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On some Brownian functionals and their applications to moments in the lognormal stochastic volatility model

by

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Abstract. We find a probabilistic representation of the Laplace transform of some special functional of geometric Brownian motion using squared Bessel and radial Ornstein–Uhlenbeck processes. Knowing the transition density functions of these processes, we obtain closed formulas for certain expectations of the relevant functional. Among other things we compute the Laplace transform of the exponent of the T transforms of Brownian motion with drift used by Donati-Martin, Matsumoto, and Yor in a variety of identities of duality type between functionals of Brownian motion. We also present links between geometric Brownian motion and Markov processes studied by Matsumoto and Yor. These results have wide applications. As an example of their use in financial mathematics we find the moments of processes representing the asset price in the lognormal volatility model.

1. Introduction. The aim of this paper is to present new results concerning some functionals of Brownian motion with drift. We also give some applications of those results to financial mathematics. The laws of many different functionals of Brownian motion have been studied in recent years (see, among others, [BS], [DY], [DGY], [MY4], [MY5]), but some of the results obtained cannot be effectively used in applications. The distribution of $\int_0^t e^{B_u^{(\mu)}}\,du$, where $B_t^{(\mu)} = B_t + \mu t$ with B a standard Brownian motion, is an example of such a situation. This distribution can be characterized by the Hartman–Watson distribution, but the oscillating nature of the latter causes difficulties in numerical calculations (see [BRY] and [G]).

We study the laws of special functionals of geometric Brownian motion, and find results convenient for numerical applications. We investigate the functionals of geometric Brownian motion

$$Y_t^{(\mu)} := \exp(B_t + (\mu - 1/2)t) \quad \text{for } \mu \in \mathbb{R}.$$

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In particular, we study properties of the functionals

$$\Gamma_t = \frac{Y_t^{(0)}}{1+\beta\int_0^t Y_s^{(0)}\,ds} \quad \text{and} \quad 1+\beta A_t^{(\mu)} \quad \text{for } \beta > 0,$$

where $A_t^{(\mu)} := \int_0^t (Y_u^{(\mu)})^2\,du$. We give a formula for the Laplace transform of Γ_t. We also present a probabilistic representation of the Laplace transform of Γ_t in terms of squared Bessel and radial Ornstein–Uhlenbeck processes. Knowing the transition density functions of these processes, we obtain computable formulas for certain expectations of the relevant functionals.

Two important advantages of the new results are, first, that they are obtained for fixed t (and not for stochastic time), and secondly, that they can be effectively used in numerical computations. Instead of using the Hartman–Watson distribution we reduce the problem of computing expectations of a functional of Brownian motion to computing several less complicated expectations.

Our results can be applied in various areas where expectations of Brownian motion functionals are calculated. As a first example we compute the Laplace transform of the exponent of T_α and $T_{\alpha/e^{B_t^{(\mu)}}}$ transforms of Brownian motion with drift, which are used by Donati-Martin, Matsumoto, and Yor in various identities of duality type between functionals of Brownian motion (see [DMY] for a detailed study). In physics, exponential functionals of Brownian motion play a crucial role in the context of one-dimensional classical diffusion in a random environment. The integral of geometric Brownian motion occurs in the study of the transport properties of disordered samples of finite length (see [CMY]).

As another example of applications, this time in financial mathematics, we compute the moments $\mathbb{E}X_t^\alpha$, for $\alpha > 0$, of the processes X_t representing the asset price in an important stochastic volatility model, the lognormal volatility model. Explicit forms of these moments have not been known so far. Computing these moments is crucial in problems of pricing derivatives (for instance, the necessity of a "convexity correction" to the forward rate price for a broad class of interest rate derivatives; for details see, e.g., [BM]). It is also important for approximations of characteristic functions of random variables with very complicated distributions. We find a duality between the Laplace transform of Γ_t and moments.

We now give a detailed plan of this paper. In Subsection 2.1, we present a method of calculating the moments $\mathbb{E}\Gamma_t^k$ for $k \in \mathbb{Z}$ (Proposition 2.3, Remark 2.4) and investigate the connection of the functional Γ with some special diffusion process (Theorem 2.5). In Subsection 2.2, we investigate various properties of $1+\beta A_t^{(\mu)}$. We find two different probabilistic representations of the Laplace transform of $(1+\beta A_t^{(\mu)})^{-1}$ (Theorems 2.9 and 2.10), and formulas for $\mathbb{E}\ln(1+\beta A_t^{(\mu)})$ and $\mathbb{E}(1+\beta A_t^{(\mu)})^{-1}$ (Corollary 2.14). It turns out that for an arbitrary strictly positive random variable we can find a representation of the Laplace transform of $(s+\xi)^{-1}$ for $s \geq 0$ in terms of a squared Bessel process (Lemma 2.6). Moreover, we find expressions for $\mathbb{E}\ln(1+\beta\xi)$, $\mathbb{E}(1+\beta\xi)^{-1}$ as well as for $\mathbb{E}f(1/\xi)$ for f being a Bernstein function (Theorems 2.13 and 2.31).

Theorem 2.11 gives identity in law of two squared Bessel processes with index -1 with different initial laws, one being the law of a squared 0-dimensional radial Ornstein–Uhlenbeck process with parameter -1. In Theorem 2.20, we express the Laplace transform of Γ_t in terms of the functions F_x introduced by Matsumoto and Yor in [MY4, Thm. 5.6]. Moreover, we find some interesting connections between $\mathbb{E}((1+\beta A_t^{(\mu)})^{-1})$ and the conditional expectation of functionals of geometric Brownian motion with opposite drift. Notice that we establish all those results for a fixed t.

In Section 3, we find the Laplace transforms of the exponent of T_α and $T_{\alpha/e^{B_t^{(\mu)}}}$ transforms and outline their applications.

Section 4 illustrates how one can use previous results in mathematical finance. Let X be the asset price process which is the unique strong solution of $dX_t = Y_t X_t dW_t$ with Y being a geometric Brownian motion (GBM) (this model is called the *lognormal stochastic volatility model* or the *Hull–White model*, see [HW]). The distribution of the asset price for the lognormal stochastic volatility model is known but its degree of complication and numerical obstacles suggest looking for simpler approximations. Jourdain [J] gave conditions on existence of moments in the lognormal stochastic volatility model, but gave no method of computing them.

In this work we find that the moment is equal to the Laplace transform of the process Γ (Theorem 4.1), so we express moments in terms of functions F_x. We find a closed formula for $\mathbb{E}X_{2T_\lambda}^\alpha$ for the model with random time T_λ which is an exponential random variable independent of the Brownian motion driving the diffusion Y (Theorem 4.6).

Summing up, we present explicit forms of some interesting functionals of Brownian motion. We compute the Laplace transform of the exponent of T_α and $T_{\alpha/e^{B_t^{(\mu)}}}$ transforms of Brownian motion with drift used by Donati-Martin et al. [DMY] in many different identities. Finally, we compute the moments of the asset price process in the lognormal stochastic volatility model.

2. Properties of some functionals of geometric Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ and Brownian motion B on it. Functionals of the process $Y_t :=$

$e^{B_t - t/2}$ play a crucial role in many problems of modern stochastic analysis. Studying the properties of the integral $\int_0^t e^{2B_u - u} du = \int_0^t Y_u^2 du$ is motivated by the problem of pricing Asian options (see [DGY], [MY]). The process $Y_t^{-2} \int_0^t Y_u^2 du$ has been considered by Matsumoto and Yor in several works concerning laws of Brownian motion functionals. Along with $Y_t^{-1} \int_0^t Y_u^2 du$ it plays a central role in a generalization of Pitman's $2M - X$ theorem (for details see for instance [MY2], [MY3], [MY5]). Here, we investigate, among other, the properties of the functional Γ defined, for $\beta > 0$, by

$$\Gamma_t = \frac{Y_t}{1 + \beta \int_0^t Y_s \, ds}. \tag{2.1}$$

It turns out that this process plays a crucial role in the problem of computing the moments of the asset price in the lognormal stochastic volatility model (see Section 4).

Remark 2.1. From the definition it follows that $\Gamma_0 = 1$. If we want to consider the process Γ which starts from a positive point x, it is enough to replace in (2.1) the process Y by $\widehat{Y}^{(x)}$, where $\widehat{Y}_t^{(x)} := x e^{B_t - t/2}$.

