STUDIA MATHEMATICA 219 (3) (2013)

Characterization of normed linear spaces with generalized Mazur intersection property

by

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Abstract. Let $\mathcal{A}$ be a compatible collection of bounded subsets in a normed linear space. We give a characterization of the following generalized Mazur intersection property: every closed convex set $A \in \mathcal{A}$ is an intersection of balls.

1. Introduction. In 1933, Mazur [10] first studied the following ball separation property in a Banach space: every closed convex bounded subset is an intersection of balls. This property is known as the *Mazur intersection property* (MIP).

In 1960, Phelps [11] gave a characterization of finite-dimensional Banach spaces with the Mazur intersection property. Then Sullivan [14] also gave a characterization of smooth spaces with the Mazur intersection property. Finally, Giles, Gregory, and Sims [5] developed Sullivan's key idea and showed that a Banach space has the Mazur intersection property if and only if the set of weak* denting points of $B(X^*)$ is norm dense in $S(X^*)$. Then Chen and Lin [2] also gave a characterization of Banach spaces with the Mazur intersection property via semidenting points. Whitfield and Zizler [16] studied the following property, called CIP, in Banach spaces: every compact convex set is an intersection of balls. They showed that if the cone generated by the extreme points of $B(X^*)$ is $\tau_{\mathcal{A}}$-dense in X^* where $\tau_{\mathcal{A}}$ denotes the topology of uniform convergence on compact subsets of X, then X has the CIP. Later on, Sersouri [12] showed that this condition is indeed equivalent to the CIP. Some other important results on the Mazur intersection property can be found in [4, 6, 8, 9, 13, 15]. We refer to Granero, Jiménez-Sevilla and Moreno's survey [7] for this topic and related matters.

Recently, Chen and Cheng [3] gave analytical characterizations of the MIP, the CIP and the MIP* via a specific class of convex functions and

2010 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B10.
Key words and phrases: Mazur intersection property, denting point, ball separation.

DOI: 10.4064/sm219-3-1

their conjugates. Thus they first established connections between the Mazur intersection property and convex functions.

In this paper, the letter X will always denote a normed space. For any bounded subset A in X, define

$$\|f\|_A = \sup\{|f(x)| : x \in A\}, \qquad f \in X^*.$$

Then $\|\cdot\|_A$ is a seminorm on X^*. Let $\operatorname{cone} A = \{\lambda h : \lambda > 0,\ h \in A\}$. For a subset $B \subset X^*$, $\operatorname{diam}_A B = \sup_{f,g\in B} \|f-g\|_A$ denotes the diameter of B under the seminorm $\|\cdot\|_A$. A *weak* slice* of B is a set $S(B,x,\delta) = \{f \in B : f(x) > \sup_{g\in B} g(x) - \delta\}$, where $x \in X$ and $\delta > 0$.

Definition 1.1. Let $\mathcal{A}$ be a collection of bounded subsets in X.

(1) We say that $f \in S(X^*)$ is an $\mathcal{A}$-*denting point* of $B(X^*)$ if for each $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists a weak* slice S of $B(X^*)$ such that $f \in S$ and $\operatorname{diam}_A S < \varepsilon$.
(2) We say that $f \in S(X^*)$ is an $\mathcal{A}$-*semidenting point* of $B(X^*)$ if for each $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists a weak* slice S of $B(X^*)$ such that $S \subset \operatorname{cone}\{g \in X^* : \|g-f\|_A < \varepsilon\}$.

The notion of $\mathcal{A}$-semidenting point will play an important role in our main theorem. A similar notion of $\mathcal{A}$-denting point was studied by Chen and Lin [1], who also introduced the following notion in the study of ball separation properties.

Definition 1.2. We say that $\mathcal{A}$ is a *compatible collection* of bounded subsets in X if:

(1) If $A \in \mathcal{A}$ and $C \subset A$, then $C \in \mathcal{A}$.
(2) For each $A \in \mathcal{A}$ and $x \in X$, we have $A + x \in \mathcal{A}$ and $A \cup \{x\} \in \mathcal{A}$.
(3) For each $A \in \mathcal{A}$, the closed absolutely convex hull of A is in $\mathcal{A}$.

We use $\tau_{\mathcal{A}}$ to denote the topology on X^* generated by $\{\|\cdot\|_A : A \in \mathcal{A}\}$.

Our purpose in this paper is to give a general characterization for several ball separation properties. In fact, for any compatible collection $\mathcal{A}$ of bounded subsets in X, we show that every closed convex set $A \in \mathcal{A}$ is an intersection of balls if and only if the cone of $\mathcal{A}$-semidenting points of $B(X^*)$ is $\tau_{\mathcal{A}}$-dense in X^*.

