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Random ε-nets and embeddings in ℓ_∞^N

by

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Abstract. We show that, given an n-dimensional normed space X, a sequence of $N = (8/\varepsilon)^{2n}$ independent random vectors $(X_i)_{i=1}^N$, uniformly distributed in the unit ball of X^*, with high probability forms an ε-net for this unit ball. Thus the random linear map $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ defined by $\Gamma x = (\langle x, X_i\rangle)_{i=1}^N$ embeds X in ℓ_∞^N with at most $1+\varepsilon$ norm distortion. In the case $X = \ell_2^n$ we obtain a random $1+\varepsilon$-embedding into ℓ_∞^N with asymptotically best possible relation between N, n, and ε.

1. Introduction. Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an arbitrary n-dimensional normed space with unit ball K. It is well known that, for any $0 < \varepsilon < 1$, X can be $1+\varepsilon$-embedded into ℓ_∞^N, for some $N = N(\varepsilon, n)$, depending on ε and n, but independent of X. In this note we investigate $1+\varepsilon$-isomorphic embeddings which are random with respect to some natural measure, depending on K. We first show that for $N = (8/\varepsilon)^{2n}$, a sequence of N independent random vectors $(X_i)_{i=1}^N$, uniformly distributed in the unit ball K^0 of the dual space X^*, forms an ε-net for K^0 with high probability. Thus, with high probability, the random linear map $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ defined by $\Gamma x = (\langle x, X_i\rangle)_{i=1}^N$ embeds X in ℓ_∞^N with at most $1+\varepsilon$ norm distortion.

The important case is $X = \ell_2^n$. In this case it is more natural to consider random vectors X_i uniformly distributed on the sphere S^{n-1}. Such vectors also form an ε-net on the sphere, hence they determine a random $1+\varepsilon$-embedding Γ of ℓ_2^n into ℓ_∞^N. We also show that $\sqrt{n/N}\,\Gamma$ is a $1+\varepsilon$-isometry from ℓ_2^n into ℓ_2^N, with high probability.

The case $X = \ell_2^n$ is connected with Dvoretzky's theorem ([D]). Milman found a new proof ([M]), using the Lévy isoperimetric inequality on the

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sphere, that there exists a function $c(\varepsilon) > 0$ such that for all $n \le c(\varepsilon) \ln N$, ℓ_2^n can be $1+\varepsilon$-embedded into any normed space Y of dimension N. His proof gives $c(\varepsilon) \sim \varepsilon^2/\ln(2/\varepsilon)$. Later a new approach was found in [G] by using random Gaussian embeddings. It implies that $c(\varepsilon) \sim \varepsilon^2$ is sufficient. Milman raised the question what is the best behavior of $c(\varepsilon)$, as $\varepsilon \to 0$, in the above estimates. Recently Schechtman showed in [S1] that one may take $c(\varepsilon) \sim \varepsilon/(\ln(2/\varepsilon))^2$, however his approach is not random.

Since in this paper we deal with embeddings into ℓ_∞^N, we shall restrict our attention to this case only. When $Y = \ell_\infty^N$, it is well known that there exists an embedding with $c(\varepsilon) \sim 1/\ln(2/\varepsilon)$. It is also known that this behavior of $c(\varepsilon)$ as $\varepsilon \to 0$ cannot be improved. The standard embedding relies on the existence of ε-nets of appropriate cardinalities. It is therefore natural to ask whether this embedding can be randomized.

In this paper we provide a positive answer to this question. Namely, we show (in Theorems 4.1, 4.3) that for the random embedding Γ determined by independent uniformly distributed vectors on S^{n-1}, with large probability one may achieve $c(\varepsilon) \sim 1/\ln(2/\varepsilon)$, which is the best possible as mentioned above. We would like to note that such a result is not valid in the setting of the Haar measure on the Grassmann manifold (equivalently, for embeddings defined by Gaussian matrices). Indeed, Schechtman recently showed ([S2]) that if "most" $n = c'(\varepsilon) \ln N$-dimensional subspaces of ℓ_∞^N are $1+\varepsilon$-Euclidean then $c'(\varepsilon) \sim \varepsilon$.

2. Notation and preliminary results. We denote by $\langle\cdot,\cdot\rangle$ the scalar product of the canonical Euclidean structure on $\mathbb{R}^n$ and by $|\cdot|$ the canonical Euclidean norm. The Euclidean ball is denoted by B_2^n and the Euclidean sphere is denoted by S^{n-1}.

