2.2, the conditions $\widetilde{T} = 0$ and $w(T) = \frac{1}{2}\|T\|$ are not equivalent: if $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $w(T) = \frac{1}{2}\|T\| = 1$, but $\widetilde{T} = 1 \oplus 0 \ne 0$.

(ii) The conditions $\widetilde{T} = 0$ and $T^2 = 0$ are equivalent. Indeed, let $T = U|T|$ be the polar decomposition. If $\widetilde{T} = 0$, then

$$T^2 = U|T|U|T| = U|T|^{1/2}\widetilde{T}|T|^{1/2} = 0.$$

Conversely, if $T^2 = 0$, then by (2.1) we have $\|\widetilde{T}\| \le \|T^2\|^{1/2} = 0$.

Corollary 2.3. *For* $T \in B(\mathcal{H})$,

$$w(T) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|\widetilde{T}_{n-1}\|.$$

Proof. By using Theorem 2.1 several times, we have

$$\begin{aligned} w(T) &\leq \tfrac{1}{2}\|T\| + \tfrac{1}{2}w(\widetilde{T}) \leq \tfrac{1}{2}\|T\| + \tfrac{1}{2}\big\{\tfrac{1}{2}\|\widetilde{T}\| + \tfrac{1}{2}w(\widetilde{T}_2)\big\} \\ &= \tfrac{1}{2}\|T\| + \tfrac{1}{4}\|\widetilde{T}\| + \tfrac{1}{4}w(\widetilde{T}_2) \\ &\leq \tfrac{1}{2}\|T\| + \tfrac{1}{4}\|\widetilde{T}\| + \tfrac{1}{8}\|\widetilde{T}_2\| + \tfrac{1}{8}w(\widetilde{T}_3) \leq \cdots \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|\widetilde{T}_{n-1}\|. \ \blacksquare \end{aligned}$$

Let

$$s(T) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|\widetilde{T}_{n-1}\|.$$

By (2.1), $\|\widetilde{A}\| \leq \|A^2\|^{1/2} \leq \|A\|$ for any $A \in B(\mathcal{H})$, and we obtain

$$r(T) \leq w(T) \leq \tfrac{1}{2}\|T\| + \tfrac{1}{2}w(\widetilde{T}) \leq s(T) \leq \tfrac{1}{2}\|T\| + \tfrac{1}{2}\|T^2\|^{1/2} \leq \|T\|, \tag{2.4}$$

where $r(T)$ means the spectral radius of T.

It is well known that T is *normaloid* (i.e., $\|T\| = r(T)$) if and only if $\|T\| = w(T)$. Here we give other conditions of normaloidity of T:

Corollary 2.4. *The following conditions are equivalent*:

(i) T *is normaloid*,
(ii) $\|T\| = s(T)$,
(iii) $r(T) = \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})$,
(iv) $s(T) = s(\widetilde{T})$.

Remark.

(i) In Corollary 2.4, condition (ii) cannot be replaced by the weaker condition $\|T\| = \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{1/2}$. For example, let

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\|T\| = \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{1/2} = 1$ but $0 = r(T) < \|T\|$.

(ii) In Corollary 2.4, condition (iii) cannot be replaced by the weaker condition $r(T) = w(T)$ either. In fact, let $T = 1 \oplus \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$. Then $1 = r(T) = w(T) < \|T\| = 2$. (We call an operator satisfying $r(T) = w(T)$ *spectraloid*.)

To prove Corollary 2.4, the following formula will be used.

Theorem B ([10]). *For any* $T \in B(\mathcal{H})$, $\lim_{n\to\infty} \|\widetilde{T}_n\| = r(T)$.

Proof of Corollary 2.4. (i)⇒ (ii), (iii) and (iv) are obvious by (2.4) and $r(T) = r(\widetilde{T}) \le s(\widetilde{T}) \le s(T) \le \|T\|$.

(ii)⇒(i). By the definition of $s(T)$,

$$s(T) = \tfrac{1}{2}\|T\| + \tfrac{1}{2}s(\widetilde{T}). \tag{2.5}$$

Hence (ii) yields $s(\widetilde{T}) = \|T\|$. Since $\|\widetilde{T}\| \le \|T\|$, this gives

$$s(\widetilde{T}) \le \|\widetilde{T}\| \le \|T\| = s(\widetilde{T}),$$

and so $s(\widetilde{T}) = \|\widetilde{T}\| = \|T\|$. By using the same technique, we have $\|T\| = \|\widetilde{T}_n\|$ for all $n \in \mathbb{N}$. Hence by Theorem B, we have

$$\|T\| = \lim_{n\to\infty} \|\widetilde{T}_n\| = r(T),$$

that is, T is normaloid.

(iii)⇒(i). Since $r(\widetilde{T}) = r(T)$, by (iii) we have

$$r(T) = \tfrac{1}{2}\|T\| + \tfrac{1}{2}w(\widetilde{T}) \ge \tfrac{1}{2}\|T\| + \tfrac{1}{2}r(\widetilde{T}) = \tfrac{1}{2}\|T\| + \tfrac{1}{2}r(T),$$

that is, $r(T) \ge \|T\|$ and so $r(T) = \|T\|$.