2. Main results. We start this section with a lemma on linear functionals due to Phelps, whose proof can be found in [11].

Lemma 2.1 (Phelps). *Let $f, g \in S(X^*)$. If $\sup f(g^{-1}(0) \cap B(X)) < \varepsilon/2$, then either $\|f-g\| < \varepsilon$ or $\|f+g\| < \varepsilon$.*

The following theorem is a local characterization of $\mathcal{A}$-semidenting points, where $\mathcal{A}$ is a compatible family of bounded sets in a Banach space. Our main theorem is then just a consequence.

THEOREM 2.2. *Suppose $\mathcal{A}$ is a compatible family of bounded sets in X. Then $f_0 \in S(X^*)$ is an $\mathcal{A}$-semidenting point of $B(X^*)$ if and only if for any $A \in \mathcal{A}$ and $x_0 \in X$, if f_0 separates A and x_0, then there is a ball B in X with $B \supset A$ and $x_0 \notin B$.*

Proof. $\Rightarrow$. We may assume that $x_0 = 0$ and $\inf f_0(A) = a > 0$. If $\|f - f_0\|_A < a/2$, then for any $x \in A$ we have $f(x) > 0$. Since f_0 is an $\mathcal{A}$-semidenting point of $B(X^*)$, there exists a weak* slice $S(B(X^*), x_1, \delta)$ (normalized by $\|x_1\| = 1$) of $B(X^*)$ such that

$$S(B(X^*), x_1, \delta) \subset \mathrm{cone}\{f \in X^* : \|f - f_0\|_A < a/2\}.$$

Hence

$$f(x) > 0 \quad \text{for all } f \in S(B(X^*), x_1, \delta) \text{ and } x \in A. \tag{2.1}$$

Fix $x \in A$ and let $f_1 \in S(X^*)$ with $f_1(x) = \|x\|$. If $f \in S(B(X^*), x_1, \delta/3)$, then

$$\begin{aligned}\left(\left(1 - \frac{\delta}{3}\right) f - \frac{\delta}{3} f_1\right)(x_1) &\geq \left(1 - \frac{\delta}{3}\right) f(x_1) - \frac{\delta}{3} \\ &> \left(1 - \frac{\delta}{3}\right)\left(1 - \frac{\delta}{3}\right) - \frac{\delta}{3} > 1 - \delta.\end{aligned} \tag{2.2}$$

Hence

$$\left(1 - \frac{\delta}{3}\right) f - \frac{\delta}{3} f_1 \in S(B(X^*), x_1, \delta).$$

It follows from (2.1) that

$$\left(\left(1 - \frac{\delta}{3}\right) f - \frac{\delta}{3} f_1\right)(x) > 0.$$

Then

$$\left(1 - \frac{\delta}{3}\right) f(x) > \frac{\delta}{3} f_1(x) = \frac{\delta}{3}\|x\| \geq \frac{\delta}{3} d(0, A).$$

Hence

$$f(x) \geq \frac{\delta d(0, A)}{3 - \delta}.$$

Now, if $f \in S(B(X^*), x_1, \delta/3)$, then

$$f(nx_1 - x) = nf(x_1) - f(x) \leq n - \frac{\delta d(0, A)}{3 - \delta}. \tag{2.3}$$

Since A is bounded, there exists a constant M with $A \subset MB(X)$. If $f \in B(X^*) \setminus S(B(X^*), x_1, \delta/3)$, then

$$\begin{aligned}f(nx_1 - x) = nf(x_1) - f(x) &\leq n\left(1 - \frac{\delta}{3}\right) + \|x\| \\ &\leq n\left(1 - \frac{\delta}{3}\right) + M.\end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4) gives

$$\|nx_1 - x\| \le \max\left\{n - \frac{\delta d(0,A)}{3-\delta}, n\left(1-\frac{\delta}{3}\right) + M\right\}.$$

It follows that, for n large enough (clearly, n is independent of x),

$$\|nx_1 - x\| \le n - \frac{\delta d(0,A)}{3-\delta}.$$

Hence

$$A \subset B\left(nx_1, n - \frac{\delta d(0,A)}{3-\delta}\right).$$

On the other hand, it is clear that

$$0 \notin B\left(nx_1, n - \frac{\delta d(0,A)}{3-\delta}\right).$$

$\Leftarrow$. Given $\varepsilon > 0$ and $A \in \mathcal{A}$, let K be the closed absolutely convex hull of A, and $K_\varepsilon = \{x \in K : f_0(x) \ge \varepsilon\}$. We may assume

$$\{x \in K : f_0(x) \ge 4\varepsilon\} \ne \emptyset, \tag{2.5}$$

and hence $K_\varepsilon \ne \emptyset$. (Otherwise, we can choose $x_0 \in X$ such that $|f_0(x_0)| = 4\varepsilon$, and let K be the closed absolutely convex hull of $A \cup x_0$.) Then f_0 separates K_ε and 0. Hence, there is a ball $B(x_1, r)$ in X with $B(x_1, r) \supset K_\varepsilon$ and $0 \notin B(x_1, r)$.