In this section we find some new properties of the exponential functional of the form

$$A_t^{(\mu)} := \int_0^t (Y_u^{(\mu)})^2 \, du, \tag{2.2}$$

where, for $\mu \in \mathbb{R}$,

$$Y_t^{(\mu)} := \exp(B_t + (\mu - 1/2)t).$$

Moreover, let

$$B_t^{(\mu)} = B_t + \mu t. \tag{2.3}$$

We also consider the random variable (often called perpetuity in the mathematical finance literature)

$$A_\infty^{(\mu)} := \int_0^\infty (Y_u^{(\mu)})^2 \, du.$$

In what follows we write $Y = Y^{(0)}$, $A = A^{(0)}$. We start by investigating Γ.

2.1. Some properties of Γ

Proposition 2.2. *Let Γ be given by* (2.1). *Then*

$$d\Gamma_t = \Gamma_t dB_t - \beta \Gamma_t^2 \, dt. \tag{2.4}$$

Proof. This follows easily from the Itô lemma. ∎

PROPOSITION 2.3. *Let Γ be given by* (2.1), *and set $p_k(t) = \int_0^t \mathbb{E}\Gamma_u^k\,du$ for $k \in \mathbb{Z}$ and $t \geq 0$. Then the sequence (p_k) satisfies the recurrence*

$$p_k'(t) = 1 + \frac{k(k-1)}{2}p_k(t) - \beta k p_{k+1}(t), \tag{2.5}$$

and

$$p_1(t) = \frac{1}{\beta}\mathbb{E}\Big(\ln\Big(1 + \beta\int_0^t Y_u\,du\Big)\Big). \tag{2.6}$$

Proof. By Proposition 2.2 and the Itô lemma we have

$$\Gamma_t^k = 1 + k\int_0^t \Gamma_u^k\,dB_u - k\beta\int_0^t \Gamma_u^{k+1}\,du + \frac{k(k-1)}{2}\int_0^t \Gamma_u^k\,du. \tag{2.7}$$

The local martingale $\int_0^t \Gamma_u^k\,dB_u$ is a true martingale. Indeed, for $k > 0$

$$\mathbb{E}\int_0^t \Gamma_u^{2k}\,du \leq \mathbb{E}\int_0^t Y_u^{2k}\,du < \infty.$$

If $-l = k < 0$, we estimate $(1/\Gamma_t)^{2l}$ using the inequality $(x+y)^{2l} \leq C(x^{2l} + y^{2l})$ for $l, x, y > 0$, and hence for $k < 0$ we have

$$\mathbb{E}\int_0^t \Gamma_u^{2k}\,du \leq C_1\mathbb{E}e^{-2kB_t} + C_2\mathbb{E}\int_0^t e^{-2kB_s}ds < \infty.$$

Taking the expectation of both sides of (2.7) we obtain (2.5). Further,

$$\begin{aligned}
p_1'(t) &= \mathbb{E}\bigg(\frac{Y_t}{1 + \beta\int_0^t Y_u\,du}\bigg) = \frac{1}{\beta}\mathbb{E}\frac{\partial}{\partial t}\Big(\ln\Big(1 + \beta\int_0^t Y_u\,du\Big)\Big)\\
&= \frac{1}{\beta}\frac{\partial}{\partial t}\mathbb{E}\Big(\ln\Big(1 + \beta\int_0^t Y_u\,du\Big)\Big),
\end{aligned}$$

as $\ln(1+\beta\int_0^t Y_u\,du) \leq \ln(1+\beta\int_0^T Y_u\,du)$ and $\mathbb{E}\ln(1+\beta\int_0^T Y_u\,du) < \infty$, which implies (2.6). ■

REMARK 2.4. Since, by (2.5),

$$\mathbb{E}\Gamma_t^k = 1 + \frac{k(k-1)}{2}p_k(t) - \beta k p_{k+1}(t), \tag{2.8}$$

Proposition 2.3 allows us to compute $\mathbb{E}\Gamma_t^k$ for $k \in \mathbb{Z}$. Taking $k = -1$, we easily deduce from (2.5) that

$$p_{-1}'(t) = 1 + \beta t + p_{-1}(t), \quad p_{-1}(0) = 0.$$

The solution of this differential equation is $p_{-1}(t) = (\beta+1)e^t - \beta t - (1+\beta)$. Notice that having p_{-1} we get, recursively from (2.5), the functions $p_{-2}, p_{-3}, \ldots$. Using p_1 we can determine $p_2, p_3, \ldots$ So, using (2.8), we can find all moments $\mathbb{E}\Gamma_t^k$ for $k \in \mathbb{Z}$, provided we know p_1. The function p_1 is given by (2.6), so we have to find $\mathbb{E}(\ln(1 + \beta \int_0^t Y_u\, du))$. Formulas for p_1 are presented in Corollaries 2.18 and 2.26 below.

Now we investigate the connection of Γ with a diffusion $V \geq 1$ given by the SDE

$$dV_t = \sqrt{V_t^2 - 1}\, dB_t. \tag{2.9}$$

We express the Laplace transform of Γ in terms of the Laplace transform of V. For a detailed discussion of the diffusion given by (2.9) see [JW].

THEOREM 2.5. *Let $\lambda \geq 0$, $\beta > 0$ and V be as above with $V_0 = 1 + \lambda/\beta$, and let Γ be given by* (2.1). *Then*

$$\mathbb{E}e^{-\lambda \Gamma_t} = \mathbb{E}e^{-\beta(V_t - 1)}.$$

Proof. Let $\theta_t = \beta\Gamma_t$. Then, by (2.4),

$$d\theta_t = \theta_t dB_t - \theta_t^2 dt$$

and $\theta_0 = \beta$. Moreover, for $x \geq 0$,

$$de^{-x\theta_t} = -e^{-x\theta_t}(x\theta_t dB_t - x\theta_t^2 dt) + \tfrac{1}{2}e^{-x\theta_t}x^2\theta_t^2 dt.$$

Taking $p(t,x) := \mathbb{E}e^{-x\theta_t}$, we deduce from the last expression that p satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \left(x + \frac{1}{2}x^2\right)\frac{\partial^2 p}{\partial x^2}, \tag{2.10}$$

with $p(0,x) = e^{-x\beta}$.

Therefore, the Laplace transform of θ_t, for $\lambda \geq 0$, is a solution of (2.10). Consider the SDE

$$dH_t = \sqrt{H_t^2 + 2H_t}\, dB_t, \qquad H_0 = \lambda/\beta \geq 0. \tag{2.11}$$

By a short calculation, for any $0 \leq y \leq x$,

$$\left|\sqrt{x^2 + 2x} - \sqrt{y^2 + 2y}\right| \leq \sqrt{(x-y)^2 + 2(x-y)},$$

so there exists a weak solution to SDE (2.11), and trajectory uniqueness holds for (2.11) (see [KS, Thm. 5.5.4] and [RW, Thm. 5.40.1]). Thus by the definition and properties of the infinitesimal generator of the process and by uniqueness of solution, the function $\bar{p}(t,x) := \mathbb{E}e^{-xH_t}$ is the solution of (2.10) with $\bar{p}(0,x) = e^{-x\lambda/\beta}$ after changing the terminal condition to the initial one (see for example [KS, Thm. 5.7.6]). Therefore,

$$p(t, \lambda/\beta) = \bar{p}(t, \beta). \tag{2.12}$$

Let us define the diffusion $V_t := H_t + 1$. It is easy to check that

$$dV_t = \sqrt{V_t^2 - 1}\, dB_t, \qquad V_0 = \lambda/\beta + 1. \tag{2.13}$$

Using the same arguments as before we see that there exists a weak solution to (2.13), and trajectory uniqueness holds for (2.13). Thus, by (2.12),

$$\mathbb{E}e^{-\lambda\Gamma_t} = \mathbb{E}e^{-\frac{\lambda}{\beta}\theta_t} = p(t,\lambda/\beta) = \mathbb{E}e^{-\beta H_t} = e^{\beta}\mathbb{E}e^{-\beta V_t}.$$

This ends the proof. ∎

2.2. Some properties of $1+\beta A_t^{(\mu)}$**.** We start by computating the Laplace transform of $(1+\beta A_t^{(\mu)})^{-1}$. It is worth remarking that we compute it for a fixed time t. It is known (see for instance [MY4]) that computing expectations for functionals of geometric Brownian motion for a fixed time is in general much more difficult than for stochastic time (see also Subsection 4.2.2).

