STUDIA MATHEMATICA 219 (3) (2013)

On the distribution of random variables corresponding to Musielak–Orlicz norms

by

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Abstract. Given a normalized Orlicz function M we provide an easy formula for a distribution such that, if X is a random variable distributed accordingly and $X_1, \ldots, X_n$ are independent copies of X, then

$$\frac{1}{C_p}\|x\|_M \leq \mathbb{E}\|(x_i X_i)_{i=1}^n\|_p \leq C_p\|x\|_M,$$

where C_p is a positive constant depending only on p. In case $p = 2$ we need the function $t \mapsto tM'(t) - M(t)$ to be 2-concave and as an application immediately obtain an embedding of the corresponding Orlicz spaces into $L_1[0,1]$. We also provide a general result replacing the ℓ_p-norm by an arbitrary N-norm. This complements some deep results obtained by Gordon, Litvak, Schütt, and Werner [Ann. Prob. 30 (2002)]. We also prove, in the spirit of that paper, a result which is of a simpler form and easier to apply. All results are true in the more general setting of Musielak–Orlicz spaces.

1. Introduction. In their outstanding work [12], Kwapień and Schütt obtained beautiful and strong combinatorial inequalities in connection with Orlicz norms that were then used to study certain invariants of Banach spaces (see also [13]). The new tool not only allowed them to compute the positive projection constant of a finite-dimensional Orlicz space, but also led to a characterization of the symmetric sublattices of $\ell_1(c_0)$ and the finite-dimensional symmetric subspaces of ℓ_1. The method was later used in [26] to determine p-absolutely summing norms, and was extended by Raynaud and Schütt to infinite-dimensional Banach spaces in [22] (see also [24] for applications to Lorentz spaces). In some special cases, the combinatorial expressions were already considered by Gluskin in [6] (see also [23]). Quite recently, in [20], the tools were generalized to obtain new results on the local structure of the classical Banach space L_1.

2010 *Mathematics Subject Classification*: Primary 46B09; Secondary 46B07, 46B45, 60B99.
Key words and phrases: Orlicz function, Orlicz norm, Musielak–Orlicz norm, random variables.

DOI: 10.4064/sm219-3-6

In the great paper [8], building upon the combinatorial results from [12] and [13], Gordon, Litvak, Schütt and Werner were able to obtain even more general results in the continuous setting. They proved that, if N is an Orlicz function and $X_1, \dots, X_n$ are independent copies of a random variable X, then $\mathbb{E}\|(x_i X_i)_{i=1}^n\|_N$ is of the order $\|x\|_M$ where M depends on N and on the distribution of X. This result, of course, is already interesting from a purely probabilistic point of view and was later used by the authors in [7] to obtain estimates for various parameters associated to the local theory of convex bodies. It also initiated further research and led to beautiful results on order statistics [10, 9]. Recently, in the series of papers [1–3], these results were also successfully applied to study geometric functionals corresponding to random polytopes.

A natural question is whether the converse is true, i.e., whether given Orlicz functions M and N, we can provide a formula for a distribution so that, if $X_1, \dots, X_n$ are independent copies of an accordingly distributed random variable X, then $\mathbb{E}\|(x_i X_i)_{i=1}^n\|_N$ is of the order $\|x\|_M$. This is part of the motivation for our work and we will answer this question in the affirmative. The "natural" candidate for the distribution is deduced from a new simpler version of a result from [8] that we prove here. In the special case of $N(t) = t^p$ we give very easy formulas for the distribution of the random variables depending on the Orlicz function M, provided M satisfies a certain condition depending on the parameter p. For $p = 2$, this condition amounts to the 2-concavity of $t \mapsto tM'(t) - M(t)$.

In his beautiful paper [25] Schütt proved that, if M is equivalent to a 2-concave Orlicz function, then the spaces ℓ_M^n, $n \in \mathbb{N}$, embed uniformly into L_1 (see also [5] and [18]). The proof is quite technical and based on combinatorial inequalities, some of which first appeared in the joint work [12, 13] with Kwapień. Given a 2-concave Orlicz function M with certain additional properties, Schütt provided an explicit formula for a sequence $a_1, \dots, a_n$ of positive real numbers such that for all $x \in \mathbb{R}^n$,

$$c_1 \|x\|_M \le \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \Big(\sum_{i=1}^n |x_i a_{\pi(i)}|^2 \Big)^{1/2} \le c_2 \|x\|_M,$$