By a *convex body* in $\mathbb{R}^n$ we always mean a compact convex set with non-empty interior. A centrally symmetric body with respect to origin will be called *symmetric*. Given a convex body K in $\mathbb{R}^n$ we denote by $|K|$ its volume and by $\|\cdot\|_K$ the Minkowski functional of K, i.e.

$$\|x\|_K = \inf\{\lambda > 0 \mid x \in \lambda K\}.$$

If K is symmetric then $\|\cdot\|_K$ is a norm with the unit ball K.

Given a finite set A we denote its cardinality by $|A|$.

Recall that if K is a symmetric convex body in $\mathbb{R}^n$ then for every $0 < \varepsilon \le 1$ there exists an ε-net Λ in K with respect to the norm $\|\cdot\|_K$ of cardinality

$$|\Lambda| \le (1+2/\varepsilon)^n \le (3/\varepsilon)^n.$$

The *polar* of a convex body $K \subset \mathbb{R}^n$ is defined by

$$K^0 = \{x \mid \langle x, y\rangle \le 1 \text{ for every } y \in K\}.$$

Let K be a convex body. We say that a vector X is *uniformly distributed* on K if $\mathbb{P}\{X \in A\} = |K \cap A|/|K|$ for every measurable $A \subset \mathbb{R}^n$.

Given a square matrix T, we denote by $\|T\|_{\mathrm{HS}}$ its Hilbert–Schmidt norm.

Below we will need the following geometric lemma. Although we will use only a particular case of the lemma, we prefer to state it in full generality for future references.

LEMMA 2.1. *Let $d > 0$ and K, L be convex bodies in $\mathbb{R}^n$ such that $K \subset -dL$. Then for every $x \in K$ and for $0 < \varepsilon \le 1$ one has*

$$|K \cap (x + \varepsilon L)| \ge \left| \frac{\varepsilon}{d+1} K \cap L \right|.$$

In particular, if $K = L = -K$ then

$$|K \cap (x + \varepsilon K)| \ge \left| \frac{\varepsilon}{2} K \right|.$$

Proof. Define

$$\alpha = 1 - \frac{\varepsilon}{d+1}, \qquad \beta = \frac{\varepsilon}{d+1}.$$

To prove the desired result it is enough to show that

$$K \cap (x + \varepsilon L) \supset \alpha x + \beta K \cap L.$$

Let $z = \alpha x + \beta y$, where $y \in K \cap L$. Clearly, $z \in K$ and $z = x + \beta(y - x)$. Since

$$y - x \in L - K \subset L + dL = (1 + d)L,$$

we obtain the result. ∎

REMARK 1. The example of the cube (when x is a vertex) shows that the estimate in the “in particular” part of Lemma 2.1 is sharp.

REMARK 2. It is known that for every convex body K in $\mathbb{R}^n$ there exists a shift such that $K - a \subset -n(K - a)$. Thus, Lemma 2.1 implies that for every convex body K in $\mathbb{R}^n$ there exists a vector $a \in \mathbb{R}^n$ such that for every $x \in K$ and for $\varepsilon > 0$ one has

$$|(K - a) \cap (x + \varepsilon(K - a))| \ge \left| \frac{\varepsilon}{n+1} K \right|.$$

The example of the regular simplex (when x is a vertex) shows that the latter estimate is sharp.

REMARK 3. It was proved in [GLMP] that if a body L is in the position of maximal volume in K (that is, $L \subset K$ and for every linear map T and every point $x \in \mathbb{R}^n$ satisfying $TL + x \subset K$ one has $|TL| \le |L|$), then there exists $a \in \mathbb{R}^n$ such that

$$L - a \subset K - a \subset -n(L - a).$$

Thus Lemma 2.1 implies that if a body L is in the position of maximal volume in K then there exists a vector $a \in \mathbb{R}^n$ such that for every $x \in K$ and for $\varepsilon > 0$ one has

$$|(K-a)\cap(x+\varepsilon(L-a))| \geq \left|\frac{\varepsilon}{n+1}\,L\right|.$$

3. Random embeddings of normed spaces in ℓ_∞^N. First we show that N vectors uniformly distributed on a symmetric convex body K form an ε-net in K.

THEOREM 3.1. *Let $n \geq 1$, $0 < \varepsilon \leq 1$, and $N = (4/\varepsilon)^{2n}$. Let $X_1, \dots, X_N$ be independent random variables uniformly distributed on a symmetric convex body K in $\mathbb{R}^n$. Then with probability larger than $1 - \exp(-(8/\varepsilon)^n/2)$ the set $\mathcal{N} = \{X_1, \dots, X_N\}$ forms an ε-net in K.*

Proof. Fix an $\varepsilon/2$-net $\Lambda \subset K$ with $|\Lambda| \leq (6/\varepsilon)^n$, and consider random vectors $X_1, \dots, X_N$ uniformly distributed on K, where N is as in the statement.

