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The essential spectrum of Toeplitz tuples with symbols in $H^\infty + C$

by

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Abstract. Let $H^2(D)$ be the Hardy space on a bounded strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. Using Gelfand theory and a spectral mapping theorem of Andersson and Sandberg (2003) for Toeplitz tuples with H^∞-symbol, we show that a Toeplitz tuple $T_f = (T_{f_1}, \ldots, T_{f_m}) \in L(H^2(\sigma))^m$ with symbols $f_i \in H^\infty + C$ is Fredholm if and only if the Poisson–Szegö extension of f is bounded away from zero near the boundary of D. Corresponding results are obtained for the case of Bergman spaces. Thus we extend results of McDonald (1977) and Jewell (1980) to systems of Toeplitz operators.

1. Introduction. Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary. Extending results of McDonald [9] for the unit ball, Jewell proved in [7] that a Toeplitz operator T_f with symbol in $H^\infty + C$ on the Bergman space or Hardy space over D is Fredholm if and only if f, or its Poisson–Szegö extension in the case of the Hardy space, is bounded away from zero near the boundary of D. A basic ingredient of the proof was the observation that, for every multiplicative linear functional ϕ of $H^\infty(D)$ belonging to the fibre of the maximal ideal space of $H^\infty(D)$ over a boundary point $\lambda \in \partial D$ and any function $f \in H^\infty(D)$, the value $\phi(f)$ belongs to the cluster set of f at λ.

In the present note we replace single Fredholm operators T_f by tuples $T_f = (T_{f_1}, \ldots, T_{f_m})$ of Toeplitz operators with symbol $f \in (H^\infty + C)^m$. If the above cluster value property of $H^\infty(D)$ were known to be true for tuples $f \in H^\infty(D)^m$ instead of single functions, then the methods from [7] could be extended in a straightforward way to calculate the essential spectrum of the essentially commuting multioperator T_f. However, the cluster value property for finite tuples in $H^\infty(D)$ is equivalent to the validity of the Corona Theorem for $H^\infty(D)$. This equivalence is well known and follows, for instance, as a direct application of Theorem 1 from [5].

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In the following we show that, in spite of this difficulty, properties of the Poisson transform and suitable results from Gelfand theory can be used to prove the spectral mapping formula

$$\sigma_e(T_f) = \bigcap (\overline{f(U \cap D)};\, U \supset \partial D \,\text{open})$$

for the essential Taylor spectrum $\sigma_e(T_f)$ of Toeplitz tuples T_f with symbol $f \in (H^\infty + C)^m$ on Hardy and Bergman spaces over strictly pseudoconvex domains. Here again, in the Hardy space case, the symbol f has to be interpreted as the Poisson–Szegö extension of f. Since, for $m = 1$, our notion of joint essential spectrum coincides with the usual essential spectrum of a single bounded linear operator, the above spectral mapping formula reduces to the cited result of Jewell in this case.

2. Preliminaries. Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary and let $H^2(D)$ be the Hardy space on D. Since the point evaluation at every point of D is continuous on $H^2(D)$, the space $H^2(D)$ is an analytic functional Hilbert space and hence it has a reproducing kernel $K : D \times D \to \mathbb{C}$. Let σ be the normalized surface measure on ∂D. We shall identify $H^2(D)$ with its image $H^2(\sigma)$ under the isometry

$$H^2(D) \to L^2(\sigma), \quad f \mapsto f^*,$$

associating with each function $f \in H^2(D)$ its non-tangential boundary value f^*. For $z \in D$, consider the function

$$P(z, \cdot) = \frac{|K(\cdot, z)^*|^2}{K(z, z)} \in L^1(\sigma).$$

As usual, we call P the *Poisson–Szegö kernel* and define the *Poisson–Szegö integral* of a function $f \in L^\infty(\sigma)$ by

$$\mathcal{P}[f] : D \to \mathbb{C}, \quad z \mapsto \int_{\partial D} f P(z, \cdot)\, d\sigma.$$

The Poisson–Szegö integral reproduces functions in $H^\infty(D)$. For $f \in C(\partial D)$ the Poisson–Szegö integral extends to a function $F \in C(\overline{D})$ with $F|\partial D = f$ (see [10] or [8] for both properties).