Let us recall that a squared δ-dimensional radial Ornstein–Uhlenbeck process with parameter λ for $\delta \ge 0$ and $\lambda \in \mathbb{R}$ is the unique solution of the SDE

$$X_t = x + \int_0^t (\delta - 2\lambda X_s)\, ds + 2\int_0^t \sqrt{X_s}\, dW_s, \tag{2.14}$$

where W is a standard Brownian motion (see [BS] and [GJY] for detailed studies of these processes). If $\lambda = 0$, then the strong solution of (2.14) is a squared δ-dimensional Bessel process (see [RY]). The number $\delta/2-1$ is called the *index* of the process.

It turns out that for an arbitrary strictly positive random variable we can find a representation of the Laplace transform of $(s+\xi)^{-1}$ for $s \ge 0$ in terms of a squared Bessel process.

LEMMA 2.6. *Let ξ be a strictly positive random variable. Then for any $x, s \ge 0$,*

$$\mathbb{E}\exp\left(-\frac{x}{s+\xi}\right) = \mathbb{E}G(R^x(s/2)), \tag{2.15}$$

where R^x is a squared Bessel process with index -1 starting at x, and

$$G(x) = \mathbb{E}\exp(-x/\xi). \tag{2.16}$$

Proof. Let us take a copy of R^x independent of ξ. By (2.16),

$$\begin{aligned}\mathbb{E}G(R^x(s/2)) &= \mathbb{E}\exp\left(-\frac{R^x(s/2)}{\xi}\right) = \mathbb{E}\mathbb{E}\big(\exp\big(-\xi^{-1}R^x(s/2)\big) \bigm| \xi\big)\\ &= \mathbb{E}\exp\left(-\frac{x\xi^{-1}}{1+\xi^{-1}s}\right) = \mathbb{E}\exp\left(-\frac{x}{s+\xi}\right),\end{aligned}$$

where we use the formula for the Laplace transform of a squared Bessel process (see [RY, Chapter XI, p. 441]). ∎

REMARK 2.7. For fixed $t > 0$ Matsumoto and Yor's result [MY4, Thm. 5.6] states that

$$\mathbb{E}\left(\exp\left(-\frac{x}{A_t^{(1/2)}}\right) \,\middle|\, B_t = y\right) = \exp\left(-\frac{\varphi_x^2(y) - y^2}{2t}\right), \tag{2.17}$$

where

$$\begin{aligned} \varphi_x(y) &= \operatorname{arcosh}(xe^{-y} + \cosh(y)) \\ &= \ln\Big(xe^{-y} + \cosh(y) + \sqrt{x^2e^{-2y} + \sinh^2(y) + 2xe^{-y}\cosh(y)}\Big). \end{aligned} \tag{2.18}$$

We use this result to find the function G from Lemma 2.6 for $\xi = A_t^{(\mu)}$.

LEMMA 2.8. *Let*

$$G_t^{(\mu)}(x) := \mathbb{E}\exp\left(-\frac{x}{A_t^{(\mu)}}\right). \tag{2.19}$$

Then

$$G_t^{(\mu)}(x) = e^{-t\mu^2/2}\mathbb{E}\exp\left(\mu B_t + \frac{1}{2t}(B_t^2 - \varphi_x^2(B_t))\right), \tag{2.20}$$

where B is a standard Brownian motion and φ_x is given by (2.18).

Proof. Define a new probability measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\mu B_t - \frac{\mu^2}{2}t\right). \tag{2.21}$$

Since B is a standard Brownian motion under $\mathbb{P}$, it follows that $\widehat{B}_t = B_t + \mu t$ is a standard Brownian motion under $\mathbb{Q}$, by the Girsanov theorem. For $\widehat{A}_t^{(1/2)} := \int_0^t e^{2\widehat{B}_u}\,du$ we have $A_t^{(\mu)} = \widehat{A}_t^{(1/2)}$, so

$$\begin{aligned} G_t^{(\mu)}(x) &= \mathbb{E}e^{-x/A_t^{(\mu)}} \\ &= \mathbb{E}_{\mathbb{Q}}(e^{-x/A_t^{(\mu)}}e^{\mu B_t + \mu^2 t/2}) = e^{-\mu^2 t/2}\mathbb{E}_{\mathbb{Q}}\exp\left(\mu\widehat{B}_t - \frac{x}{\widehat{A}_t^{(1/2)}}\right). \end{aligned}$$

Now, we use the result of Matsumoto and Yor recalled in Remark 2.7 to obtain

$$G_t^{(\mu)}(x) = e^{-t\mu^2/2}\mathbb{E}\exp\left(\mu B_t + \frac{1}{2t}(B_t^2 - \varphi_x^2(B_t))\right). \ \blacksquare$$

THEOREM 2.9. *Fix $\beta > 0, \mu \in \mathbb{R}$ and $t > 0$. Then, for any $\lambda \geq 0$,*

$$\mathbb{E}\exp\left(-\frac{\lambda}{1 + \beta A_t^{(\mu)}}\right) = \mathbb{E}G_t^{(\mu)}(R^{\lambda/\beta}(1/(2\beta))),$$

where $R^{\lambda/\beta}$ is a squared Bessel process with index -1 starting at λ/β, and $G_t^{(\mu)}$ is defined by (2.19).

Proof. This is a direct corollary of Lemmas 2.6 and 2.8. ■

THEOREM 2.10. *Let $\beta_0 > 0$, $\beta \in (0, \beta_0]$, $\mu \in \mathbb{R}$ and $t > 0$. Let R^x, $x > 0$, be a squared Bessel process with index -1 starting at x. Then, for any $\lambda \geq 0$,*

$$\mathbb{E}\exp\biggl(-\frac{\lambda}{1+\beta A_t^{(\mu)}}\biggr) = \mathbb{E}\phi_t(\theta^\lambda(-\ln\sqrt{\beta/\beta_0})),$$

where $\theta^\lambda(t)$ is a squared 0-dimensional radial Ornstein–Uhlenbeck process with parameter -1 such that $\theta^\lambda(0) = \lambda$ and, for $x > 0$,

$$\phi_t(x) = \mathbb{E}G_t^{(\mu)}(R^{x/\beta_0}(1/(2\beta_0))),$$

with $G_t^{(\mu)}$ given by (2.20).

Proof. Let B be a standard Brownian motion under $\mathbb{P}$. Set

$$p(s,x) := \mathbb{E}\exp\biggl(-\frac{x}{1+\beta_0 e^{-2s}A_t^{(\mu)}}\biggr)$$

for $x, s \geq 0$. Observe that $p(s,x) \leq 1$. The function p belongs to the class $C^{1,2}([0,\infty) \times [0,\infty))$ by the Lebesgue theorem and satisfies the partial differential equation

$$\frac{\partial p}{\partial s} = 2x\biggl(\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x^2}\biggr), \tag{2.22}$$

for $s, x \geq 0$ and $p(0,x) = \phi_t(x)$.

Observe that the right-hand side of (2.22) is now the infinitesimal generator of a 0-dimensional radial Ornstein–Uhlenbeck process with parameter -1. Thus by the definition and properties of infinitesimal generators (see for example [KS, Thm. 5.7.6]) one obtains

$$\mathbb{E}\exp\biggl(-\frac{\lambda}{1+\beta A_t^{(\mu)}}\biggr) = \mathbb{E}\phi_t(\theta^\lambda(-\ln\sqrt{\beta/\beta_0}))$$

for

$$\phi_t(x) = \mathbb{E}G_t^{(\mu)}(R^{x/\beta_0}(1/(2\beta_0))),$$

which follows from Theorem 2.9. The form of the function $G_t^{(\mu)}$ follows from Lemma 2.8. This finishes the proof. ■

Theorems 2.9 and 2.10 give two probabilistic representations for $\mathbb{E}\exp\bigl(-\frac{\lambda}{1+\beta A_t^{(\mu)}}\bigr)$, which leads to a new interesting equality in law stated in the next theorem.

