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Weak-star point of continuity property and Schauder bases

by

Ginés López-Pérez and José A. Soler-Arias (Granada)

Abstract. We characterize the weak-star point of continuity property for subspaces of dual spaces with separable predual and we deduce that the weak-star point of continuity property is determined by subspaces with a Schauder basis in the natural setting of dual spaces of separable Banach spaces. As a consequence of the above characterization we show that a dual space has the Radon–Nikodym property if, and only if, every seminormalized topologically weak-star null tree has a boundedly complete branch, which improves some results of Dutta and Fonf (2008) obtained for the separable case. Also, as a consequence of the above characterization, the following result of Rosenthal (2007) is deduced: every seminormalized basic sequence in a Banach space with the point of continuity property has a boundedly complete subsequence.

1. Introduction. We recall (see [2] for background) that a bounded subset C of a Banach space has the *Radon–Nikodym property* (RNP) if every subset of C is dentable, that is, every subset of C has slices of arbitrarily small diameter. A Banach space is said to have RNP whenever its closed unit ball has RNP. It is well known that separable dual spaces have RNP, and spaces with RNP contain many subspaces which are themselves separable dual spaces. (Note that containing many separable dual subspaces is equivalent to containing many boundedly complete basic sequences.)

As RNP is separably determined, that is, a Banach space X has RNP whenever every separable subspace of X does, it seems natural to look for a sequential characterization of RNP in terms of boundedly complete basic sequences. In [3] it is proved that the space B_∞ (which fails to have RNP) still has the property that any w-null normalized sequence has a boundedly complete basic subsequence. However, it has been proved in [3] that the dual space of a separable Banach space X has RNP if, and only if, every w^*-null tree in the unit sphere of X^* has some boundedly complete basic branch. It then seems natural to look for a characterization of RNP for general dual Banach spaces in terms of boundedly complete basic sequences,

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extending the result in [3] proved for duals of separable Banach spaces. For this, we introduce the concept of topologically w^*-null tree, which is a weaker condition than being a w^*-null tree, and we characterize in terms of trees the RNP for w^*-compact subsets of general dual Banach spaces (Proposition 2.1). As a consequence, we prove (Theorem 2.6) that a dual Banach space X has RNP if, and only if, every seminormalized and topologically w^*-null tree in the unit sphere of X has some boundedly complete branch, which immediately implies the aforementioned result of [3].

We recall that a closed and bounded subset of a Banach space X has the *point of continuity property* (PCP) if every closed subset of C has some point of weak continuity, that is, the weak and the norm topologies agree at this point. Also, when X is a dual space, C is said to have the *weak-star point of continuity property* (w^*-PCP) if every closed subset of C has some point of w^*-continuity, equivalently every nonempty subset of C has relatively w^*-open subsets of arbitrarily small diameter. The space X has PCP (resp. w^*-PCP when X is a dual space) if B_X, the closed unit ball of X, has PCP (resp. w^*-PCP). Also, a subspace X of a dual space Y^* is said to have w^*-PCP if B_X, as a subset of Y^*, has w^*-PCP. It is well known that RNP implies PCP, the converse being false, and it is clear that w^*-PCP implies PCP. Moreover, RNP and w^*-PCP are equivalent for convex w^*-compact sets in a dual space (see [2, Theorem 4.2.13]. We will use this last fact freely in the future. We refer to [9] for background about PCP and w^*-PCP.

It is a well known open problem [1] whether PCP (resp. RNP) is determined by subspaces with a Schauder basis. Our goal is characterize w^*-PCP for closed and bounded subsets of dual spaces of separable Banach spaces and conclude (Theorem 2.10) that, in fact, w^*-PCP is determined by subspaces with a Schauder basis in the natural setting of subspaces of dual spaces with a separable predual. As an easy consequence we also deduce from the above characterization of w^*-PCP that every seminormalized basic sequence in a Banach space with PCP has a boundedly complete basic subsequence. This last result was obtained in [8].

We begin with some notation and preliminaries. Let X be a Banach space and let B_X, respectively S_X, be the closed unit ball, respectively sphere, of X. The weak-star topology in X, when it is a dual space, will be denoted by w^*. If A is a subset in X, $\overline{A}^{w^*}$ stands for the w^*-closure of A in X. Given a basic sequence $\{e_n\}$ in X, the closed linear span of $\{e_n\}$ is denoted by $[e_n]$. Moreover, $\{e_n\}$ is said to be: *seminormalized* if $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty$; *boundedly complete* if whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < \infty$, then $\sum_n \lambda_n e_n$ converges; and *shrinking* if $[e_n]^* = [e_n^*]$, where $\{e_n^*\}$ denotes the sequence of biorthogonal functionals associated to $\{e_n\}$.

A boundedly complete basic sequence $\{e_n\}$ in a Banach space X spans a dual space. In fact, $[e_n] = [e_n^*]^*$ [7]. Following [11], a sequence $\{e_n\}$ in a

Banach space is said to be *type P* if the set $\{\sum_{k=1}^{n} e_k : n \in \mathbb{N}\}$ is bounded. Observe, from the definitions, that type P seminormalized basic sequences fail to be always boundedly complete basic sequences.