For $f \in L^\infty(\sigma)$, we define the *Toeplitz operator* $T_f \in L(H^2(\sigma))$ and the *Hankel operator* $H_f \in L(H^2(\sigma), L^2(\sigma))$ with symbol f by

$$T_f = PM_f|H^2(\sigma) \quad \text{and} \quad H_f = (1 - P)M_f|H^2(\sigma).$$

Here $P : L^2(\sigma) \to H^2(\sigma)$ denotes the orthogonal projection and $M_f : L^2(\sigma) \to L^2(\sigma)$, $g \mapsto fg$, is the operator of multiplication f. For $z \in D$, let $k_z = K(\cdot, z)^*/\|K(\cdot, z)^*\|_{H^2(\sigma)}$ be the normalized reproducing kernel vector at the point z. The *Berezin transform* of an operator $T \in L(H^2(\sigma))$ is the function

$$\Gamma(T) : D \to \mathbb{C}, \qquad z \mapsto \langle Tk_z, k_z\rangle.$$

The Berezin transform of the Toeplitz operator T_f with symbol $f \in L^\infty(\sigma)$ coincides with its Poisson–Szegö integral:

$$\Gamma(T_f)(z) = \langle T_f k_z, k_z\rangle_{H^2(\sigma)} = \int_{\partial D} fP(z,\cdot)\, d\sigma.$$

It is well known (see for instance [5, Proposition 6]) that the Berezin transform of a compact operator $K \in L(H^2(\sigma))$ vanishes on the boundary of D in the sense that

$$\lim_{z\to\partial D} \Gamma(K)(z) = 0.$$

For a given subset $S \subset L^\infty(\sigma)$, the *Toeplitz algebra with symbol class* S is the closed subalgebra of $L(H^2(\sigma))$ defined by

$$\mathcal{T}(S) = \overline{\mathrm{alg}}\,\{T_f;\, f \in S\}.$$

Important choices for S are the set of all bounded analytic functions (or better their boundary values), which will be denoted by $H^\infty = H^\infty(\sigma)$ in what follows, and the class $C = C(\partial D)$ consisting of all complex-valued continuous functions on ∂D. A result of Aytuna and Chollet [2], generalizing a corresponding observation of Rudin for the unit ball, shows that $H^\infty + C = H^\infty(\sigma) + C(\partial D) \subset L^\infty(\sigma)$ is a closed subalgebra. It is known (see for instance [4]) that the Toeplitz algebra $\mathcal{T}(H^\infty + C)$ contains the set $\mathcal{K}(H^2(\sigma))$ of all compact operators and that the map

$$\tau : H^\infty + C \to \mathcal{T}(H^\infty + C)/\mathcal{K}(H^2(\sigma)), \qquad f \mapsto T_f + \mathcal{K}(H^2(\sigma)),$$

is an isometric isomorphism of Banach algebras. In particular, Toeplitz tuples $T_f = (T_{f_1}, \ldots, T_{f_m})$ with symbols $f_i \in H^\infty + C$ essentially commute in the sense that the commutators

$$[T_{f_i}, T_{f_j}] = T_{f_i}T_{f_j} - T_{f_j}T_{f_i} \quad (1 \le i, j \le m)$$

are compact.

The Koszul complex (cf. [6, Section 2.2])

$$K^\bullet(T,H) : 0 \to \Lambda^0(H) \xrightarrow{\delta_T^0} \Lambda^1(H) \xrightarrow{\delta_T^1} \cdots \xrightarrow{\delta_T^{m-1}} \Lambda^m(H) \to 0$$

of an essentially commuting tuple $T \in L(H)^m$ of bounded operators on a Hilbert space H is an essential complex of Hilbert spaces in the sense that $\delta_T^{i+1} \circ \delta_T^i$ is compact for every i. The tuple T is called *Fredholm* if the Koszul complex $K^\bullet(T,H)$ has an *essential homotopy*, that is, there are bounded operators $\epsilon^i : \Lambda^i(H) \to \Lambda^{i-1}(H)$ with

$$\epsilon^{i+1}\delta_T^i - \delta_T^{i-1}\epsilon^i - 1_{\Lambda^i(H)} \in \mathcal{K}(\Lambda^i(H))$$

for all i. One can show [6, Lemma 2.6.10 and Theorem 10.2.5] that the tuple T is Fredholm if and only if the Koszul complex $K^\bullet(L_T, \mathcal{C}(H))$ of the commuting tuple $L_T = (L_{T_1}, \ldots, L_{T_m})$ consisting of the left multiplication operators

$$L_{T_i} : \mathcal{C}(H) \to \mathcal{C}(H), \quad [A] \mapsto [T_i A],$$

on the Calkin algebra $\mathcal{C}(H) = L(H)/\mathcal{K}(H)$ is exact. The essential spectrum of an essentially commuting tuple $T \in L(H)^m$ is defined as

$$\sigma_e(T) = \{z \in \mathbb{C}^m;\ K^\bullet(z - T, H) \text{ is not Fredholm}\} = \sigma(L_T, \mathcal{C}(H)),$$

where $\sigma(L_T, \mathcal{C}(H))$ denotes the Taylor spectrum [11] of the commuting tuple $L_T \in L(\mathcal{C}(H))^m$.