THEOREM 2.11. *Fix $\lambda \geq 0$, $\beta > 0$ and $s \geq 0$. Let θ^λ be a squared 0-dimensional radial Ornstein–Uhlenbeck process with parameter -1 such that $\theta^\lambda(0) = \lambda$, and R^x be a squared Bessel process with index -1 starting* *at x and independent of θ^λ. Then $R^{e^{2s}\lambda/\beta}(e^{2s}/(2\beta))$ and $R^{\theta^\lambda(s)/\beta}(1/(2\beta))$ have the same law.*

Proof. Taking $\beta_0 \le \beta$ such that $s = -\ln\sqrt{\beta_0/\beta}$ we infer from Theorems 2.9 and 2.10 that

$$\mathbb{E}\phi_t(\theta^\lambda(s)) = \mathbb{E}G_t^{(\mu)}(R^{e^{2s}\lambda/\beta}(e^{2s}/(2\beta))).$$

Hence the definition of ϕ_t yields

$$\mathbb{E}G_t^{(\mu)}(R^{\theta^\lambda(s)/\beta}(1/(2\beta))) = \mathbb{E}G_t^{(\mu)}(R^{e^{2s}\lambda/\beta}(e^{2s}/(2\beta))). \tag{2.23}$$

Let us denote

$$\psi_1 := R^{\theta^\lambda(s)/\beta}(1/(2\beta)), \qquad \psi_2 := R^{e^{2s}\lambda/\beta}(e^{2s}/(2\beta)).$$

Then (2.23) takes the form

$$\mathbb{E}G_t^{(\mu)}(\psi_1) = \mathbb{E}G_t^{(\mu)}(\psi_2).$$

Recalling the definition of $G_t^{(\mu)}$ we obtain

$$\mathbb{E}e^{\mu B_t + \frac{1}{2t}(B_t^2 - \varphi_{\psi_1}^2(B_t))} = \mathbb{E}e^{\mu B_t + \frac{1}{2t}(B_t^2 - \varphi_{\psi_2}^2(B_t))}.$$

Observe that the last equality holds for any $\mu \in \mathbb{R}$, so for any $z \in \mathbb{R}$ and $t > 0$,

$$\mathbb{E}e^{-\frac{1}{2t}\varphi_{\psi_1}^2(z)} = \mathbb{E}e^{-\frac{1}{2t}\varphi_{\psi_2}^2(z)}.$$

Hence for any $z \in \mathbb{R}$,

$$\varphi_{\psi_1}^2(z) \overset{\text{(law)}}{=} \varphi_{\psi_2}^2(z). \tag{2.24}$$

Observe that

$$\varphi_{\psi_1}(0) = \ln\Big(\psi_1 + 1 + \sqrt{\psi_1^2 + 2\psi_1}\Big).$$

As the function $x \mapsto x + 1 + \sqrt{x^2 + 2x}$ is monotone for $x > 0$ we conclude that $\psi_1 \overset{\text{(law)}}{=} \psi_2$, which finishes the proof. ∎

From the last theorem we immediately obtain

COROLLARY 2.12. *Let θ^λ be a squared 0-dimensional radial Ornstein–Uhlenbeck process with parameter -1 such that $\theta^\lambda(0) = \lambda > 0$. Fix $t \ge 0$. Then for $\beta > 0$ and $\gamma \ge 0$ we have*

$$\mathbb{E}\exp\Big(-\frac{\gamma}{\gamma+\beta}\theta^\lambda(t)\Big) = \exp\Big(-\frac{\lambda\gamma e^{2t}}{\beta + \gamma e^{2t}}\Big). \tag{2.25}$$

Proof. Observe that for R^x as in Theorem 2.11 and $x > 0$,

$$\mathbb{E}e^{-\gamma R^x(t)} = e^{-\frac{x\gamma}{1+2\gamma t}}.$$

Consequently,

$$\mathbb{E}\exp\big(-\gamma R^{\theta^\lambda(s)/\beta}(1/(2\beta))\big) = \mathbb{E}\exp\Big(-\frac{\gamma}{\gamma+\beta}\theta^\lambda(s)\Big)$$

and

$$\mathbb{E}\exp\big(-\gamma R^{e^{2s}\lambda/\beta}(e^{2s}/(2\beta))\big) = \exp\Big(-\frac{\lambda\gamma e^{2s}}{\beta+\gamma e^{2s}}\Big).$$

The assertion now follows from Theorem 2.11. ∎

THEOREM 2.13. *For a strictly positive and integrable random variable* ξ *and* $\beta \geq 0$,

$$\mathbb{E}\ln(1+\beta\xi) = \int_0^\infty \frac{G(y)}{y}(1-e^{-y\beta})\,dy, \tag{2.26}$$

$$\mathbb{E}\Big(\frac{1}{1+\beta\xi}\Big) = 1-\beta\int_0^\infty G(y)e^{-y\beta}\,dy, \tag{2.27}$$

with G *given by* (2.16).

Proof. Observe that $\lambda/(\lambda+\beta) = \beta\int_0^\infty(1-e^{-\lambda s})e^{-\beta s}\,ds$. Taking $\lambda = 1/\xi$ we have

$$\frac{1}{1+\beta\xi} = \beta\int_0^\infty(1-e^{-y/\xi})e^{-\beta y}\,dy.$$

Next, we take the expectation and use Fubini's theorem to obtain

$$\mathbb{E}\Big(\frac{1}{1+\beta\xi}\Big) = 1-\beta\int_0^\infty G(y)e^{-y\beta}\,dy, \tag{2.28}$$

so (2.27) holds. Since

$$\mathbb{E}\ln(1+\beta\xi) \leq 1+\beta\mathbb{E}\xi < \infty,$$

we can differentiate (2.28) under the expectation with respect to β and

$$\frac{\partial}{\partial\beta}\mathbb{E}\ln(1+\beta\xi) = \frac{1}{\beta}-\frac{1}{\beta}\mathbb{E}\Big(\frac{1}{1+\beta\xi}\Big) = \int_0^\infty G(y)e^{-y\beta}\,dy.$$

This implies (2.26). ∎

Using the above results we can obtain the expectations of $\ln(1+\beta A_t^{(\mu)})$ and $(1+\beta A_t^{(\mu)})^{-1}$.

COROLLARY 2.14. *Fix* $\beta > 0$, $\mu \in \mathbb{R}$ *and* $t \geq 0$. *Then*

$$\mathbb{E}\ln(1+\beta A_t^{(\mu)}) = \int_0^\infty \frac{G_t^{(\mu)}(y)}{y}(1-e^{-y\beta})\,dy, \tag{2.29}$$

$$\mathbb{E}\Big(\frac{1}{1+\beta A_t^{(\mu)}}\Big) = 1-\beta\int_0^\infty G_t^{(\mu)}(y)e^{-y\beta}\,dy, \tag{2.30}$$

where $G_t^{(\mu)}$ *is given by* (2.19).

Proof. Apply Theorem 2.13 for $\xi = A_t^{(\mu)}$. ∎

REMARK 2.15. Formula (2.30) gives a closed form of $\mathbb{E}((1+\beta A_t^{(\mu)})^{-1})$ for $\beta > 0$. The density of $A_t^{(\mu)}$ is known, but due to the complicated nature of the Hartman–Watson distribution, it can hardly be used for numerical computations (see for instance [BRY] and [MY4]). Since the simple form of $G_t^{(\mu)}$ is given explicitly, the formulas (2.29) and (2.30) allow one to obtain $\mathbb{E}\ln(1+\beta A_t^{(\mu)})$ and $\mathbb{E}((1+\beta A_t^{(\mu)})^{-1})$ numerically.

It turns out that the above arguments for $f(\lambda) = (1+\beta/\lambda)^{-1}$ can be generalized to Bernstein functions. Recall that $f : (0,\infty) \to \mathbb{R}$ is a *Bernstein function* if it is nonnegative, C^∞ and $(-1)^{n-1} f^{(n)}(\lambda) \ge 0$ for all $n \ge 1$ and $\lambda > 0$ (see [SSV, Definition 3.1]). Each Bernstein function f admits a representation

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda y})\,\mu(dy) \tag{2.31}$$

for $\lambda > 0$, where $a, b \ge 0$ and μ is a measure on $(0,\infty)$ such that $\int_{(0,\infty)} (1 \wedge y)\,\mu(dy) < \infty$ (for details see [SSV, Theorem 3.2]). The function $f(\lambda) = (1+\beta/\lambda)^{-1}$ is a Bernstein function for which $a = b = 0$ and μ is an exponential distribution. For simplicity we only consider Bernstein functions for which $a = b = 0$.