A sequence $\{x_n\}$ in a Banach space X is said to be *strongly summing* if whenever $\{\lambda_n\}$ is a sequence of scalars with $\sup_n \|\sum_{k=1}^{n} \lambda_k x_k\| < \infty$, then the series of scalars $\sum_n \lambda_n$ converges. The remarkable c_0-theorem [10] ensures that in a Banach space not containing subspaces isomorphic to c_0, every weak-Cauchy sequence that is not weakly convergent has a strongly summing basic subsequence.

$\mathbb{N}^{<\omega}$ stands for the set of all ordered finite sequences of natural numbers, including the empty sequence denoted by $\emptyset$. We consider the natural order in $\mathbb{N}^{<\omega}$, that is, given $\alpha = (\alpha_1, \dots, \alpha_p), \beta = (\beta_1, \dots, \beta_q) \in \mathbb{N}^{<\omega}$, one has $\alpha \leq \beta$ if $p \leq q$ and $\alpha_i = \beta_i$ for all $1 \leq i \leq p$. If $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^{<\omega}$ we set $\alpha- = (\alpha_1, \dots, \alpha_{p-1})$. Also $|\alpha|$ denotes the *length* of α, and $\emptyset$ is the minimum of $\mathbb{N}^{<\omega}$ with the above partial order.

A *tree* in a Banach space X is a family $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ of vectors in X indexed by $\mathbb{N}^{<\omega}$. The tree will be called *seminormalized* if $0 < \inf_A \|x_A\| \leq \sup_A \|x_A\| < \infty$. When X is a dual space, we will say that the tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is *w^*-null* if the sequence $\{x_{(A,n)}\}_n$ is w^*-null for every $A \in \mathbb{N}^{<\omega}$. The tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is *topologically w^*-null* if $0 \in \overline{\{x_{(A,n)} : n \in \mathbb{N}\}}^{w^*}$ for every $A \in \mathbb{N}^{<\omega}$.

A sequence $\{x_{A_n}\}_{n\geq 0}$ is called a *branch* if $\{A_n\}$ is a maximal totally ordered subset of $\mathbb{N}^{<\omega}$, that is, there exists a sequence $\{\alpha_n\}$ of natural numbers such that $A_n = (\alpha_1, \dots, \alpha_n)$ for every $n \in \mathbb{N}$ and $A_0 = \emptyset$. Given a tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ in a Banach space, a *full subtree* is a new tree $\{y_A\}_{A\in\mathbb{N}^{<\omega}}$ defined by $y_\emptyset = x_\emptyset$ and $y_{(A,n)} = x_{(A,\sigma_A(n))}$ for all $A \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$, where for every $A \in \mathbb{N}^{<\omega}$, σ_A is a strictly increasing map, equivalently when every branch of $\{y_A\}$ is also a branch of $\{x_A\}$. The tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is said to be *uniformly type P* if every branch of the tree is type P and the partial sums of every branch are uniformly bounded. The tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is said to be *basic* if the countable set $\{x_A : A \in \mathbb{N}^{\omega}\}$ is a basic sequence for some rearrangement.

Whenever $\{x_n\}$ is a sequence in a Banach space X, we will view it also as a tree setting $x_A = x_{\max(A)}$ for every $A \in \mathbb{N}^{<\omega}$. Furthermore the branches of this tree are the subsequences of $\{x_n\}$.

Finally, we recall that a *boundedly complete skipped blocking finite-dimensional decomposition* (BCSBFDD) in a separable Banach space X is a sequence $\{F_j\}$ of finite-dimensional subspaces in X such that:

(1) $X = [F_j : j \in \mathbb{N}]$.
(2) $F_k \cap [F_j : j \neq k] = \{0\}$ for every $k \in \mathbb{N}$.
(3) For every sequence $\{n_j\}$ of nonnegative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$ and for every $f \in [F_{(n_j,n_{j+1})} : j \in \mathbb{N}]$ there exists a

unique sequence $\{f_j\}$ with $f_j \in F_{(n_j,n_{j+1})}$ for all $j \in \mathbb{N}$ such that $f = \sum_{j=1}^{\infty} f_j$.

(4) Whenever $f_j \in F_{(n_j,n_{j+1})}$ for all $j \in \mathbb{N}$ and $\sup_n \|\sum_{j=1}^{n} f_j\| < \infty$ then $\sum_{j=1}^{\infty} f_j$ converges.

If X is a subspace of Y^* for some Y, a BCSBFDD $\{F_j\}$ in X will be called *w^*-continuous* if $F_i \cap \overline{[F_j : j \neq i]}^{w^*} = \{0\}$ for every i. Here, $[A]$ denotes the closed linear span of A in X and, for every nonempty interval I of nonnegative integers, we denote by F_I the linear span of the F_j's for $j \in I$.