We want to show that the probability

$$\mathbb{P}\{\forall x \in K\ \exists i \leq N \text{ such that } \|x - X_i\|_K < \varepsilon\} \tag{1}$$

is large. Clearly this probability is larger than

$$\mathbb{P}\{\forall x \in \Lambda\ \exists i \leq N \text{ such that } \|x - X_i\|_K < \varepsilon/2\}. \tag{2}$$

We denote by A the event considered in (2), and estimate the probability of its complement A^{c}. We have

$$\begin{aligned}\mathbb{P}(A^{\mathrm{c}}) &= \mathbb{P}\{\exists x \in \Lambda\ \forall i \leq N \text{ one has } \|x - X_i\|_K \geq \varepsilon/2\}\\ &\leq |\Lambda|(\mathbb{P}\{\|x_0 - X_1\|_K \geq \varepsilon/2\})^N\\ &\leq |\Lambda|(1 - \mathbb{P}\{\|x_0 - X_1\|_K < \varepsilon/2\})^N,\end{aligned}$$

where $x_0 \in \Lambda$ satisfies

$$\mathbb{P}\{\|x_0 - X_i\|_K \geq \varepsilon/2\} = \max_{x\in\Lambda} \mathbb{P}\{\|x - X_i\|_K \geq \varepsilon/2\}.$$

Note that

$$\mathbb{P}\{\|x_0 - X_1\|_K < \varepsilon/2\} = \mathbb{P}\{X_1 \in x_0 + (\varepsilon/2)K\} = |K \cap (x_0 + (\varepsilon/2)K)|/|K|.$$

Applying Lemma 2.1 we obtain

$$\mathbb{P}\{\|x_0 - X_1\|_K < \varepsilon/2\} \geq (\varepsilon/2)^n.$$

This implies

$$\begin{aligned}\mathbb{P}(A^{\mathrm{c}}) &\leq (6/\varepsilon)^n(1 - (\varepsilon/2)^n)^N \leq (6/\varepsilon)^n \exp(-(\varepsilon/2)^n N)\\ &= \exp(n\ln(6/\varepsilon) - (\varepsilon/2)^n(4/\varepsilon)^{2n}) \leq \exp(-(8/\varepsilon)^n/2),\end{aligned}$$

which yields the result. ∎

To prove the next theorem we need the following standard lemma. We provide its proof for the sake of completeness.

LEMMA 3.2. *Let X be a Banach space and K be its unit ball. Let $\mathcal{N}$ be an ε-net in the unit ball K^0 (or in the unit sphere ∂K^0) of the dual space. Then for every $x \in X$ we have*

$$\sup_{y\in\mathcal{N}} \langle x, y\rangle \le \|x\|_K \le (1-\varepsilon)^{-1} \sup_{y\in\mathcal{N}} \langle x, y\rangle.$$

Proof. The left hand side estimate is obvious. Now let $\|x\|_X = 1$ and consider $z \in \partial K^0$ such that $\langle x, z\rangle = 1$. Then for an appropriate $y \in \mathcal{N}$ we have $1 = \langle x, y\rangle + \langle x, z-y\rangle \le \sup_{y\in\mathcal{N}}\langle x, y\rangle + \varepsilon$, which implies the required estimate. ■

Combining Theorem 3.1 with Lemma 3.2 we deduce that a random matrix whose rows are independent random vectors uniformly distributed on the polar of a symmetric convex body provides a random embedding of the body into ℓ_∞^N.

THEOREM 3.3. *Let $0 < \varepsilon < 1$ and $n \le (\ln N)/2\ln(4/\varepsilon)$. Let K be a symmetric convex body in $\mathbb{R}^n$. Let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on K^0. Consider the matrix $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ whose rows are $X_1, \ldots, X_N$ (i.e. $\Gamma x = (\langle x, X_i\rangle)_{i=1}^N$). Then with probability larger than $1 - \exp(-(8/\varepsilon)^n/2)$ we have*

$$(1-\varepsilon)\|x\|_K \le \|\Gamma x\|_\infty \le \|x\|_K \quad \textit{for all } x \in \mathbb{R}^n.$$

4. The Euclidean case. In this section we discuss the embedding of ℓ_2^n into ℓ_∞^N. Here it is more natural to work with random vectors uniformly distributed on the Euclidean sphere S^{n-1}. Accordingly, in the rest of the paper $X_1, \ldots, X_N$ stand for independent random vectors uniformly distributed on the Euclidean sphere S^{n-1} and $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ is the matrix whose rows are $X_1, \ldots, X_N$ (that is, $\Gamma x = (\langle x, X_i\rangle)_{i=1}^N$).