THEOREM 2.16. *For a strictly positive random variable ξ and a Bernstein function f having representation* (2.31) *with $a = b = 0$ the following identity holds:*

$$\mathbb{E} f(1/\xi) = \int_0^\infty (1 - G(y))\,\mu(dy),$$

where G is given by (2.16) *and μ is a measure on $(0,\infty)$ such that $\int_{(0,\infty)} (1 \wedge y)\mu(dy) < \infty$.*

Proof. The proof goes in the same way as that of Theorem 2.13. We use representation (2.31) and Fubini's theorem. ∎

REMARK 2.17. If we know the function G we can use the above theorem to compute the expectation $\mathbb{E} f(1/\xi)$ for a Bernstein function f. For f one can take $f(\lambda) = (\lambda/(1+\lambda))^\alpha$ for $\alpha \in (0,1)$, $f(\lambda) = \sqrt{\lambda}(1 - e^{-2a\sqrt{\lambda}})$ for $a > 0$, $f(\lambda) = \ln(1+\lambda^\alpha)$ for $\alpha \in (0,1)$ and many, many others—see [SSV, Chapter 15], where for each f the corresponding measure μ is given. In particular, Theorem 2.16 gives a closed form of $\mathbb{E} f((A_t^{(\mu)})^{-1})$ for a Bernstein function f. For example, taking the Bernstein function

$$f(\lambda) = \frac{\lambda(1 - e^{-2\sqrt{\lambda + a}})}{\sqrt{\lambda + a}}$$

(see [SSV, Chapter 15, point 20]) we have, for $a, t > 0$,

$$\mathbb{E}\bigg(\frac{1-e^{-2\sqrt{(A_t)^{-1}+a}}}{\sqrt{A_t + aA_t^2}}\bigg) = \int_0^\infty (1-G(y))\frac{1}{2\sqrt{\pi y^5}}\big(e^{-1/y-ay}(2+y(e^{1/y}-1)(1+2y))\big)\,dy,$$

where G is given by (2.16).

Corollary 2.14 allows us to find the first function $p_1(\cdot)$ for the recurrence of Proposition 2.3.

COROLLARY 2.18. *Let p_1 be given by* (2.6). *Then*

$$p_1(t) = \frac{1}{\beta}\int_0^\infty \frac{G_{t/4}^{(-1/2)}(y)}{y}(1-e^{-4\beta y})\,dy, \tag{2.32}$$

where $G^{(\mu)}$ is defined by (2.19).

Proof. Since $Z_u = \frac{1}{2}B_{4u}$ is a standard Brownian motion, we infer that

$$\beta p_1(4t) = \mathbb{E}\ln\Big(1+\beta\int_0^{4t} Y_u\,du\Big) = \mathbb{E}\ln\Big(1+4\beta\int_0^t e^{B_{4u}-2u}\,du\Big)$$
$$= \mathbb{E}\ln\Big(1+4\beta\int_0^t e^{2(Z_u-u)}\,du\Big) = \mathbb{E}\ln(1+4\beta A_t^{(-1/2)}).$$

Hence, (2.32) follows from (2.29). ∎

To find an exact formula for the Laplace transform of $\mathit{\Gamma}_t$, for fixed $t \ge 0$, we need the following theorem.

THEOREM 2.19. *For $\lambda, \alpha > 0$, $\mu \in \mathbb{R}$ and $a \in \mathbb{R}$,*

$$\mathbb{E}\bigg[\exp\bigg(-\frac{\lambda e^{aB_t^{(\mu)}}}{1+\alpha A_t^{(\mu)}}\bigg)\bigg] = \mathbb{E}\big[F_{B_t^{(\mu)}}\big(\alpha^{-1}R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)\big)\big],$$

where $F_x(\cdot)$ for $x \in \mathbb{R}$ is given by

$$F_x(z) = \exp\bigg(-\frac{\varphi_z(x)^2 - x^2}{2t}\bigg) \tag{2.33}$$

for $z > 0$, φ_x is given by (2.18), *and R^x is a squared Bessel process with index -1 starting at x and independent of $B^{(\mu)}$ given by* (2.3).

Proof. For each x, let R^x be as in the statement. Using the Laplace transform of a squared Bessel process (see [RY, Chapter XI, p. 441]) and the density function of $R^x_{1/2}$ we obtain

$$\mathbb{E}\exp\Big(-\frac{\lambda e^{aB_t^{(\mu)}}}{1+\alpha A_t^{(\mu)}}\Big) = \mathbb{E}\exp\Big(-\frac{R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)}{\alpha A_t^{(\mu)}}\Big)$$
$$= \mathbb{E}F_{B_t^{(\mu)}}\Big(\frac{1}{\alpha}R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)\Big),$$

where

$$F_x(z) = \mathbb{E}(e^{-z/A_t^{(\mu)}} \mid B_t = x).$$

Indeed, let $\mathbb{Q}$ be the measure defined by (2.21), so $B_t^{(\mu)}$ is a standard Brownian motion under $\mathbb{Q}$. Then

$$\mathbb{E}\exp\Big(-\frac{R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)}{\alpha A_t^{(\mu)}}\Big) = \mathbb{E}_{\mathbb{Q}}\Big[e^{\mu B_t^{(\mu)}-\mu^2 t}\exp\Big(-\frac{R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)}{\alpha A_t^{(\mu)}}\Big)\Big]$$
$$= \mathbb{E}_{\mathbb{Q}}\Big[e^{\mu B_t^{(\mu)}-\mu^2 t}\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\frac{R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)}{\alpha A_t^{(\mu)}}\Big)\ \Big|\ B_t^{(\mu)}, R\Big]\Big]$$
$$= \mathbb{E}_{\mathbb{Q}}\Big[e^{\mu B_t^{(\mu)}-\mu^2 t}F_{B_t^{(\mu)}}\Big(\frac{1}{\alpha}R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)\Big)\Big] = \mathbb{E}\Big[F_{B_t^{(\mu)}}\Big(\frac{1}{\alpha}R^{(\lambda e^{aB_t^{(\mu)}})}(1/2)\Big)\Big].$$

Using Remark 2.7 we finish the proof. ■

Now, we give a formula for the Laplace transform of Γ.

THEOREM 2.20. *For $\lambda > 0$, we have*

$$\mathbb{E}e^{-\lambda\Gamma_t} = \mathbb{E}\big[F_{B_t^{(-1/2)}}\big((4\beta)^{-1}R^{(\lambda e^{2B_t^{(-1/2)}})}(1/2)\big)\big],$$

where F_x is given by (2.33), *and R^x is a squared Bessel process with index -1 starting at x and independent of $B^{(-1/2)}$ given by* (2.3).

Proof. From the definition of Γ we have

$$\Gamma_{4t} = \frac{Y_{4t}}{1+\beta\int_0^{4t} Y_u\,du} = \frac{e^{2(B_{4t}/2-t)}}{1+4\beta\int_0^t e^{2(B_{4t}/2-t)}\,du}.$$

Since $B_{4t}/2$ is a Brownian motion, the statement follows from Theorem 2.19 with $\alpha = 4\beta$, $a = 2$ and $\mu = -1/2$. ■

Hence and from Theorem 2.5 we have

COROLLARY 2.21. *Let V be a diffusion defined by SDE* (2.9) *with $V_0 \geq 1$. For $\beta > 0$, we have*

$$\mathbb{E}e^{-\beta V_t} = e^{-\beta}\mathbb{E}\big[F_{B_t^{(-1/2)}}\big((4\beta)^{-1}R^{(\lambda e^{2B_t^{(-1/2)}})}(1/2)\big)\big],$$

where $\lambda = \beta(V_0 - 1)$.

Now, we use formula (2.30) and the results of Matsumoto and Yor to obtain some interesting connections between $\mathbb{E}((1+\beta A_t^{(\mu)})^{-1})$ and the conditional expectation of functionals of geometric Brownian motion with opposite drift.

PROPOSITION 2.22. *For $\mu > 0$ and $\beta > 0$, we have*

$$\mathbb{E}\bigg(\frac{1}{1+2\beta A_t^{(\mu)}}\bigg) = 1 - 2\beta\mathbb{E}\big(A_t^{(-\mu)} \mid A_\infty^{(-\mu)} = 1/(2\beta)\big).$$

Proof. By the result of Matsumoto and Yor [MY1, Thm. 2.2] the process $\{B_t^{(-\mu+1/2)}, t \ge 0\}$ given $A_\infty^{(-\mu)} = 1/(2\beta)$ has the same distribution as the process $\{B_t^{(\mu-1/2)} - \log(1+2\beta A_t^{(\mu)}), t \ge 0\}$ for $\mu > 0$. From that we obtain

$$\begin{aligned}\mathbb{E}\big(A_t^{(-\mu)} \mid A_\infty^{(-\mu)} = 1/(2\beta)\big) &= \mathbb{E}\int_0^t \frac{e^{2B_s^{(\mu-1/2)}}}{(1+2\beta A_s^{(\mu)})^2}\, ds \\ &= \frac{1}{2\beta}\bigg(1 - \mathbb{E}\bigg(\frac{1}{1+2\beta A_t^{(\mu)}}\bigg)\bigg). \quad \blacksquare\end{aligned}$$

PROPOSITION 2.23. *Let $\beta > 0$ and $\mu > 0$. Then*

$$\mathbb{E}\big(A_t^{(-\mu)} \mid A_\infty^{(-\mu)} = 1/(2\beta)\big) = \frac{1}{2}\int_0^\infty G_t^{(\mu)}(y) e^{-y\beta}\, dy, \tag{2.34}$$

where $G^{(\mu)}$ is defined by (2.19).