3. Main result. To prove the spectral mapping theorem for the essential spectrum of Toeplitz tuples with symbol in $H^\infty + C$, we need a result on the asymptotic multiplicativity of the Poisson–Szegö transform.

LEMMA 3.1. *For $f, g \in H^\infty + C$, the Poisson–Szegö transform satisfies*

$$\lim_{z \to \partial D} \left|\mathcal{P}[fg](z) - \mathcal{P}[f](z)\mathcal{P}[g](z)\right| = 0.$$

Proof. We need some results on the Berezin transform that are implicitly contained in [3]. Since every point $z \in \partial D$ is a peak point for the Banach algebra $A(D) = \{f \in C(\overline{D});\ f|D \text{ holomorphic}\}$, it follows that $A(D)$ is a pointed function algebra in the sense of [3, Definition 2.1 and Theorem 2.3]. It is elementary to check that the Hardy space $H^2(D)$ is a quasi-free Hilbert module over $A(D)$ as defined in [3].

For $z \in D$, consider the isometry

$$V_z : \mathbb{C} \to H^2(\sigma), \quad t \mapsto t k_z.$$

The mapping $P_z = V_z V_z^*$ is the orthogonal projection onto the one-dimensional subspace of $H^2(\sigma)$ spanned by k_z. For given operators S, T in $L(H^2(\sigma))$, the estimate

$$\begin{aligned} |\Gamma(ST)(z) - \Gamma(S)(z)\Gamma(T)(z)| &= \left|(V_z^* S T V_z - V_z^* S P_z T V_z)(1)\right| \\ &= \left|V_z^* S [T, P_z] V_z(1)\right| \le \|S\| \, \|[T, P_z]\| \end{aligned}$$

holds for every point $z \in D$. For $\alpha \in \partial D$, the set of all operators T in $L(H^2(\sigma))$ with the property that

$$\lim_{z \to \alpha} \|[T, P_z]\| = 0$$

is a C^*-algebra containing the Toeplitz algebra

$$\mathcal{T}(C) = C^*(\{T_f;\ f \in A(D)|\partial D\})$$

(see the proof of Theorem 3.2 in [3]). An elementary compactness argument shows that $\lim_{z\to\partial D}\|[T,P_z]\|=0$ for every operator $T\in\mathcal{T}(C)$. Therefore

$$\lim_{z\to\partial D}\left|\varGamma(ST)(z)-\varGamma(S)(z)\varGamma(T)(z)\right|=0$$

for any pair of operators $S\in L(H^2(\sigma))$, $T\in\mathcal{T}(C)$. Since for $g\in C$ and $f\in L^\infty(\sigma)$, the semicommutator $T_{fg}-T_fT_g=PM_fH_g$ is compact [12, Theorem 4.2.17], it follows that

$$\begin{aligned}|\mathcal{P}[fg](z)&-\mathcal{P}[f](z)\mathcal{P}[g](z)|\\&\le\left|\varGamma(T_{fg}-T_fT_g)(z)\right|+\left|\varGamma(T_fT_g)(z)-\varGamma(T_f)(z)\varGamma(T_g)(z)\right|\end{aligned}$$

tends to zero as $z\to\partial D$. Using in addition the fact that $\mathcal{P}[fg]=\mathcal{P}[f]\mathcal{P}[g]$ for $f,g\in H^\infty$, one easily deduces the assertion. ∎

We begin by proving one half of our spectral mapping theorem in a particular situation. For simplicity, we use the notation

$$F=\mathcal{P}[f]=(\mathcal{P}[f_1],\ldots,\mathcal{P}[f_m])$$

for the Poisson–Szegö transform of a tuple $f=(f_1,\ldots,f_m)\in L^\infty(\sigma)^m$.