If $\{F_j\}$ is a BCSBFDD in a separable Banach space X and $\{x_j\}$ is a sequence in X such that $x_j \in F_{(n_j,n_{j+1})}$ for some sequence $\{n_j\}$ of nonnegative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$, we say that $\{x_j\}$ is a *skipped block sequence* of $\{F_n\}$. It is standard to prove that there is a positive constant K such that every skipped block sequence $\{x_j\}$ of $\{F_n\}$ with $x_j \neq 0$ for every j is a boundedly complete basic sequence with constant at most K.

From [4], we know that the family of separable Banach spaces with PCP is exactly the family of separable Banach spaces with a BCSBFDD.

2. Main results. We begin with a characterization of RNP for w^*-compact subsets of general dual spaces. This result can be seen as a w^*-version of results in [6].

PROPOSITION 2.1. *Let X be a Banach space and let K be a w^*-compact and convex subset of X^*. Then the following assertions are equivalent:*

(i) *K fails RNP.*

(ii) *There is a seminormalized topologically w^*-null tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ in X^* such that $\{\sum_{C\le A} x_C : A \in \mathbb{N}^{<\omega}\} \subset K$.*

Proof. (i)⇒(ii). Assume that K fails RNP. Then, from [2, Theorem 2.3.6] there is a nondentable and countable subset D of K. Now $\overline{\mathrm{co}}^{w^*}(D)$ is a w^*-compact and w^*-separable subset of K failing w^*-PCP. So there is a relatively w^*-separable subset B of $\overline{\mathrm{co}}^{w^*}(D)$ and $\delta > 0$ such that every relatively w^*-open subset of B has diameter greater than 2δ. Hence $b \in \overline{B \setminus B(b,\delta)}^{w^*}$ for every $b \in B$, where $B(b,\delta)$ stands for the open ball with center b and radius δ. Note then that since B is relatively w^*-separable, for every $b \in B$ there is a countable subset $C_b \in B \setminus B(b,\delta)$ such that $b \in \overline{C_b}^{w^*}$.

First, we construct a tree $\{y_A\}_{A\in\mathbb{N}^{<\omega}}$ in B satisfying:

(a) $y_A \in \overline{B \setminus B(y_A,\delta)}^{w^*}$ for every $A \in \mathbb{N}^{<\omega}$.

(b) $\|y_A - y_{(A,i)}\| > \delta$ for every $A \in \mathbb{N}^{<\omega}$.

(c) $y_A \in \overline{\{y_{(A,i)} : i \in \mathbb{N}\}}^{w^*}$ for every $A \in \mathbb{N}^{<\omega}$.

Pick $y_\emptyset \in B$. As $y_\emptyset \in \overline{B \setminus B(y_\emptyset,\delta)}^{w^*}$ there is a countable set $C_{y_\emptyset} = \{y_{(i)} :$

$i \in \mathbb{N}\} \subset B \setminus B(y_\emptyset, \delta)$ such that $y_\emptyset \in \overline{C_{y_\emptyset}}^{w^*}$. Then (a)–(c) are satisfied. By iterating this process we construct a tree $\{y_A\}_{A\in\mathbb{N}^{<\omega}}$ satisfying (a)–(c).

Now we define a new tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ by $x_\emptyset = y_\emptyset$ and $x_{(A,i)} = y_{(A,i)} - y_A$ for every $i \in \mathbb{N}$ and $A \in \mathbb{N}^{<\omega}$. From (b) we see that $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is a seminormalized tree, since B is bounded. From (c), $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is topologically w^*-null. Furthermore, if $A \in \mathbb{N}^{<\omega}$ then $\sum_{C\le A} x_C = y_A$, by the definition of the tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$. So $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is a uniformly type P tree, since B is bounded and $y_A \in B$ for every $A \in \mathbb{N}^{<\omega}$. This finishes the proof of (i)⇒(ii).

(ii)⇒(i). Let $\{x_A\}$ be a seminormalized topologically w^*-null tree such that $B = \{\sum_{C\le A} x_C : A \in \mathbb{N}^{<\omega}\} \subset K$ and let $\delta > 0$ be such that $\|x_A\| > \delta$ for every $A \in \mathbb{N}^{<\omega}$. For every $A \in \mathbb{N}^{<\omega}$ and for every $n \in \mathbb{N}$ we have $\sum_{C\le(A,n)} x_C = \sum_{C\le A} x_C + x_{(A,n)}$, but $0 \in \overline{\{x_{(A,n)} : n \in \mathbb{N}\}}^{w^*}$, since the tree $\{x_A\}$ is topologically w^*-null. So $\sum_{C\le A} x_C \in \overline{\{\sum_{C\le(A,n)} x_C : n \in \mathbb{N}\}}^{w^*}$ and $\|\sum_{C\le(A,n)} x_C - \sum_{C\le A} x_C\| > \delta$. This proves that B has no points where the identity map is continuous from the w^*-topology to the norm topology. In fact, we have proved that every relatively w^*-open subset of B has diameter greater than δ. Now, $\overline{B}^{\|\cdot\|}$ is a closed and bounded subset of K such that every relatively w^*-open subset of $\overline{B}^{\|\cdot\|}$ has diameter greater than δ, and so K fails w^*-PCP. As K is w^*-compact, it fails RNP. ∎

Essentially, the fact that RNP is separably determined has allowed us to get the above result in the setting of general dual spaces. The next theorem characterizes w^*-PCP for subsets of dual spaces with a separable predual in terms of w^*-null trees, since in this case the w^*-topology is metrizable on bounded sets. It seems natural, then, to think that a characterization of w^*-PCP for subsets in general dual spaces in terms of topologically w^*-null trees has to be true; however, we do not know if w^*-PCP is separably determined in general. This is the difference between the above proposition and the next one, which is now obtained easily.