Proof. This follows from Proposition 2.22 and (2.30). ■

PROPOSITION 2.24. *For $\beta > 0$ and $\mu \in \mathbb{R}$, we have*

$$\mathbb{E}\bigg(\frac{e^{2\mu B_t^{(-\mu)}}}{1+2\beta A_t^{(-\mu)}}\bigg) = 1 - 2\beta\int_0^\infty G_t^{(\mu)}(y) e^{-y\beta}\, dy.$$

Proof. Fix $\mu \in \mathbb{R}$. Define a new probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{-2\mu B_t - 2\mu^2 t}.$$

The process $B_t^{(2\mu)} = B_t + 2\mu t$ is a standard Brownian motion under $\mathbb{Q}$, so

$$\mathbb{E}\bigg(\frac{1}{1+2\beta A_t^{(\mu)}}\bigg) = \mathbb{E}_{\mathbb{Q}}\bigg(\frac{e^{2\mu(B_t^{(2\mu)} - \mu t)}}{1+2\beta\int_0^t e^{2(B_t^{(2\mu)} - \mu u)}\, du}\bigg) = \mathbb{E}\bigg(\frac{e^{2\mu B_t^{(-\mu)}}}{1+2\beta A_t^{(-\mu)}}\bigg).$$

Now, the statement follows from Corollary 2.14. ■

We go back, for a moment, to the function p_1 given by (2.6).

PROPOSITION 2.25. *For $t \geq 0$ we have*

$$p_1(t) = t - 4\beta\int_0^t \mathbb{E}\big(A_{s/4}^{(-1/2)} \mid A_\infty^{(-1/2)} = 1/(4\beta)\big)\,ds.$$

Proof. We have $p_1'(t) = \mathbb{E}\mathit{\Gamma}_t$, $p_1(0) = 0$, and

$$\mathit{\Gamma}_{4t} = \frac{e^{2\hat{B}_t^{(-1/2)}}}{1 + 4\beta\hat{A}_t^{(-1/2)}},$$

where $\hat{B}_t = B_{4t}/2$ is a standard Brownian motion, and $\hat{A}^{(-1/2)}$ is defined by (2.2) with $\hat{B}$ instead of B. Since

$$\mathbb{E}\big(A_{t/4}^{(-1/2)} \mid A_\infty^{(-1/2)} = 1/(4\beta)\big) = \mathbb{E}\big(\hat{A}_{t/4}^{(-1/2)} \mid \hat{A}_\infty^{(-1/2)} = 1/(4\beta)\big),$$

the assertion follows from (2.34) with $\mu = 1/2$ and Proposition 2.24. ■

Proposition 2.23 now gives immediately

COROLLARY 2.26. *For $t \geq 0$ we have*

$$p_1(t) = t - 2\beta\int_0^t\int_0^\infty G_{s/4}^{(1/2)}(y)e^{-2y\beta}\,dy\,ds,$$

where $G_t^{(1/2)}$ is defined by (2.19) with $\mu = 1/2$.

REMARK 2.27. Notice that we have established all the results for fixed t. In several papers (for instance [MY], [MY4]) integral functionals of a geometric Brownian motion with random time given by a random variable independent of the Brownian motion and with exponential distribution were investigated. To give an example of such a functional. Let E_λ be an exponential random variable with parameter $\lambda > 0$. If $\zeta_{1,a}$ is a random variable with beta distribution with parameters 1 and $a = \frac{\sqrt{2\lambda+1/4}-1/2}{2}$, γ_b is a random variable with gamma distribution with parameter $b = \frac{\sqrt{2\lambda+1/4}+1/2}{2}$, and $\zeta_{1,a}$ and γ_b are independent, then

$$\mathbb{E}\ln\Big(1 + \beta\int_0^{E_\lambda} Y_u^2\,du\Big) = \mathbb{E}\ln\Big(1 + \beta\frac{\zeta_{1,a}}{\gamma_b}\Big), \tag{2.35}$$

because $\int_0^{E_\lambda} Y_u^2\,du \overset{(\text{law})}{=} \zeta_{1,a}/\gamma_b$ (see [MY4]). In Corollary 2.14 we have calculated the RHS of (2.35) for a fixed time t instead of a random time given by E_λ.

Later, we will be making use of random time. In Subsection 4.2.2 we show how to compute the moments in a lognormal stochastic volatility model with random time which is exponentially distributed and independent of the Brownian motion driving the model.

3. The Laplace transform of the exponent of the anticipative T_α transform. In this section we consider the exponent of the T_α transform of Brownian motion with drift $B_t^{(\mu)}$ studied by Donati-Martin et al. [DMY]. The T_α *transform* of $B^{(\mu)}$, for $\alpha \geq 0$ and $\mu \in \mathbb{R}$, is defined by

$$T_\alpha(B^{(\mu)})_t = B_t^{(\mu)} - \ln(1 + \alpha A_t^{(\mu)}).$$

We also consider an anticipative version of the T_α transform, for $s \leq t$,

$$T_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s = B_s^{(\mu)} - \ln(1 + \alpha A_s^{(\mu)}/e^{B_t^{(\mu)}}).$$

Its importance is established in [DMY], where the transforms T_α and $T_{\alpha/e^{B_t^{(\mu)}}}$ appear in many identities between functionals of Brownian motion. These identities are mostly of duality type, and the distributions of $T_\alpha(B^{(\mu)})_t$ and $T_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s$ for $\alpha > 0$ are not established there. Using the methods from the previous section we find, for fixed $t \geq 0$ and $s \leq t$, the Laplace transforms of the random variables

$$\hat{T}_\alpha(B^{(\mu)})_t := \exp(T_\alpha(B^{(\mu)})_t) = \frac{e^{B_t^{(\mu)}}}{1 + \alpha A_t^{(\mu)}},$$

$$\hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s := \exp(T_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s) = \frac{e^{B_s^{(\mu)}}}{1 + \alpha A_s^{(\mu)}/e^{B_t^{(\mu)}}}.$$

THEOREM 3.1. *For $\lambda > 0$ and $\mu \in \mathbb{R}$ we have*

$$\mathbb{E}\exp(-\lambda \hat{T}_\alpha(B^{(\mu)})_t) = \mathbb{E}F_{B_t^{(\mu)}}\big(\alpha^{-1} R^{(\lambda e^{B_t^{(\mu)}})}(1/2)\big),$$

where F_x is given by (2.33), *and for each x, R^x is a squared Bessel process with index -1 starting at x and independent of $B^{(\mu)}$ given by* (2.3).

Proof. Follows from Theorem 2.19 with $a = 1$. ∎

THEOREM 3.2. *For $\lambda > 0$, $\mu \in \mathbb{R}$ and $s \leq t$, we have*

$$\mathbb{E}\exp\big(-\lambda \hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s\big) = \mathbb{E}F_{B_s^{(\mu)}}\bigg(\frac{e^{B_t^{(\mu)} - B_s^{(\mu)}}}{\alpha} R^{(\lambda e^{B_s^{(\mu)}})}(e^{B_s^{(\mu)}}/2)\bigg), \tag{3.1}$$

where F_x is given by (2.33), *and for each x, R^x is a squared Bessel process with index -1 starting at x and independent of $B^{(\mu)}$ given by* (2.3).