We consider the weak* slice

$$S\left(B(X^*), \frac{x_1}{\|x_1\|}, 1 - \frac{r}{\|x_1\|}\right) = \{f \in B(X^*) : f(x_1) > r\}.$$

If $f \in S(B(X^*), x_1/\|x_1\|, 1 - r/\|x_1\|)$, then

$$f(x) = f(x_1) - f(x_1 - x) > r - \|x_1 - x\| \ge 0 \tag{2.6}$$

for all $x \in K_\varepsilon \subset B(x_1, r)$. Hence $f^{-1}(0) \cap K_\varepsilon = \emptyset$. It follows that

$$\sup f_0(f^{-1}(0) \cap K) < 2\varepsilon.$$

Applying Lemma 2.1 in the normed space $Y = \operatorname{span} K$ with K as the unit ball, we have either

$$\left\|\frac{f}{\|f\|_K} - \frac{f_0}{\|f_0\|_K}\right\|_K < \frac{4\varepsilon}{\|f_0\|_K}$$

or

$$\left\|\frac{f}{\|f\|_K} + \frac{f_0}{\|f_0\|_K}\right\|_K < \frac{4\varepsilon}{\|f_0\|_K}.$$

However, by (2.5) and (2.6), we have

$$\begin{aligned}\left\|\frac{f}{\|f\|_K}+\frac{f_0}{\|f_0\|_K}\right\|_K &= \sup\left(\frac{f}{\|f\|_K}+\frac{f_0}{\|f_0\|_K}\right)(K)\\ &\ge \sup\left(\frac{f}{\|f\|_K}+\frac{f_0}{\|f_0\|_K}\right)(K_\varepsilon)\\ &\ge \sup\left(\frac{f_0}{\|f_0\|_K}\right)(K_\varepsilon)=1\ge\frac{4\varepsilon}{\|f_0\|_K}.\end{aligned}$$

Therefore

$$\left\|\frac{f}{\|f\|_K}-\frac{f_0}{\|f_0\|_K}\right\|_K<\frac{4\varepsilon}{\|f_0\|_K}.$$

It follows that

$$\left\|f_0-\frac{\|f_0\|_K}{\|f\|_K}f\right\|_A\le\left\|f_0-\frac{\|f_0\|_K}{\|f\|_K}f\right\|_K<4\varepsilon.$$

Then

$$S\left(B(X^*),\frac{x_1}{\|x_1\|},1-\frac{r}{\|x_1\|}\right)\subset\operatorname{cone}\{f\in X^*:\|f-f_0\|_A<4\varepsilon\}.$$

Hence f_0 is an $\mathcal{A}$-semidenting point of $B(X^*)$. ∎

We are now ready to prove the main result of this paper.

THEOREM 2.3. *Suppose $\mathcal{A}$ is a compatible family of bounded sets in X. Then the following conditions are equivalent:*

(1) *The cone of $\mathcal{A}$-semidenting points of $B(X^*)$ is $\tau_{\mathcal{A}}$-dense in X^*.*
(2) *Any $f\in S(X^*)$ is an $\mathcal{A}$-semidenting point of $B(X^*)$.*
(3) *Every closed convex set $A\in\mathcal{A}$ is an intersection of balls.*

Proof. (1)⇒(3). Suppose that $A\in\mathcal{A}$ is a closed convex set and $x_0\notin A$. We need to show that there exists a ball $B\subset X$ such that $A\subset B$ and $x_0\notin B$. We can assume that $x_0=0$. By the Hahn–Banach theorem there exists $f\in S(X^*)$ such that $\inf f(A)>0$. By (1), there exist $\lambda>0$ and $f_0\in S(X^*)$ which is an $\mathcal{A}$-semidenting point of $B(X^*)$ such that

$$\|f-\lambda f_0\|_A<\inf f(A).$$

Hence $\inf f_0(A)>0$. It follows from Theorem 2.2 that there exists a ball $B\subset X$ such that $A\subset B$ and $0\notin B$.

(3)⇒(2). Use Theorem 2.2.

(2)⇒(1). Trivial. ∎

3. Remarks. First, we recall the definition of a semidenting point and the characterization of Banach spaces with the Mazur intersection property from [2].

Definition 3.1. We say $f \in S(X^*)$ is a *semidenting point* of $B(X^*)$ if for every $\varepsilon > 0$ there exists a weak* slice S of $B(X^*)$ such that $S \subset \{g \in X^* : \|g - f\| < \varepsilon\}$.