PROPOSITION 2.2. *Let X be a separable Banach space and let K be a closed and bounded subset of X^*. Then the following assertions are equivalent:*

(i) *K fails w^*-PCP.*
(ii) *There is a seminormalized w^*-null tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ in X^* such that $\{\sum_{C\le A} x_C : A \in \mathbb{N}^{<\omega}\} \subset K$.*

Proof. (i)⇒(ii). If K fails w^*-PCP there is a subset B of K and $\delta > 0$ such that every relatively w^*-open subset of B has diameter greater than 2δ. So $b \in \overline{B \setminus B(b,\delta)}^{w^*}$ for every $b \in B$. Note then that since X is separable the w^*-topology in X^* is metrizable on bounded sets, and so for every $b \in B$ there is a countable subset $C_b \in B \setminus B(b,\delta)$ such that $b \in \overline{C_b}^{w^*}$. Hence we can assume that C_b is a sequence w^*-converging to b. Now we can construct, exactly as in the proof of (i)⇒(ii) of the above proposition, the desired w^*-null tree satisfying (ii).

(ii)⇒(i). If we assume (ii) we can repeat the proof of (ii)⇒(i) in the above proposition to deduce that K fails w^*-PCP. ∎

Remark 2.3. If X is a separable subspace of a dual space Y^* with X satisfying w^*-PCP, it is shown in [9] (see (1) implies (8) of Theorem 2.4 together with the comments on p. 276) that there is a separable subspace Z of Y such that X is isometric to a subspace of Z^* and X has w^*-PCP as a subspace of Z^*. Thus, in order to study the w^*-PCP of a subspace of Y^*, it is more natural to assume that Y is separable.

We now prove our characterization of w^*-PCP in terms of boundedly complete basic sequences in a general setting. A similar characterization for PCP can be found in [6], but the proof of the following result strongly uses the concept of w^*-continuous boundedly complete skipped blocking finite-dimensional decomposition and assumes separability in the predual space.

Theorem 2.4. *Let X, Y be Banach spaces with Y separable and X a subspace of Y^*. Then the following assertions are equivalent:*

(i) *X has w^*-PCP.*
(ii) *No w^*-null tree in S_X is uniformly type P.*
(iii) *No w^*-null tree in S_X has a type P branch.*
(iv) *Every w^*-null tree in S_X has a boundedly complete branch.*

We need the following easy

Lemma 2.5. *Let X, Y be Banach spaces with X a subspace of Y^*, and let M be a finite-codimensional subspace of X. Assume that $\varepsilon > 0$ and $\{x_n^*\}$ is a sequence in X such that $0 \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$. If $P : X \to N$ is a linear and relatively w^*-continuous projection onto some finite-dimensional subspace N of X with kernel M then there is $n_0 \in \mathbb{N}$ such that $\operatorname{dist}(x_{n_0}^*, M) < \varepsilon$.*

Proof. From $0 \in \overline{\{x_n^* : n \in \mathbb{N}\}}^{w^*}$ we deduce that $0 \in \overline{\{P(x_n^*) : n \in \mathbb{N}\}}^{\|\cdot\|}$, since N is a finite-dimensional subspace of X. Now, pick $n_0 \in \mathbb{N}$ with $\|P(x_{n_0}^*)\| < \varepsilon$. Then

$$\operatorname{dist}(x_{n_0}^*, M) = \|x_{n_0}^* + M\| = \|P(x_{n_0}^*) + M\| \le \|P(x_{n_0}^*)\| < \varepsilon. \ \blacksquare$$

Proof of Theorem 2.4. (iv)⇒(iii) is a consequence of the fact that no boundedly complete basic sequence is type P, discussed in the introduction, and (iii)⇒(ii) is trivial.

For (ii)⇒(i) it is enough to apply Theorem 2.2 for $K = B_X$ by assuming that X fails w^*-PCP and normalizing.

(i)⇒(iv). Assume that X has w^*-PCP and pick a w^*-null tree $\{x_A\}$ in S_X.

From [9] (see (b) of Theorem 3.10 together with the equivalence between (1) and (3) of Corollary 2.6) we know that every separable subspace of Y^* with w^*-PCP has a w^*-continuous boundedly complete skipped blocking finite-dimensional decomposition. As the subspace generated by the tree $\{x_A\}$ is separable we can assume that X has such a decomposition, that is, there is a sequence $\{F_j\}$ of finite-dimensional subspaces in X such that:

(1) $X = [F_j : j \in \mathbb{N}]$.
(2) $F_k \cap [F_j : j \neq k] = \{0\}$ for every $k \in \mathbb{N}$.
(3) For every sequence $\{n_j\}$ of nonnegative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$ and for every $f \in [F_{(n_j, n_{j+1})} : j \in \mathbb{N}]$ there exists a unique sequence $\{f_j\}$ with $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ such that $f = \sum_{j=1}^{\infty} f_j$.
(4) Whenever $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ and $\sup_n \|\sum_{j=1}^{n} f_j\| < \infty$ then $\sum_{j=1}^{\infty} f_j$ converges.
(5) $F_i \cap \overline{[F_j : j \neq i]}^{w^*} = \{0\}$ for every i.