where $\mathfrak{S}_n$ is the set of all permutations of the numbers $\{1, \dots, n\}$, and c_1, c_2 are absolute constants (see Theorem 2 in [25]). Khinchin's inequality then implies that these Orlicz spaces embed uniformly into L_1. Unfortunately, the formula is rather complicated and it is non-trivial to calculate the Orlicz function. This, in fact, is the other part of our motivation. The converse result that we obtain for $p = 2$, where we need $t \mapsto tM'(t) - M(t)$ to be 2-concave, immediately implies that these Orlicz spaces ℓ_M^n, $n \in \mathbb{N}$, are uniformly isomorphic to subspaces of L_1. Although it seems we need a somewhat stronger assumption on M, the inversion formula we obtain is much simpler and easier to apply. The result might also be useful in finding new and easily verifiable characterizations for more general classes of subspaces of L^1.

We provide here two different approaches to prove the converse results (for ℓ_p-norms and general N-norms); in each, conditions on M naturally appear. Even more, if $p = 2$ and we do not assume the 2-concavity of $t \mapsto tM'(t) - M(t)$, but only the equivalence of $\mathbb{E}\|(x_iX_i)_{i=1}^n\|_2$ and $\|x\|_M$, then it is not hard to see that $t \mapsto tM'(t) - M(t)$ already has to be 2-concave (see Proposition 7.1). Therefore, it seems that the condition is natural and "not too far" from the 2-concavity of M.

Our main result is the following:

THEOREM 1.1. *Let $1 < p < \infty$ and $M \in \mathcal{C}^3$ be an Orlicz function with $M'(0) = 0$ and $M''(T) = 0$ for $T = M^{-1}(1)$. Assume the normalization $\int_0^\infty x\,dM'(x) = 1$ and that $M|_{[T,\infty)}$ is linear. Moreover, assume that for all $x > 0$,*

$$f_X(x) = \left(1 - \frac{2}{p}\right)\frac{1}{x^3}M''\left(\frac{1}{x}\right) - \frac{1}{px^4}M'''\left(\frac{1}{x}\right) \geq 0.$$

Then f_X is a probability density and for all $x \in \mathbb{R}^n$,

$$c_1(p-1)^{1/p}\|x\|_M \leq \mathbb{E}\|(x_iX_i)_{i=1}^n\|_p \leq c_2\|x\|_M,$$

where c_1, c_2 are positive absolute constants and $X_1, \ldots, X_n$ are iid with density f_X.

If M is not normalized, we can divide the function f_X by $\int_0^\infty x\,dM'(x)$ to obtain a probability density and the statement of the theorem is true with constants depending on p and M. Due to the definition of the Orlicz norm, its value is uniquely determined by the values of the function M on the interval $[0, M^{-1}(1)]$. Hence, it is no restriction to extend M linearly. If $p = 2$, this immediately yields the desired embedding of Orlicz spaces into L_1 (see Corollary 6.1). In fact, we will prove the case $p = \infty$ first, which will then imply the result for arbitrary ℓ_p-norms.

2. Preliminaries and notation. A convex function $M : [0,\infty) \to [0,\infty)$ where $M(0) = 0$ and $M(t) > 0$ for $t > 0$ is called an *Orlicz function*. The n-dimensional *Orlicz space* ℓ_M^n is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_M = \inf\Big\{\rho > 0 : \sum_{i=1}^n M(|x_i|/\rho) \leq 1\Big\}. \tag{2.1}$$

In case $M(t) = t^p$, $1 \leq p < \infty$, we just have $\ell_M^n = \ell_p^n$, i.e., $\|\cdot\|_M = \|\cdot\|_p$. Given Orlicz functions $M_1, \ldots, M_n$, we define the corresponding *Musielak–Orlicz function* as $\mathbb{M} = (M_1, \ldots, M_n)$, and the n-dimensional *Musielak–*

Orlicz space $\ell_{\mathbb{M}}^n$ is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_{\mathbb{M}} = \inf\Big\{\rho > 0 : \sum_{i=1}^{n} M_i(|x_i|/\rho) \le 1\Big\}.$$