Proof. In the first step, we prove (3.1) for $s = t$. Using the Laplace transform of a squared Bessel process and arguing as in the proof of Theorem

2.19 we obtain

$$\mathbb{E}\exp\big(-\lambda \hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_t\big) = \mathbb{E}\exp\bigg(-\frac{\lambda e^{B_t^{(\mu)}}}{1+\alpha A_t^{(\mu)} e^{-B_t^{(\mu)}}}\bigg)$$

$$= \mathbb{E}\exp\bigg(-\frac{\lambda e^{2B_t^{(\mu)}}}{e^{B_t^{(\mu)}}+\alpha A_t^{(\mu)}}\bigg) = \mathbb{E}\exp\bigg(-\frac{R^{(\lambda e^{2B_t^{(\mu)}})}(e^{B_t^{(\mu)}}/2)}{\alpha A_t^{(\mu)}}\bigg)$$

$$= \mathbb{E}F_{B_t^{(\mu)}}\bigg(\frac{1}{\alpha}R^{(\lambda e^{2B_t^{(\mu)}})}(e^{B_t^{(\mu)}}/2)\bigg).$$

In the second step, we prove (3.1) for $s < t$. By definition of $\hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s$,

$$\mathbb{E}\exp\big(-\lambda \hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s\big) = \mathbb{E}\exp\bigg(-\frac{\lambda e^{B_s^{(\mu)}}}{1+\alpha A_s^{(\mu)} e^{-B_t^{(\mu)}}}\bigg)$$

$$= \mathbb{E}\exp\bigg(-\frac{\lambda e^{B_s^{(\mu)}}}{1+\alpha A_s^{(\mu)} e^{-B_s^{(\mu)}} e^{-(B_t^{(\mu)}-B_s^{(\mu)})}}\bigg).$$

Since $e^{-(B_t^{(\mu)}-B_s^{(\mu)})}$ is independent of $\sigma(B_u^{(\mu)}, u \le s)$, taking the conditional expectation with respect to $\sigma(B_t^{(\mu)} - B_s^{(\mu)})$ and using the result from the first step with $\alpha e^{-(B_t^{(\mu)}-\, B_s^{(\mu)})}$ in place of α, we obtain

$$\mathbb{E}\exp\big(-\lambda \hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s\big) = \mathbb{E}F_{B_s^{(\mu)}}\bigg(\frac{e^{B_t^{(\mu)}-B_s^{(\mu)}}}{\alpha}R^{(\lambda e^{2B_s^{(\mu)}})}(e^{B_s^{(\mu)}}/2)\bigg),$$

which completes the proof. ∎

Remark 3.3. Theorems 3.1 and 3.2 enable us to find the Laplace transform of the T-transforms $\hat{T}_\alpha(B^{(\mu)})_t$ and $\hat{T}_{\alpha/e^{B_t^{(\mu)}}}(B^{(\mu)})_s$. Indeed, we know the function F_x (see (2.33)) and the density function of R^x, and since $B^{(\mu)}$ is independent of R^x, we can compute the expectations $\mathbb{E}F_{B_t^{(\mu)}}\big(\alpha^{-1}R^{\lambda e^{B_t^{(\mu)}}}(1/2)\big)$ and $\mathbb{E}F_{B_s^{(\mu)}}\big(\frac{e^{B_t^{(\mu)}-\,B_s^{(\mu)}}}{\alpha}R^{\lambda e^{B_s^{(\mu)}}}(e^{B_s^{(\mu)}}/2)\big)$.

4. Moments of the asset price in the lognormal stochastic volatility model

4.1. Model of market. We consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T < \infty$, satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Without loss of generality we assume the savings account to be constant and identically equal to one. Moreover, we assume that the price X_t of the underlying asset at time t has a stochastic volatility Y_t which is a geometric Brownian motion, so the dynamics of the proces X is given by

$$dX_t = Y_t X_t dW_t, \tag{4.1}$$

where $X_0 = 1$. The dynamics of the vector (X, Y) is given by a system of SDEs consisting of (4.1) and

$$dY_t = Y_t dZ_t, \qquad Y_0 = 1. \tag{4.2}$$

The processes W, Z are correlated Brownian motions, $d\langle W, Z\rangle_t = \rho dt$ with $\rho \in [-1, 1]$. The process X has the form

$$X_t = \exp\Big(\int_0^t Y_u dW_u - \frac{1}{2}\int_0^t Y_u^2\, du\Big), \tag{4.3}$$

and this is a unique strong solution of SDE (4.1) on $[0, T]$. The existence and uniqueness follow directly from the assumptions on Y and the well known properties of stochastic exponents (see, e.g., Revuz and Yor [RY]). Since the process X is a local martingale, there is no arbitrage on the market so defined.

Notice that we can represent W as

$$W_t = \rho Z_t + \sqrt{1-\rho^2}\, V_t, \tag{4.4}$$

where (V, Z) is a standard two-dimensional Wiener process. Using (4.3) and (4.4) we can express the moment of order α of X as

$$\begin{aligned} \mathbb{E}X_t^\alpha &= \mathbb{E}\exp\Big(\alpha\rho\int_0^t Y_u\, dZ_u + \alpha\sqrt{1-\rho^2}\int_0^t Y_u\, dV_u - \frac{\alpha}{2}\int_0^t Y_u^2\, du\Big) \\ &= \mathbb{E}\exp\Big(\alpha\rho\int_0^t Y_u\, dZ_u + \frac{\alpha^2(1-\rho^2)-\alpha}{2}\int_0^t Y_u^2\, du\Big). \end{aligned} \tag{4.5}$$

Therefore the calculation of moments reduces to calculating Brownian functionals.

4.2. Moments of the asset price in the lognormal stochastic volatility model

4.2.1. *Moments of order $\alpha > 0$.* Jourdain [J] proved that for $\alpha \in (1, (1-\rho^2)^{-1})$ and $\rho \neq 0$ the moments EX^α exist, but he did not find their values. We calculate the moments for $\alpha > 0$, $\alpha(1-\rho^2) < 1$ and $\rho \in [-1, 1]$. Note that Sin [S] has established that the process X is a true martingale if and only if $\rho \le 0$.

First, we prove that the moment of order α of the strong solution of (4.1) is equal to the Laplace transform of the process Γ.

THEOREM 4.1. *Let $t \in [0,T]$, $\alpha > 0$, $\alpha(1-\rho^2) < 1$ and Γ be given by* (2.1) *with*

$$\beta = \sqrt{\alpha - \alpha^2(1-\rho^2)}. \tag{4.6}$$

If X is given by (4.1), *then*

$$\begin{aligned} \mathbb{E}X_t^\alpha &= e^{-(\beta+\rho\alpha)}\mathbb{E}\exp((\beta+\rho\alpha)\Gamma_t) \\ &= e^{-(\beta+\rho\alpha)}\mathbb{E}F_{B_t^{(-1/2)}}\big((4\beta)^{-1}R^{(-(\beta+\rho\alpha)e^{2B_t^{(-1/2)}})}(1/2)\big), \end{aligned} \tag{4.7}$$

where F_x is given by (2.33), *and for each x, R^x is a squared Bessel process with index -1 starting at x and independent of $B^{(\mu)}$ given by* (2.3).

Proof. Define a measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = \exp\bigg(-\beta\int_0^T Y_u\,dZ_u - \frac{\beta^2}{2}\int_0^T Y_u^2\,du\bigg),$$

with β given by (4.6). Then $\mathbb{Q}$ is a probability measure since, by (4.2),

$$\exp\bigg(-\beta\int_0^T Y_u\,dZ_u - \frac{\beta^2}{2}\int_0^T Y_u^2\,du\bigg) = \exp\bigg(-\beta(Y_T-1) - \frac{\beta^2}{2}\int_0^T Y_u^2\,du\bigg) \le e^\beta.$$

Using (4.5) and the definition of $\mathbb{Q}$ we infer

$$\mathbb{E}X_t^\alpha = \mathbb{E}\exp\bigg(\alpha\rho\int_0^t Y_u\,dZ_u + \frac{\alpha^2(1-\rho^2)-\alpha}{2}\int_0^t Y_u^2\,du\bigg) = \mathbb{E}_{\mathbb{Q}}e^{(\rho\alpha+\beta)(Y_t-1)}. \tag{4.8}$$

By the Girsanov theorem, $B_t = Z_t + \int_0^t \beta Y_s\,ds$ is a standard Brownian motion under $\mathbb{Q}$ and, by (4.2),

$$dZ_t = dB_t - \beta Y_t dt = dB_t - \beta\exp(Z_t - t/2)dt, \qquad Z_0 = 0. \tag{4.9}$$

We know, by the result of Alili, Matsumoto, and Shiraishi [AMY, Lem. 3.1], that the unique strong solution of (4.9) is given by

$$Z_t = \frac{t}{2} + \ln\bigg(\frac{U_t}{1+\beta\int_0^t U_s ds}\bigg), \quad \text{where} \quad U_t = e^{B_t - t/2}.$$

Therefore,

$$Y_t = \exp(Z_t - t/2) = \frac{U_t}{1+\beta\int_0^t U_s ds}.$$

Thus, the law of Y under $\mathbb{Q}$ is equal to the law of Γ under $\mathbb{P}$, since the law of U under $\mathbb{Q}$ is equal to the law of Y under $\mathbb{P}$. Hence (4.8) yields the first equality in (4.7). The second follows from Proposition 2.20. ∎

REMARK 4.2. (a) From Theorem 4.1, we immediately see that all moments exist provided $\rho^2 = 1$.