Let K be a positive constant such that every skipped block sequence $\{x_j\}$ of $\{F_n\}$ with $x_j \neq 0$ for every j is a boundedly complete basic sequence with constant at most K.

Observe that for every $n \in \mathbb{N}$ there is a linear surjective projection $\widetilde{P_n} : \overline{[F_i : i \geq n]}^{w^*} \oplus [F_i : i < n] \to [F_i : i < n]$ with kernel $\overline{[F_i : i \geq n]}^{w^*}$ and so $\widetilde{P_n}$ is w^*-continuous, since $\overline{[F_i : i \geq n]}^{w^*} \oplus [F_i : i < n]$ is a w^*-closed subspace of Y^* and hence a dual Banach space, and the closed graph theorem applies to P_n because its kernel is w^*-closed and its range is finite-dimensional. Then the restriction of $\widetilde{P}_n$ to X, say P_n, is a linear and relatively w^*-continuous projection from X onto $[F_i : i < n]$ with kernel $[F_i : i \geq n]$.

We have to construct a boundedly complete branch of the tree $\{x_A\}$. For this, fix a sequence $\{\varepsilon_j\}$ of positive real numbers with $\sum_{j=0}^{\infty} \varepsilon_j < 1/(2K)$, where K is the constant of the decomposition $\{F_j\}$. Now we construct a sequence $\{f_j\}$ in X with $f_j \in F_{(n_j, n_{j+1})}$ for all j, for some increasing sequence $\{n_j\}$ of integers, and a branch $\{x_{A_j}\}$ of the tree such that $\|x_{A_j} - f_j\| < \varepsilon_j$ for all j. Put $n_0 = 0$. Then there exist $n_1 > 2$ and $f_0 \in F_{(n_0, n_1)}$ such that $\|x_{A_0} - f_0\| < \varepsilon_0$, where $A_0 = \emptyset$.

Now, assume that $n_1, \ldots, n_{j+1}$, $f_1, \ldots, f_j$ and $A_1, \ldots, A_j$ have been constructed. Put $A_k = (p_1, \ldots, p_k)$ for all $1 \leq k \leq j$. As the tree is w^*-null we have $0 \in \overline{\{x_{(A_j, p)} : p \in \mathbb{N}\}}^{w^*}$. Then, by Lemma 2.5, we deduce that there is some $p_{j+1} \in \mathbb{N}$ such that $\operatorname{dist}(x_{(A_j, p_{j+1})}, [F_{[n_{j+1}+1, \infty)}]) < \varepsilon_{j+1}$ since $[F_{[n_{j+1}+1, \infty)}]$ is a finite-codimensional subspace in X and $P_{n_{j+1}+1}$ is relatively

w^*-continuous. Then there exist $n_{j+2} > n_{j+1}+1$ and $f_{j+1} \in F_{(n_{j+1},n_{j+2})}$ such that $\|x_{A_{j+1}} - f_{j+1}\| < \varepsilon_{j+1}$, where $A_{j+1} = (A_j, p_{j+1})$.

This finishes the inductive construction of the branch $\{x_{A_j}\}$ satisfying $\|x_{A_j} - f_j\| < \varepsilon_j$ for all j. Finally we get $\sum_{j=1}^{\infty} \|x_{A_j} - f_j\| < 1/(2K)$. Thus $\{x_{A_j}\}$ is a branch of the tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ which is a basic sequence equivalent to $\{f_j\}$, since $\{f_j\}$ is a skipped block sequence of $\{F_n\}$; hence $\{x_{A_j}\}$ is a boundedly complete sequence and the proof of Theorem 2.4 is finished.

Now we can get a characterization of RNP for dual spaces, following the above proof.

Theorem 2.6. *Let X be a Banach space. Then the following assertions are equivalent:*

(i) *X^* has RNP.*
(ii) *No topologically w^*-null tree in S_X is uniformly type P.*
(iii) *No topologically w^*-null tree in S_X has a type P branch.*
(iv) *Every topologically w^*-null tree in S_X has a boundedly complete branch.*

Proof. (iv)⇒(iii) is a consequence of the fact that no boundedly complete basic sequence is type P, as discussed in the introduction; and (iii)⇒(ii) is trivial.

For (ii)⇒(i) it is enough applying Theorem 2.1 for $K = B_{X^*}$ by assuming that X^* fails RNP and normalizing.