Theorem 3.2 (Chen and Lin). *Given a Banach space X, the following conditions are equivalent:*

(1) *X has the Mazur intersection property.*
(2) *Any $f \in S(X^*)$ is a semidenting point of $B(X^*)$.*
(3) *The set of semidenting points of $B(X^*)$ is norm dense in $S(X^*)$.*

Theorem 2.3 gives a characterization of the ball separation property described in (3). This property depends on $\mathcal{A}$. For example, when $\mathcal{A}$ consists of all bounded subsets of X, this is just the Mazur intersection property. When $\mathcal{A}$ is the compatible family generated by all compact subsets of X, this property is just the CIP.

If $\mathcal{A}$ consists of all bounded subsets of X, it is clear that $\tau_{\mathcal{A}}$ is the norm topology, and the following result shows the that $\mathcal{A}$-semidenting points are precisely the semidenting points. Hence, when $\mathcal{A}$ consists of all bounded subsets of X, Theorem 2.3 is just Theorem 3.2.

Proposition 3.3. *If $\mathcal{A}$ consists of all bounded subsets of X, then every $\mathcal{A}$-semidenting point of $B(X^*)$ is a semidenting point of $B(X^*)$ and vice versa.*

Proof. Suppose that f_0 is an $\mathcal{A}$-semidenting point of $B(X^*)$. For the bounded subset $B(X^*)$ and $0 < \varepsilon < 1/2$, there exists a weak* slice

$$S(B(X^*), x_0, \delta) \subset \text{cone}\{g \in X^* : \|g - f_0\| < \varepsilon\}.$$

It is clear that we can choose $\delta \leq \varepsilon$ and $\|x_0\| = 1$.

If $f \in S(B(X^*), x_0, \delta)$, there exist $g \in X^*$ and $\lambda > 0$ such that $\|f_0 - g\| < \varepsilon$ and $f = \lambda g$. Since $\|f_0\| = 1$,

$$1 - \varepsilon < \|g\| < 1 + \varepsilon. \tag{3.1}$$

Clearly $1 - \delta \leq \|f\| \leq 1$, and hence

$$1 - \varepsilon \leq 1 - \delta \leq \|\lambda g\| \leq 1. \tag{3.2}$$

Combining (3.1) with (3.2) gives

$$\frac{1 - \varepsilon}{1 + \varepsilon} < \lambda < \frac{1}{1 - \varepsilon}.$$

Noting that $\varepsilon < 1/2$, we have

$$-2\varepsilon < \frac{-2\varepsilon}{1 + \varepsilon} < \lambda - 1 < \frac{\varepsilon}{1 - \varepsilon} < 2\varepsilon.$$

Then

$$\|f - f_0\| = \|\lambda g - \lambda f_0 + \lambda f_0 - f_0\| = \lambda\|g - f_0\| + |\lambda - 1|$$
$$< \frac{\varepsilon}{1-\varepsilon} + 2\varepsilon < 4\varepsilon.$$

It follows that

$$S(B(X^*), x_0, \delta) \subset \{g \in X^* : \|g - f_0\| < 4\varepsilon\}.$$

Hence f_0 is also a semidenting point of $B(X^*)$.

Conversely, suppose f_0 is a semidenting point of $B(X^*)$. For every $\varepsilon > 0$, there exists a weak* slice S of $B(X^*)$ such that

$$S \subset \{g \in X^* : \|g - f_0\| < \varepsilon\}.$$

Let A be any bounded subset of X. Without loss of generality, we may assume $A \subset B(X)$. Hence $\|g - f_0\|_A \le \|g - f_0\|$ for every $g \in X^*$. Now we have

$$S \subset \{g \in X^* : \|g - f_0\| < \varepsilon\} \subset \{g \in X^* : \|g - f_0\|_A < \varepsilon\}$$
$$\subset \mathrm{cone}\{g \in X^* : \|g - f_0\|_A < \varepsilon\}.$$

It follows that f_0 is also an $\mathcal{A}$-semidenting point of $B(X^*)$. ∎

Acknowledgements. We want to thank the referee for his helpful comments, suggestions and careful corrections to the previous version of this manuscript. We are also grateful to the Editor, Professor Tadeusz Figiel, for a helpful conversation. We thank Professor Cheng Lixin for helpful conversations and encouragement. The first named author also wants to thank Professor Niels J. Laustsen for his helpful suggestions.

Research of Y. B. Dong was supported by the Natural Science Foundation of China, grant 11201353. Research of Q. J. Cheng was supported by the Natural Science Foundation of China, grant 11001231.

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Received September 22, 2012
Revised version July 14, 2013 (7624)