Lemma 3.2. *For given $g\in(H^\infty)^r$, $h\in C^s$ and $f=(g,h)$, the following spectral inclusion holds:*

$$\bigcap(\overline{F(U\cap D)};\,U\supset\partial D\,\text{open})\subset\sigma_e(T_f).$$

Proof. Suppose that T_f is Fredholm. It suffices to show that $F=\mathcal{P}[f]$ is bounded away from zero close to the boundary of D. Since T_f is Fredholm, the row multiplication

$$H^2(\sigma)^m\xrightarrow{T_f}H^2(\sigma)$$

with $m=r+s$ has finite-codimensional range. The orthogonal projection $Q\in L(H^2(\sigma))$ to the kernel of the operator $T_fT_f^*$ has finite rank and $T_fT_f^*+Q$ is bounded below. Hence there is a constant $c>0$ with

$$T_fT_f^*+Q\ge c1_{H^2(\sigma)}.$$

Since the Berezin transform $\varGamma(Q)(z)$ tends to zero as z approaches the boundary of D, there is an open neighbourhood U of ∂D such that

$$\sum_{i=1}^m\varGamma(T_{f_i}T_{f_i}^*)(z)=\varGamma(T_fT_f^*)\ge c/2$$

for all $z\in U\cap D$. An elementary calculation [5, Lemma 7] yields

$$\varGamma(T_{g_i}T_{g_i}^*)=|G_i|^2\qquad(i=1,\ldots,r)$$

on D. Since $T_{h_i}T_{h_i}^*-T_{|h_i|^2}$ is compact and since by Lemma 3.1,

$$\mathcal{P}[|h_i|^2](z)-|\mathcal{P}[h_i](z)|^2\xrightarrow{z\to\partial D}0,$$

it follows that

$$\Gamma(T_{h_i}T_{h_i}^*)(z) - |H_i(z)|^2 \xrightarrow{z\to\partial D} 0 \qquad (i = 1, \ldots, s).$$

Summarizing we obtain

$$\sum_{i=1}^{m} |F_i(z)|^2 - \Gamma(T_f T_f^*)(z) = \sum_{i=1}^{s} \bigl(|H_i(z)|^2 - \Gamma(T_{h_i}T_{h_i}^*)(z)\bigr) \to 0$$

as z approaches the boundary of D. Thus the assertion follows. ∎

To prepare the proof of the opposite inclusion, we recall some results from Gelfand theory. Consider a unital algebra homomorphism $\Phi : \mathcal{M} \to L(X)$ from a unital commutative Banach algebra $\mathcal{M}$ into the algebra of all bounded operators on a Banach space X. A *spectral system* on $B = \overline{\Phi(\mathcal{M})}$ is a rule σ that assigns to each finite tuple $a \in B^r$ a compact subset $\sigma(a) \subset \mathbb{C}^r$ which is contained in the joint spectrum

$$\sigma_B(a) = \Bigl\{z \in \mathbb{C}^r;\ 1 \notin \sum_{i=1}^{r} (z_i - a_i)B\Bigr\}$$

of a in B and which is compatible with projections in the sense that

$$p(\sigma(a,b)) = \sigma(a) \quad \text{and} \quad q(\sigma(a,b)) = \sigma(b)$$

for $a \in B^r$ and $b \in B^s$, where p and q are the projections of $\mathbb{C}^{r+s}$ onto its first r and last s coordinates.

For a given set M, let us denote by $c(M)$ the set of all finite tuples of elements in M. Standard results going back to J. L. Taylor (see, e.g., [6, Proposition 2.6.1]) show that, for a spectral system σ as above, the set

$$\Delta_{\Phi,\sigma} = \{\lambda \in \Delta_{\mathcal{M}};\ \hat{f}(\lambda) \in \sigma(\Phi(f)) \text{ for all } f \in c(\mathcal{M})\}$$

is the unique closed subset of the maximal ideal space $\Delta_{\mathcal{M}}$ of $\mathcal{M}$ with $\hat{f}(\Delta_{\Phi,\sigma}) = \sigma(\Phi(f))$ for all $f \in c(\mathcal{M})$. Here $\Phi(f) = (\Phi(f_1), \ldots, \Phi(f_r))$ and the Gelfand transforms $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_r)$ are formed componentwise for $f \in \mathcal{M}^r$.