If $M_i = M$ for all $i = 1, \dots, n$, then $\ell_{\mathbb{M}}^n = \ell_M^n$. We say that two Orlicz functions M and N are *equivalent* if there are positive constants a and b such that for all $t \ge 0$,

$$a^{-1} M(b^{-1} t) \le N(t) \le a M(bt).$$

If two Orlicz functions are equivalent, so are their norms. An Orlicz function is said to be *p-concave* for some $1 \le p < \infty$ if $t \mapsto M(t^{1/p})$ is a concave function. We say that an Orlicz function M is *normalized* if

$$\int_0^\infty x \, dM'(x) = 1.$$

Note also that, if two Orlicz functions are equivalent in a neighborhood of zero, then the corresponding sequence spaces already coincide [14, Proposition 4.a.5]. For a detailed and thorough introduction to the theory of Orlicz spaces we refer the reader to [11], [21] or [14, 15] and to [16] in the case of Musielak–Orlicz spaces.

Let X and Y be isomorphic Banach spaces. We say that they are *C-isomorphic* if there is an isomorphism $T : X \to Y$ with $\|T\| \, \|T^{-1}\| \le C$. We define the *Banach–Mazur distance* of X and Y by

$$d(X, Y) = \inf\{\|T\| \, \|T^{-1}\| : T \in L(X, Y) \text{ isomorphism}\}.$$

Let $(X_n)_n$ be a sequence of n-dimensional normed spaces and let Z also be a normed space. If there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ there exists a normed space $Y_n \subseteq Z$ with $\dim(Y_n) = n$ and $d(X_n, Y_n) \le C$, then we say $(X_n)_n$ *embeds uniformly* into Z. The beautiful monograph [27] gives a detailed introduction to the concept of Banach–Mazur distances.

We will use the notation $A \sim B$ to indicate the existence of two positive absolute constants c_1, c_2 such that $c_1 A \le B \le c_2 A$. Similarly, we define the symbol $\lesssim$. We write $\sim_p$, with some positive constant p, to indicate that the constants c_1 and c_2 depend on p. The symbols $c_1, c_2, c, C, \dots$ will always denote positive absolute constants whose value may change from line to line.

By L_1 we denote the L_1 space on the unit interval $[0, 1]$ with Lebesgue measure.

We write $f \in \mathcal{C}^k$, for some $k \in \mathbb{N}$, whenever the function f is k times continuously differentiable; and $\mathcal{C}^k(a, b)$ means $\mathcal{C}^k((a, b))$.

The following theorem was obtained in [10] and provides a formula for the Orlicz function M provided that we know the distribution of X:

THEOREM 2.1 ([10, Lemma 5.2]). *Let $X_1, \ldots, X_n$ be iid integrable random variables. For all $s \geq 0$ define*

$$M(s) = \int_0^s \int_{1/t \leq |X_1|} |X_1| \, d\mathbb{P} \, dt.$$

Then, for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$,

$$c_1 \|x\|_M \leq \mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_M,$$

where c_1, c_2 are absolute constants independent of the distribution of X_1.

Obviously, the function

$$M(s) = \int_0^s \int_{1/t \leq |X_1|} |X_1| \, d\mathbb{P} \, dt \tag{2.2}$$

is non-negative and convex, since $\int_{1/t \leq |X|} |X| \, d\mathbb{P}$ is increasing in t. Furthermore, M is continuous, differentiable and $M(0) = M'(0) = 0$.

Note that, in fact, Theorem 2.1 is true for Musielak–Orlicz spaces when we do not assume the random variables to be identically distributed:

THEOREM 2.2. *Let $X_1, \ldots, X_n$ be independent integrable random variables. For all $s \geq 0$ and all $j = 1, \ldots, n$ define*

$$M_j(s) = \int_0^s \int_{1/t \leq |X_j|} |X_j| \, d\mathbb{P} \, dt.$$

Then, for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$,

$$c_1 \|x\|_{\mathbb{M}} \leq \mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_{\mathbb{M}},$$

where c_1, c_2 are absolute constants and $\mathbb{M} = (M_1, \ldots, M_n)$.

A proof in the case of averages over permutations can be found in [17] and can be generalized to our setting by a straightforward adaption of the proof of Theorem 2.1.

REMARK 2.3. Because of Theorem 2.2, all results presented in this paper hold in the more general setting of Musielak–Orlicz spaces, but for notational convenience we state them only for Orlicz spaces.

REMARK 2.4. If M is an Orlicz function such that $M \in \mathcal{C}^3$, then for $t \mapsto tM'(t) - M(t)$, being 2-concave is equivalent to $M''' \leq 0$. Therefore, and for the sake of convenience, we will later assume $M''' \leq 0$, but might still talk about the 2-concavity of $t \mapsto tM'(t) - M(t)$ at the same time.