One can easily check that Theorem 3.1 holds for S^{n-1} and such vectors $X_1, \ldots, X_N$. Indeed, this follows from the same argument as before with minor modifications. We need only observe that given $y \in S^{n-1}$ the normalized Lebesgue measure of a cap

$$\{x \in S^{n-1} \mid |x - y| \le \varepsilon\}$$

is larger than or equal to $(\varepsilon/2)^n$ (cf. e.g. [P, Chapter 6]), as well as the fact that in S^{n-1} there exists an ε-net of cardinality $(3/\varepsilon)^n$. Therefore Theorem 3.3 holds with $K = B_2^n$ and with the matrix $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ defined above. We formally state both facts for future reference.

THEOREM 4.1. *Let $0 < \varepsilon < 1$ and let $n \le (\ln N)/2\ln(4/\varepsilon)$. Let $\mathcal{N} = \{X_1, \ldots, X_N\}$ where X_i $(i = 1, \ldots, N)$ are independent random vectors* *uniformly distributed on* S^{n-1}. *Then, with probability larger than* $1-\exp(-(8/\varepsilon)^n/2)$, $\mathcal{N}$ *forms an* ε*-net on* S^{n-1}. *Furthermore, with the same probability, the matrix* Γ *as above satisfies*

$$(1-\varepsilon)|x| \le \|\Gamma x\|_\infty \le |x| \quad \textit{for all } x \in \mathbb{R}^n.$$

Denote by Q the unit ball of ℓ_∞^N (i.e. the N-dimensional cube). Theorem 4.1 shows that Q has an n-dimensional section (which can be realized as $E := \Gamma\mathbb{R}^n$) which is almost Euclidean, i.e.

$$\Gamma B_2^n \subset Q \cap E \subset (1-\varepsilon)^{-1}\Gamma B_2^n.$$

Below we show that in fact the ellipsoid ΓB_2^n is, up to $(1+\varepsilon)/(1-\varepsilon)$, equivalent to the standard Euclidean ball of radius $\sqrt{N/n}$. In other words, a random subspace $E = \Gamma\mathbb{R}^n$ of ℓ_∞^N is nearly Euclidean with respect to the canonical Euclidean structure on $\mathbb{R}^N$. Namely, Theorem 4.3 below shows that

$$(1-\varepsilon)\sqrt{N/n}\,\Gamma B_2^n \subset Q \cap E \subset \frac{1+\varepsilon}{1-\varepsilon}\sqrt{N/n}\,\Gamma B_2^n.$$

We need the following lemma, which shows that $\sqrt{n/N}\,\Gamma$ almost preserves the Euclidean norm of a vector.

LEMMA 4.2. *Let* $0 < \varepsilon < 1$ *and let* $N \ge n^3/\varepsilon^4$. *Let* $X_1, \dots, X_N$ *be independent random points on the sphere* S^{n-1}. *Then with probability larger than* $1 - n^2/(\varepsilon^4 N)$ *we have*

$$(1-\varepsilon)|x| \le |\Gamma x|\sqrt{n/N} \le (1+\varepsilon)|x| \quad \textit{for all } x \in \mathbb{R}^n.$$

REMARK. One can get better estimates using a theorem of Bourgain [B]. For instance, the above inequalities are satisfied with probability larger than $1-\delta$ as far as $N \ge c(\delta)n(\ln n)^3/\varepsilon^2$ (instead of $N \ge n^3/\varepsilon^4$) for some function $c(\delta) > 0$. However, we prefer to present here a simpler proof, which provides estimates good enough for our purposes.

Proof of Lemma 4.2. Set $A := \|\Gamma^*\Gamma - (N/n)I\|_{\mathrm{HS}}$. Using the fact that $\|T\|_{\mathrm{HS}}^2 = \mathrm{tr}(T^*T)$ for every operator $T : \mathbb{R}^n \to \mathbb{R}^N$, we get

$$\begin{aligned} A^2 &= \sum_{i,j} |\langle X_i, X_j\rangle|^2 + (N^2/n^2)n - (2N/n)\|\Gamma\|_{\mathrm{HS}}^2 \\ &= \sum_{i,j} |\langle X_i, X_j\rangle|^2 + (N^2/n^2)n - 2N^2/n \\ &= \sum_{i} |X_i|^4 + \sum_{i\ne j} |\langle X_i, X_j\rangle|^2 - N^2/n. \end{aligned}$$

Therefore,

$$\mathbb{E}A^2 = N + N(N-1)\mathbb{E}|\langle X_1, X_2\rangle|^2.$$

Since $\mathbb{E}|\langle X_1, X_2\rangle|^2 = 1/n$, we finally obtain $\mathbb{E}A^2 = N(1-1/n)$.