Let $\Phi_0 : \mathcal{M}_0 \to L(X)$ be the restriction of Φ to a unital closed subalgebra $\mathcal{M}_0 \subset \mathcal{M}$, and let σ_0 denote the spectral system on $B_0 = \overline{\Phi(\mathcal{M}_0)}$ obtained by restricting σ. An elementary exercise, using the uniqueness property of Δ_{Φ_0,σ_0}, shows that the restriction map

$$r : \Delta_{\Phi,\sigma} \to \Delta_{\Phi_0,\sigma_0}, \qquad \lambda \mapsto \lambda|\mathcal{M}_0,$$

is well defined, surjective and continuous (relative to the Gelfand topologies).

As before, let $H^2(\sigma)$ be the Hardy space on a bounded strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. We apply the above remarks to the Banach algebras $\mathcal{M}_0 = H^\infty$, $\mathcal{M} = H^\infty + C$ and the algebra homomorphism $\Phi : \mathcal{M} \to L(\mathcal{C}(H^2(\sigma)))$, $f \mapsto L_{T_f}$, mapping $f \in \mathcal{M}$ to the operator L_{T_f} of left multiplication with T_f on the Calkin algebra $\mathcal{C}(H^2(\sigma))$. Let σ be the spectral system on $B = \overline{\Phi(\mathcal{M})}$ associating with each tuple $a \in B^r$ its Taylor spectrum as a commuting tuple of bounded operators on $\mathcal{C}(H^2(\sigma))$. We write σ_0 for the restriction of σ to $B_0 = \overline{\Phi(\mathcal{M}_0)}$.

Recall that, for a tuple $f \in c(L^\infty(\sigma))$, we write $F = \mathcal{P}[f]$ for its Poisson–Szegö transform. As usual we shall identify functions $f \in H^\infty(\sigma)$ with their Poisson–Szegö transforms $F = \mathcal{P}[f] \in H^\infty(D)$. It was shown by Andersson and Sandberg [1, Theorem 1.2] that the spectral mapping formula

$$\sigma(\Phi(f)) = \sigma_e(T_f) = \bigcap(\overline{f(U \cap D)};\ U \supset \partial D \text{ open})$$

holds for every tuple $f \in c(H^\infty)$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be the tuple of coordinate functions. Using Theorem 1 in [5] we obtain

$$\hat{f}(\lambda) \in \bigcap(\overline{f(U \cap D)};\ U \text{ open neighbourhood of } \hat{\pi}(\lambda))$$

for $f \in c(H^\infty)$ and every functional $\lambda \in \Delta_{\Phi_0,\sigma_0}$.

Proposition 3.3. *For $g \in (H^\infty)^r$, $h \in C^s$ and $f = (g, h)$, the following spectral inclusion formula holds:*

$$\sigma_e(T_f) \subset \bigcap(\overline{F(U \cap D)};\ U \supset \partial D \textit{ open}).$$

Proof. Suppose that $0 \in \sigma_e(T_f)$. It suffices to show that 0 is contained in the intersection on the right. By the remarks preceding the proposition there is a functional $\lambda \in \Delta_{\Phi,\sigma}$ with $0 = \hat{f}(\lambda) = (\hat{g}(\lambda), \hat{h}(\lambda))$. Since $\lambda|C \in \Delta_C$, there is a point $z_0 \in \partial D$ with

$$\lambda(\phi) = \phi(z_0) \qquad (\phi \in C).$$

In particular, it follows that $\lim_{z \to z_0} H(z) = h(z_0) = 0$. The above-cited results from [1] and [5] imply that

$$0 = \hat{g}(\lambda) \in \bigcap(\overline{g(U \cap D)};\ U \text{ open neighbourhood of } z_0).$$

Hence there is a sequence $(z_k)_{k \ge 1}$ in D with $\lim_{k \to \infty} z_k = z_0$ and

$$\lim_{k \to \infty} (g(z_k), H(z_k)) = 0.$$

This observation completes the proof. ∎

Our next aim is to show that Lemma 3.2 and Proposition 3.3 remain true for arbitrary symbols $f \in (H^\infty + C)^m$.