(i)⇒(iv). Assume that X^* has RNP and pick a topologically w^*-null tree $\{x_A\}$ in S_{X^*}. Denote by Y the closed linear span of the tree $\{x_A\}$. Now Y is a separable subspace of X^* and so there is a separable subspace Z of X norming Y so that Y is isometric to a subspace of Z^*. As X^* has RNP, so does Z^*. Hence Y is a separable subspace of Z^*, since Z is a separable space, and Z^* has w^*-PCP since Z^* has RNP. Observe that the tree $\{x_A\}$ is now a topologically w^*-null tree in S_Z^*, so we can select a full w^*-null subtree of $\{y_A\}$, since Z is separable and so the w^*-topology in Z^* is metrizable for bounded sets. We apply the proof of (i)⇒(iv) in the above result with $X = Y = Z^*$ to get a boundedly complete branch of $\{y_A\}$. As $\{y_A\}$ is a full subtree of $\{x_A\}$, the branches of $\{y_A\}$ are branches of $\{x_A\}$, and so $\{x_A\}$ has a boundedly complete branch. ∎

In case X is a separable Banach space, the above result can be written in terms of w^*-null trees. Then we immediately get the following corollary, obtained in [3] in a different way.

Corollary 2.7. *Let X be a separable Banach space. Then X^* is separable (equivalently, X^* has RNP) if, and only if, every w^*-null tree in S_{X^*} has a boundedly complete branch.*

Proof. When X is separable, the w^*-topology in X^* is metrizable on bounded sets and so every topologically w^*-null tree in S_{X^*} has a full subtree which is w^*-null. With this in mind, it is enough to apply the above theorem to conclude, since X^* is separable if and only if X^* has RNP, whenever X is separable. ■

The following consequence, obtained in a different way in [8], shows how many separable and dual subspaces contains every Banach space with PCP.

COROLLARY 2.8. *Let X be a Banach space with PCP. Then every seminormalized basic sequence in X has a boundedly complete subsequence.*

Proof. Pick a seminormalized basic sequence $\{x_n\}$ in X. Then either $\{x_n\}$ has a subsequence equivalent to the unit vector basis of ℓ_1, and hence boundedly complete, or it has a weakly Cauchy subsequence, which we denote again $\{x_n\}$.

If $\{x_n\}$ is weakly convergent it is weakly null, being a basic sequence. Now, $\{x_n\}$ is a seminormalized weakly null tree in X and hence $\{x_n\}$ is a seminormalized w^*-null tree in X^{**}. As $[x_n]$ is a separable subspace of X^{**} with w^*-PCP, from Remark 2.3 there is a separable subspace $Z \subset X^*$ such that $[x_n]$ is isometric to a subspace of Z^* with w^*-PCP. Therefore $\{x_n\}$ is a seminormalized w^*-null tree in $[x_n]$, which is a subspace of Z^* with w^*-PCP, since Z is separable. From Theorem 2.4, we get a boundedly complete branch and so a boundedly complete subsequence.

If $\{x_n\}$ is not weakly convergent we can apply the c_0-theorem [10] to get a strongly summing subsequence, denoted again by $\{x_n\}$, since X has PCP and so does not contain c_0. Let $x^{**} = w^*\text{-}\lim_n \ x_n \in X^{**}$. Now $\{x_n - x^{**}\}$ is a w^*-null sequence in $X \oplus [x^{**}] \subset X^{**}$. As X has PCP, it has w^*-PCP as a subspace of X^{**}, and then it is easy to see that $X \oplus [x^{**}] \subset X^{**}$ has w^*-PCP. Now $[x_n - x^{**}]$ is a separable subspace of X^{**} with w^*-PCP and so, from Remark 2.3, there is a separable subspace Z of X^* such that $[x_n - x^{**}]$ is isometric to a subspace of Z^* with w^*-PCP, as Z is separable. From Theorem 2.4, we get a boundedly complete branch and so a boundedly complete subsequence, denoted again by $\{x_n - x^{**}\}$. So $\{x_n - x^{**}\}$ is boundedly complete and $\{x_n\}$ is strongly summing.

Let us see that $\{x_n\}$ is boundedly complete. Indeed, if for some sequence of scalars $\{\lambda_n\}$ we have $\sup_n \|\sum_{k=1}^n \lambda_n x_n\| < \infty$, then the series $\sum_n \lambda_n$ is convergent, since $\{x_n\}$ is strongly summing. Now it is clear that $\sup_n \|\sum_{k=1}^n \lambda_n(x_n - x^{**})\| < \infty$ and hence $\sum_n \lambda_n(x_n - x^{**})$ converges, since $\{x_n - x^{**}\}$ is boundedly complete. So $\sum_n \lambda_n x_n$ converges, since $\sum_n \lambda_n$ is convergent, and $\{x_n\}$ is boundedly complete. ■

The converse of the above result is false, even for Banach spaces not containing ℓ_1 (see [5]).

Now we turn to some consequences relating to the problem of the determination of w^*-PCP by subspaces with a basis. We begin by proving that every seminormalized w^*-null tree has a basic full w^*-null subtree. The same is then true for the weak topology, by considering X as a subspace of X^{**}. We do not know an exact reference for this result, so we give a proof based on Mazur's proof of the known result that every seminormalized sequence in a dual Banach space such that 0 belongs to its w^*-closure has a basic subsequence.