(b) The condition $\alpha(1-\rho^2) < 1$ is not necessary for the existence of moments since for $\rho = 0$ the process X is a martingale, so EX_t exists. However, for $\rho = 0$, $\mathbb{E}X_t^\alpha = \infty$ for $\alpha > 1$ ([J]).

We can also express the moments of order $\alpha > 1$ in terms of the diffusion given by SDE (2.9).

THEOREM 4.3. *Assume that* $\alpha > 1$, $\alpha(1-\rho^2) < 1$ *and* $\beta = \sqrt{\alpha - \alpha^2(1-\rho^2)}$. *Let* X *be given by* (4.1) *and* V *be the diffusion given by SDE* (2.9) *such that* $V_0 = -\rho\alpha/\beta$. *Then*

$$\mathbb{E}X_t^\alpha = e^{-\rho\alpha}\mathbb{E}e^{-\beta V_t}.$$

Proof. Let $\lambda = -(\beta + \rho\alpha)$. Since $\lambda > 0$ provided $\alpha > 1$, the assertion follows from Theorems 4.1 and 2.5. ∎

4.2.2. *Moments with independent random time.* In this subsection, we find closed formulas for the moments in the lognormal stochastic volatility model when the time is an exponential random variable independent of the Brownian motion driving the diffusion Y. The idea of considering such time is not new and can be found in many studies of Asian options (see for instance [MY], [MY4]).

PROPOSITION 4.4. *Let* E_λ *be a random variable with exponential distribution with parameter* $\lambda > 0$. *Assume that* E_λ *is independent of a standard Brownian motion* B. *Let* $Z_t = 2B_{t/4}$, $Y_t = e^{-t/2+Z_t}$, *and* $U_t = e^{B_t - t}$. *Then*

$$\mathbb{E}\Big(\ln\Big(1+\beta\int_0^{4E_\lambda} Y_u\,du\Big)\Big) = \frac{4\beta}{\lambda} - 4\beta^2\int_0^\infty \mathbb{E}\Big(\int_0^{E_\lambda} U_s^2 ds - K/4\Big)^+ (1+\beta K)^{-2} dK.$$

Proof. It is obvious that $Y_{4t} = e^{-2t+Z_{4t}} = e^{-2t+2B_t} = U_t^2$. The Taylor theorem with integral remainder applied to the function $f(x) = \ln(1+\beta x)$ gives

$$\ln(1+\beta x) = \beta x - \beta^2\int_0^\infty (x-K)^+(1+\beta K)^{-2}\,dK.$$

Hence replacing x by $\int_0^{4E_\lambda} Y_u\,du$ and taking the expectation we get

$$\begin{aligned}
(4.10)\qquad & \mathbb{E}\Big(\ln\Big(1+\beta\int_0^{4E_\lambda} Y_u\,du\Big)\Big) \\
&= \beta\mathbb{E}\Big(\int_0^{4E_\lambda} Y_u\,du\Big) - \beta^2\int_0^\infty \mathbb{E}\Big(\int_0^{4E_\lambda} Y_u\,du - K\Big)^+ (1+\beta K)^{-2}\,dK \\
&= 4\beta\mathbb{E}\Big(\int_0^{E_\lambda} U_s^2\,ds\Big) - 4\beta^2\int_0^\infty \mathbb{E}\Big(\int_0^{E_\lambda} U_s^2\,ds - K/4\Big)^+ (1+\beta K)^{-2}\,dK.
\end{aligned}$$

Let $A_t = \int_0^t U_s^2\, ds$. It is known (see [MY4]) that $\mathbb{E}A_{E_\lambda} = 1/\lambda$. By (4.10) this completes the proof. ∎

REMARK 4.5. The RHS of (4.4) can be computed due to a result of Mansuy and Yor [MY, Thm. 6.1] which gives

$$\mathbb{E}\Big(\int_0^{E_\lambda} U_s^2\, ds - K/4\Big)^+ = \frac{1}{\lambda\Gamma\big(\frac{\sqrt{2\lambda+1}-1}{2}\big)} \int_0^{2/K} e^{-u} u^{\frac{\sqrt{2\lambda+1}-3}{2}} (1-Ku/2)^{\frac{\sqrt{2\lambda+1}+1}{2}}\, du.$$

Here and below, Γ denotes the gamma function.

In the next theorem we establish an explicit formula for the moments of $X_{E_{2\lambda}}$.

THEOREM 4.6. *Let $\alpha > 0$, $\alpha(1-\rho^2) < 1$ and E_λ be a random variable with exponential distribution with parameter $\lambda > 0$. Assume that E_λ is independent of the Brownian motions V and Z driving the process X. Then*

$$\begin{aligned}\mathbb{E}X^\alpha_{2E_\lambda} &= \lambda e^{-(\alpha\rho+\beta)} \frac{\Gamma((1+\sqrt{4\lambda+1})/2)}{\Gamma(1+\sqrt{4\lambda+1})} \\ &\quad \times \bigg(\phi_1\Big(\frac{1}{2}\beta\Big) \int_0^{1/(2\beta)} e^{\frac{\alpha\rho-\beta}{2\beta}\frac{1}{y}} \phi_2(y)\, dy + \phi_2\Big(\frac{1}{2}\beta\Big) \int_{1/2\beta}^{\infty} e^{\frac{\alpha\rho-\beta}{2\beta}\frac{1}{y}} \phi_1(y)\, dy\bigg),\end{aligned}$$

where $\beta = \sqrt{\alpha - \alpha^2(1-\rho^2)}$,

$$\phi_1(x) = x^{-(1+\sqrt{1+4\lambda})/2} \Phi\big((1+\sqrt{1+4\lambda})/2, 1+\sqrt{1+4\lambda}, x^{-1}\big),$$
$$\phi_2(x) = x^{-(1+\sqrt{1+4\lambda})/2} \Psi\big((1+\sqrt{1+4\lambda})/2, 1+\sqrt{1+4\lambda}, x^{-1}\big),$$

and Φ, Ψ denote the confluent hypergeometric functions of the first and second kind, respectively,

$$\Phi(\alpha,\gamma,z) = \sum_{k=0}^\infty \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!},$$
$$\Psi(\alpha,\gamma,z) = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \Phi(\alpha,\gamma,z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(1+\alpha-\gamma, 2-\gamma, z),$$

where $(\alpha)_0 = 1$ *and*

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad \textit{for } k = 1,2,\ldots.$$

Proof. Let us define

$$\eta_t := \frac{1}{2\beta\Gamma_{2t}}.$$

Observe that

$$\eta_t = \exp(\sqrt{2}\,\hat{B}_t + t)\bigg(\frac{1}{2\beta} + \int_0^t \exp(-\sqrt{2}\,\hat{B}_u - u)\, du\bigg),$$

where $\hat{B}_t = -\frac{1}{\sqrt{2}} Z_{2t}$. We know that η_t is a Markov process with resolvent

$$U_\lambda f(x) = \frac{\Gamma((1+\sqrt{4\lambda+1})/2)}{\Gamma(1+\sqrt{4\lambda+1})} \times \Big(\phi_1(x) \int_0^x e^{-1/y}\phi_2(y)f(y)\,dy + \phi_2(x)\int_x^\infty e^{-1/y}\phi_1(y)f(y)\,dy\Big)$$

(for details see [DGY, Thm. 3.1]), so we conclude by Theorem 4.1 and the definition of η that

$$\mathbb{E}X^\alpha_{2E_\lambda} = e^{-(\alpha\rho+\beta)}\mathbb{E}\exp\left\{\frac{\alpha\rho+\beta}{2\beta}\,\frac{1}{\eta_{E_\lambda}}\right\} = \lambda e^{-(\alpha\rho+\beta)}U_\lambda f\left(\frac{1}{2\beta}\right),$$

with $f(x) = \exp\{\frac{\alpha\rho+\beta}{2\beta}\frac{1}{x}\}$. ■

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