By Chebyshev's inequality we get, for any $\varepsilon_1 > 0$,

$$\mathbb{P}\{A > \varepsilon_1\} \le \mathbb{E}A^2/\varepsilon_1^2 \le N/\varepsilon_1^2.$$

Thus

$$\mathbb{P}\Big\{\Big\|\frac{n}{N}\,\Gamma^*\Gamma - I\Big\| < \varepsilon_1\Big\} \ge \mathbb{P}\Big\{\Big\|\frac{n}{N}\,\Gamma^*\Gamma - I\Big\|_{\mathrm{HS}} < \varepsilon_1\Big\}$$
$$\ge 1 - \mathbb{P}\Big\{A > \frac{N}{n}\,\varepsilon_1\Big\} \ge 1 - \frac{Nn^2}{\varepsilon_1^2 N^2} = 1 - \frac{n^2}{\varepsilon_1^2 N}.$$

The last estimate implies that, for any $\varepsilon_1 > 0$, with probability larger than or equal to $1 - n^2/(\varepsilon_1^2 N)$, we have the following estimates for singular numbers of the matrix Γ:

$$|\sqrt{n/N}\, s_j(\Gamma) - 1| < \sqrt{\varepsilon_1} \quad \text{for } j = 1, \dots, n.$$

In particular,

$$1 - \sqrt{\varepsilon_1} < \sqrt{n/N}\, s_n(\Gamma) \le \sqrt{n/N}\, s_1(\Gamma) < 1 + \sqrt{\varepsilon_1}.$$

Setting $\varepsilon_1 = \varepsilon^2$ immediately implies the desired conclusion. ∎

Combining Theorem 3.3 with Lemma 4.2 we obtain

THEOREM 4.3. *Let* $0 < \varepsilon < 1$ *and* $2 \le n \le (\ln N)/2\ln(4/\varepsilon)$. *Let* $X_1, \dots, X_N$ *be independent random vectors uniformly distributed on* S^{n-1}. *Consider the matrix* $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ *whose rows are* $X_1, \dots, X_N$. *Then with probability larger than* $1 - n^2\varepsilon^{2n-4}/16^n - \exp(-(8/\varepsilon)^n/2) \ge 1 - e^{-n}$ *we have*

$$\frac{1-\varepsilon}{1+\varepsilon}\,|\Gamma x| \le \sqrt{\frac{N}{n}}\,\|\Gamma x\|_\infty \le \frac{1}{1-\varepsilon}\,|\Gamma x| \quad \textit{for all } x \in \mathbb{R}^n.$$

Finally, we would like to emphasize the differences between the randomness given by the matrix Γ and a standard Gaussian matrix G (i.e., with independent $N(0,1)$ entries). Fix N and $0 < \varepsilon < 1$. As already mentioned in the introduction, Γ gives a random embedding with $n_1 \sim (\ln N)/\ln(2/\varepsilon)$ (which is best possible in general), while G provides a random embedding with $n_2 \sim \varepsilon \ln N$, which is best possible if one requires high probability ([S2]).

Another observation is that Euclidean sections of the cube determined by Γ and G, and taken in the appropriate dimensions n_1 and n_2 (or smaller), will have different radii. Indeed, the conclusion of Theorem 4.3 implies that, with high probability defined by Γ, for every non-zero $y \in \Gamma\mathbb{R}^{n_1}$,

$$\frac{\|y\|_\infty}{|y|} \sim \sqrt{\frac{n_1}{N}} \sim \sqrt{\frac{\ln N}{N \ln(2/\varepsilon)}}.$$

On the other hand, with high probability defined by G for every non-zero $y = Gx \in G\mathbb{R}^{n_2}$ one has

$$\frac{\|y\|_\infty}{|y|} \sim \frac{\mathbb{E}\,\|Gx\|_\infty}{\mathbb{E}\,|Gx|} = \frac{\mathbb{E}\,\|Ge_1\|_\infty}{\mathbb{E}\,|Ge_1|} \sim \sqrt{\frac{\ln N}{N}}.$$

These two expectations are not comparable uniformly in ε (as $\varepsilon \to 0$).

Added in proof. Theorem 3.1 should be compared with Proposition 5.3 of [GM].

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