LEMMA 2.9. *Let X be a Banach space and $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ a seminormalized w^*-null tree in S_{X^*}. Then for every $\varepsilon > 0$ there is a full basic subtree still w^*-null with basic constant less than $1+\varepsilon$.*

Proof. Let $\phi : \mathbb{N}^{<\omega} \to \mathbb{N}\cup\{0\}$ be a fixed bijective map such that $\phi(\emptyset) = 0$, $\phi(A) \le \phi(B)$ whenever $A \le B \in \mathbb{N}^{<\omega}$, and $\phi(A,i) \le \phi(A,j)$ whenever $A \in \mathbb{N}^{<\omega}$ and $i \le j$. Fix also $\varepsilon > 0$ and a sequence $\{\varepsilon_n\}_{n\ge 0} \subset (0,1)$ such that

$$\frac{1+\sum_{n=0}^{\infty}\varepsilon_n}{\prod_{n=0}^{\infty}(1-\varepsilon_n)} < 1+\varepsilon.$$

Now we construct the desired subtree $\{y_A\}_{A\in\mathbb{N}^{<\omega}}$ by induction, following the order given by ϕ to define y_A for every $A \in \mathbb{N}^{<\omega}$ and thus get the "full" condition. That is, we have to prove that for every $n \in \mathbb{N}\cup\{0\}$ we can construct $y_{\phi^{-1}(n)}$ such that $\{y_A\}_{A\in\mathbb{N}^{<\omega}}$ is a w^*-null full subtree with the property that for every $n \in \mathbb{N}\cup\{0\}$ there is a finite set $\{f_1^n,\dots,f_{k_n}^n\} \subset S_X$ such that:

(i) $\{f_1^n,\dots,f_{k_n}^n\}$ is a $(1-\varepsilon_n)$-norming set for $Y_n = [y_{\phi^{-1}(0)},\dots,y_{\phi^{-1}(n)}]$.
(ii) $|f_i^n(y_{\phi^{-1}(n+1)})| < \varepsilon_n$ for every i.
(iii) For every $A \in \mathbb{N}^{<\omega}$ there is an increasing map $\sigma_A : \mathbb{N} \to \mathbb{N}$ such that $y_{(A,i)} = x_{(A,\sigma_A(i))}$ for every i.

For $n = 0$, we have $\phi^{-1}(0) = \emptyset$ and we define $y_\emptyset = x_\emptyset$. Now take $f_1^0 \in S_X$ $(1-\varepsilon_0)$-norming the subspace $Y_0 = [y_\emptyset]$. As the tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is w^*-null there is $p_0 \in \mathbb{N}$ such that $|f_1^0(x_{(p)})| < \varepsilon_0$ for every $p \ge p_0$. Then we set $y_{\phi^{-1}(1)} = x_{(p_0)}$. As $\phi(A,i) \le \phi(A,j)$ whenever $A \in \mathbb{N}^{<\omega}$ and $i \le j$, we deduce that $\phi^{-1}(1) = (1)$ and define $\sigma_\emptyset(1) = p_0$ so that $y_{(\emptyset,1)} = x_{(\emptyset,\sigma_\emptyset(1))}$.

Assume $n \in \mathbb{N}$ and that we have already defined $y_{\phi^{-1}(0)},\dots,y_{\phi^{-1}(n-1)}$. Now $\phi^{-1}(n)- < \phi^{-1}(n)$, hence $\phi(\phi^{-1}(n)-) < n$ and so $y_{\phi^{-1}(n)-}$ has already been constructed, by induction hypothesis. Put $\phi^{-1}(n) = (A,h)$ for some $h \in \mathbb{N}$, where $A = \phi^{-1}(n)-$. As $\phi(A,k) \le \phi(A,h)$ for $k \le h$, we see that $y_{(A,k)}$ has been constructed with $y_{(A,k)} = x_{(A,\sigma_A(k))}$ whenever $k < h$, and $\sigma_A(k)$ has been constructed strictly increasing for $k < h$. Put $Y_{n-1} = [y_{\phi^{-1}(0)},\dots,y_{\phi^{-1}(n-1)}]$ and pick elements $f_1^{n-1},\dots,f_{k_{n-1}}^{n-1}$ in $S_{Y_{n-1}}$ $(1-\varepsilon_{n-1})$-norming Y_{n-1}. As the tree $\{x_A\}_{A\in\mathbb{N}^{<\omega}}$ is w^*-null there is $p_{n-1} > \max_{k<h} \sigma_A(k)$ such that $|f_i^{n-1}(x_{(A,p)})| < \varepsilon_{n-1}$ for all $1 \le i \le k_{n-1}$ and $p \ge p_{n-1}$. Then we set $y_{\phi^{-1}(n)} = x_{(A,p_{n-1})}$ and $\sigma_A(h) = p_{n-1}$. Thus $\sigma_A(k)$ is strictly increasing for $k \le h$ and $y_{\phi^{-1}(n)} = y_{(A,h)} = x_{(A,\sigma_A(h))}$. Now the construction of the subtree $\{y_A\}$ is complete and satisfies (i)–(iii). From the construction we see that $\{y_A\}$ is a full and w^*-null subtree.