Theorem 3.4. *For $f \in (H^\infty + C)^m$,*

$$\sigma_e(T_f) = \bigcap(\overline{F(U \cap D)};\ U \supset \partial D \textit{ open}).$$

Proof. Let $f = g + h \in (H^\infty + C)^m$ be given with $g \in (H^\infty)^m$ and $h \in C^m$. Using a particular case of the analytic spectral mapping theorem for the Taylor spectrum, we obtain

$$\begin{aligned}\sigma_e(T_f) &= \sigma_e(T_g + T_h) = \sigma(L_{T_g} + L_{T_h}) \\ &= \{z + w;\ (z,w) \in \sigma(L_{T_g}, L_{T_h})\} = \{z + w;\ (z,w) \in \sigma_e(T_g, T_h)\}.\end{aligned}$$

If $(z,w) \in \sigma_e(T_g, T_h)$, then by Proposition 3.3 there is a sequence (u_k) in D converging to some point $u \in \partial D$ such that

$$(z,w) = \lim_{k\to\infty} (G,H)(u_k).$$

But then

$$z + w = \lim_{k\to\infty} (G+H)(u_k) = \lim_{k\to\infty} F(u_k).$$

Hence $\sigma_e(T_f)$ is contained in the intersection on the right-hand side.

Conversely, if ξ is a point in the intersection on the right-hand side, then there is a sequence (u_k) in D converging to a point $u \in \partial D$ such that

$$\xi = \lim_{k\to\infty} F(u_k) = \lim_{k\to\infty} (G(u_k) + H(u_k)).$$

But then $w = \lim_{k\to\infty} H(u_k) = h(u)$ exists and hence also $z = \lim_{k\to\infty} G(u_k)$ exists. By Lemma 3.2 we know that $(z,w) \in \sigma_e(T_g, T_h)$. Hence $\xi = z + w$ belongs to $\sigma_e(T_f)$ as was to be shown. ∎

For a tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ of operators on a Hilbert space H, the *right essential spectrum* $\sigma_{re}(T)$ is usually defined as the set of all points $z \in \mathbb{C}^n$ for which the range of the row multiplication

$$H^n \xrightarrow{(z_1 - T_1, \ldots, z_n - T_n)} H$$

is not finite codimensional, or equivalently, the row multiplication

$$\mathcal{C}(H)^n \xrightarrow{(z_1 - L_{T_1}, \ldots, z_n - L_{T_n})} \mathcal{C}(H)$$

is not onto (see e.g. Lemma 2.6.10 in [6] for the equivalence). Hence the right essential spectrum $\sigma_{re}(T)$ of T coincides with the right spectrum $\sigma_r(L_T, \mathcal{C}(H))$ of the multiplication tuple L_T on the Calkin algebra. Since Lemma 3.2 remains true with $\sigma_e(T_f)$ replaced by $\sigma_{re}(T_f)$ (see the proof of the lemma) and since the analytic spectral mapping formula used in the proof of Theorem 3.4 also holds for the right Taylor spectrum [6, Corollary 2.6.8], we obtain the following consequence.

Corollary 3.5. *For $f \in (H^\infty + C)^m$,*

$$\sigma_e(T_f) = \sigma_{re}(T_f) = \bigcap (\overline{F(U \cap D)};\ U \supset \partial D \text{ open}).\ ∎$$

Our main result (Theorem 3.4) can also be proved for Toeplitz tuples $T_f \in L(L_a^2(D))^m$ with symbol $f \in (H^\infty(D) + C(\overline{D})|D)^m$ on the Bergman space $L^2_a(D) = \{f \in \mathcal{O}(D);\ \|f\|^2 = \int_D |f|^2\, d\lambda < \infty\}$ formed with respect to the volume measure λ on a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. It suffices to replace the spectral mapping formula of Andersson and Sandberg [1] for Toeplitz tuples with H^∞-symbol by the corresponding spectral mapping formula for the Bergman space (Theorem 8.2.6 in [6]) and to replace the Poisson–Szegö transform by the Poisson–Bergman transform. All properties needed for the Poisson–Bergman integral can be found in [8]. We only state the corresponding result in the Bergman case.

THEOREM 3.6. *Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary. Then for $f \in (H^\infty(D) + C(\overline{D}))^m$, the essential spectrum of the Toeplitz tuple $T_f \in L(L^2_a(D))^m$ on the Bergman space $L^2_a(D)$ is given by*

$$\sigma_e(T_f) = \sigma_{re}(T_f) = \bigcap(\overline{f(U \cap D)};\ U \supset \partial D \text{ open}). \ \blacksquare$$

The reader should observe that, since the Poisson–Bergman transform of a continuous function $h \in C(\overline{D})^m$ extends to a continuous function $H \in C(\overline{D})^m$ with $H|\partial D = h|\partial D$, the intersection on the right-hand side does not change when f is replaced by its Poisson–Bergman transform F.

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