(iv)⇒(ii). Evident by (2.5). ∎

In [2], Ando shows that the equality $W(T) = W(\widetilde{T})$ of numerical ranges is equivalent to $\operatorname{co}\sigma(T) = W(T)$ (i.e., T is convexoid) for any matrix T, where $\operatorname{co}\sigma(T)$ means the convex hull of the spectrum of T. We think that this result is parallel to the equivalence between (i) and (iv). So we expect that $s(T)$ has some interesting properties.

3. Condition equivalent to $w(T) = \frac{1}{2}\|T\|$**.** In Corollary 2.2, we have obtained a sufficient condition for $w(T) = \frac{1}{2}\|T\|$ to hold. Some (necessary or sufficient) conditions for $w(T) = \frac{1}{2}\|T\|$ to hold are given in [5, Theorems 1.3-4 and 1.3-5]. But no condition equivalent to $w(T) = \frac{1}{2}\|T\|$ has been known. In this section, we give such a condition:

THEOREM 3.1. *Let $T \in B(\mathcal{H})$. The following conditions are equivalent*:

(i) $w(T) = \frac{1}{2}\|T\|$,

(ii) $\|T\| = \|\operatorname{Re}(e^{i\theta}T)\| + \|\operatorname{Im}(e^{i\theta}T)\|$ *for all* $\theta \in \mathbb{R}$.

We remark that (ii) cannot be replaced by "$\|T\| = \|\operatorname{Re}(e^{i\theta}T)\| + \|\operatorname{Im}(e^{i\theta}T)\|$ for *some* $\theta \in \mathbb{R}$," because if T is a non-zero self-adjoint operator, then $\|T\| = \|\operatorname{Re} T\| + \|\operatorname{Im} T\| = \|\operatorname{Re} T\|$, but $w(T) = \|T\| > \frac{1}{2}\|T\|$.

To prove Theorem 3.1, we need the following theorem:

THEOREM C ([3]). *Let $A, B \in B(\mathcal{H})$ be non-zero. Then the equality $\|A + B\| = \|A\| + \|B\|$ holds if and only if $\|A\|\,\|B\| \in \overline{W(A^*B)}$.*

Proof of Theorem 3.1. Let $e^{i\theta}T = H_\theta + iK_\theta$ be the Cartesian decomposition of $e^{i\theta}T$. We remark that

$$K_\theta = H_{\theta-\pi/2}, \tag{3.1}$$

because $e^{i(\theta-\pi/2)}T = -ie^{i\theta}T = K_\theta - iH_\theta$.

(i)⇒(ii). Since $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\| = \sup_{\theta\in\mathbb{R}} \|K_\theta\|$ by (2.3) and (3.1), we have

$$\|T\| = \|e^{i\theta}T\| = \|H_\theta + iK_\theta\| \le \|H_\theta\| + \|K_\theta\| \le w(T) + w(T) = \|T\|,$$

proving (ii).

(ii)⇒(i). For any $\theta \in \mathbb{R}$, (ii) ensures $\|H_\theta\|\,\|K_\theta\| \in \overline{W(H_\theta^*(iK_\theta))}$ by Theorem C, i.e., $-i\|H_\theta\|\,\|K_\theta\| \in \overline{W(H_\theta K_\theta)}$. Since $-i\|H_\theta\|\|K_\theta\|$ is a purely imaginary number and $\mathrm{Im}(H_\theta K_\theta) = \mathrm{Im}(H_0K_0)$ for all $\theta \in \mathbb{R}$, we have

$$\|H_\theta\|\,\|K_\theta\| = w(H_\theta K_\theta) = \|\mathrm{Im}(H_\theta K_\theta)\| = \|\mathrm{Im}(H_0K_0)\|.$$

Thus for all $\theta \in \mathbb{R}$,

$$\|H_\theta\| + \|K_\theta\| = \|T\|, \qquad \|H_\theta\|\,\|K_\theta\| = \|\mathrm{Im}(H_0K_0)\|,$$

that is,

$$\|H_\theta\| = \frac{\|T\| + \sqrt{\|T\|^2 - 4\|\mathrm{Im}(H_0K_0)\|}}{2},$$
$$\|K_\theta\| = \frac{\|T\| - \sqrt{\|T\|^2 - 4\|\mathrm{Im}(H_0K_0)\|}}{2},$$

or the other way round. We remark that these values do not depend on $\theta \in \mathbb{R}$. So the function $\|H_\theta\|$ of $\theta \in \mathbb{R}$ takes only two values by (3.1). By an easy calculation, we have

$$H_\theta = H_0\cos\theta - K_0\sin\theta.$$

Hence by the continuity of the operator norm, the function $\|H_\theta\|$ is continuous on $\theta \in \mathbb{R}$. Therefore it must be constant, i.e.,

$$\|H_\theta\| = \|K_\theta\| = \tfrac{1}{2}\|T\|.$$

Hence we have (i). ∎

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