Let us see that $\{y_{\phi^{-1}(n)}\}$ is a basic sequence in X. Put $z_n = y_{\phi^{-1}(n)}$, fix $p < q \in \mathbb{N}$ and compute $\| \sum_{i=1}^q \lambda_i z_i\|$, where $\{\lambda_i\}$ is a scalar sequence. Assume that $\| \sum_{i=1}^q \lambda_i z_i\| \le 1$. From (i) pick j such that $|f_j^{q-1}(\sum_{i=1}^{q-1} \lambda_i z_i)| > (1-\varepsilon_{q-1})\| \sum_{i=1}^{q-1} \lambda_i z_i\|$. Then from (ii) we have $|f_j^{q-1}(z_q)| < \varepsilon_{q-1}$ and so

$$\Big\| \sum_{i=1}^q \lambda_i z_i \Big\| \ge \Big| f_j^{q-1}\Big(\sum_{i=1}^q \lambda_i z_i\Big)\Big| > (1-\varepsilon_{q-1})\Big\|\sum_{i=1}^{q-1} \lambda_i z_i\Big\| - \varepsilon_{q-1}.$$

By repeating this computation we get

$$\Big\| \sum_{i=1}^q \lambda_i z_i \Big\| \ge \Big(\prod_{i=p+1}^{q} (1-\varepsilon_{i-1})\Big)\Big\|\sum_{i=1}^{p} \lambda_i z_i\Big\| - \sum_{i=p+1}^{q} \varepsilon_{i-1},$$

and so

$$\Big\| \sum_{i=1}^p \lambda_i z_i \Big\| \le \frac{1+\sum_{i=p+1}^q \varepsilon_{i-1}}{\prod_{i=p+1}^q (1-\varepsilon_{i-1})} < 1+\varepsilon.$$

The last inequality proves that $\{z_n\}$ is a basic sequence in X with basic constant less than $1+\varepsilon$, and the proof is complete. ∎

We do not know if the above result is still true with "w^*-null" replaced by "topologically w^*-null".

The following result shows that w^*-PCP is determined by subspaces with a Schauder basis in the natural setting of dual spaces of separable Banach spaces.

Corollary 2.10. *Let X, Y be Banach spaces such that Y is separable and X is a subspace of Y^*. Then X has w^*-PCP if, and only if, every subspace of X with a Schauder basis has w^*-PCP.*

Proof. Assume that X fails w^*-PCP. Then, from Theorem 2.4, there is a w^*-null tree in the unit sphere of X without boundedly complete branches. Now, by Lemma 2.9, we can extract a w^*-null full basic subtree. The subspace Z generated by this subtree is a subspace of X with a Schauder basis containing a w^*-null tree in S_X without boundedly complete branches, from the "full" condition, so Z fails w^*-PCP, by Theorem 2.4. ∎

As a consequence we deduce, for example, that a subspace of ℓ_∞, the space of bounded scalar sequences with the sup norm, failing w^*-PCP (or failing PCP) contains a further subspace with a Schauder basis failing w^*-PCP.

If we take $X = Y^*$ in the above corollary we can deduce the following

Corollary 2.11. *Let X be a separable Banach space. Then X^* has RNP if, and only if, every subspace of X^* with a Schauder basis has w^*-PCP.*

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References

[1] J. Bourgain, *Dentability and finite-dimensional decompositions*, Studia Math. 67 (1980), 135–148.
[2] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, Berlin, 1983.
[3] S. Dutta and V. P. Fonf, *On tree characterizations of G_δ-embeddings and some Banach spaces*, Israel J. Math. 167 (2008), 27–48.
[4] N. Ghoussoub and B. Maurey, *G_δ-embeddings in Hilbert spaces*, J. Funct. Anal. 61 (1985), 72–97.
[5] G. López-Pérez, *Banach spaces with many boundedly complete basic sequences failing PCP*, J. Funct. Anal. 259 (2010), 2139–2146.
[6] G. López-Pérez and J. Soler-Arias, *A tree characterization of the point of continuity property in general Banach spaces*, Proc. Amer. Math. Soc. 140 (2012), 4243–4245.
[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.
[8] H. P. Rosenthal, *Boundedly complete weak-Cauchy basic sequences in Banach spaces with PCP*, J. Funct. Anal. 253 (2007), 772–781.
[9] H. P. Rosenthal, *Weak*-Polish Banach Spaces*, J. Funct. Anal. 76 (1988), 267–316.
[10] H. P. Rosenthal, *A subsequence principle characterizing Banach spaces containing c_0*, Bull. Amer. Math. Soc. 30 (1994), 227–233.
[11] I. Singer, *Bases in Banach Spaces I*, Springer, Berlin, 1970.

Ginés López-Pérez, José A. Soler-Arias
Universidad de Granada
Facultad de Ciencias
Departamento de Análisis Matemático
18071 Granada, Spain
E-mail: glopezp@ugr.es
jasoler@ugr.es

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