We will also need a result from [19] about the generating distribution of ℓ_p-norms. We recall that the density of a $\log \gamma_{1,p}$ distributed random variable

ξ with parameter $p > 0$ is given by

$$f_\xi(x) = px^{-p-1}\mathbb{1}_{[1,\infty)}(x).$$

Note also that for all $x > 0$,

$$\mathbb{P}(\xi \geq x) = \min(1, x^{-p}).$$

THEOREM 2.5 ([19, Theorem 3.1]). *Let $p > 1$ and $\xi_1, \ldots, \xi_n$ be iid copies of a $\log\gamma_{1,p}$ distributed random variable ξ. Then, for all $x \in \mathbb{R}^n$,*

$$c_1\|x\|_p \leq \mathbb{E}\max_{1\leq i\leq n}|x_i\xi_i| \leq \frac{c_2}{(p-1)^{1/p}}\|x\|_p,$$

where c_1, c_2 are positive absolute constants.

Recall the following well-known theorem about the existence of independent random variables corresponding to given distributions:

THEOREM 2.6 ([4, Theorem 20.4]). *Let $(\mu_j)_j$ be a finite or infinite sequence of probability measures on the real line. Then there exists an independent sequence of random variables $(\xi_j)_j$ defined on the probability space $([0,1], \mathfrak{B}_\mathbb{R}, \lambda)$, with Borel σ-algebra $\mathfrak{B}_\mathbb{R}$ and Lebesgue measure λ, so that the distribution of ξ_j is μ_j.*

3. A simple representation result. In this section we prove a result of the same spirit as Theorem 2.1, where we replace the ℓ_∞-norm by some ℓ_p-norm for $1 < p < \infty$. This is a special case of Theorem 1 in [8] with $N(t) = t^p$. There it seems unclear how to determine the "precise" form of the Orlicz function that appears. Of course, this is somehow unsatisfactory, and therefore we provide a result that produces a "simple" representation of this Orlicz function. Observe also that the following result, which is a consequence of Theorems 2.1 and 2.5, corresponds to the discrete results recently obtained in [20].

THEOREM 3.1. *Let $1 < p < \infty$, and $X_1, \ldots, X_n$ be iid integrable random variables. For all $s \geq 0$ define*

$$M(s) = \frac{p}{p-1}\int_0^s \Big(\int_{|X_1|\leq 1/t} t^{p-1}|X_1|^p\, d\mathbb{P} + \int_{|X_1|>1/t} |X_1|\, d\mathbb{P}\Big)\, dt.$$

Then, for all $x \in \mathbb{R}^n$,

$$c_1(p-1)^{1/p}\|x\|_M \leq \mathbb{E}\|(x_iX_i)_{i=1}^n\|_p \leq c_2\|x\|_M,$$

where c_1, c_2 are positive absolute constants.

Proof. Let $X_1, \ldots, X_n$ be defined on $(\Omega_1, \mathbb{P}_1)$ and let $\xi_1, \ldots, \xi_n$ be independent copies of a $\log\gamma_{1,p}$ distributed random variable ξ, say on $(\Omega_2, \mathbb{P}_2)$. Then, by Theorem 2.5,

$$\mathbb{E}_{\Omega_1}\|(x_iX_i)_{i=1}^n\|_p \lesssim \mathbb{E}_{\Omega_1}\mathbb{E}_{\Omega_2}\max_{1\le i\le n}|x_iX_i\xi_i| \lesssim (p-1)^{-1/p}\mathbb{E}_{\Omega_1}\|(x_iX_i)_{i=1}^n\|_p$$

for all $x \in \mathbb{R}^n$. On the other hand, by Theorem 2.1,

$$\mathbb{E}_{\Omega_1}\mathbb{E}_{\Omega_2}\max_{1\le i\le n}|x_iX_i\xi_i| \sim \|x\|_M$$

for all $x \in \mathbb{R}^n$, where

$$M(s) = \int_0^s \int_{1/t\le|X_1\xi|} |X_1\xi|\,d\mathbb{P}\,dt.$$

For $t > 0$ and $\omega_1 \in \Omega_1$ define

$$I_{\omega_1} := \{\omega_2 \in \Omega_2 : t|\xi(\omega_2)X_1(\omega_1)| \ge 1\}.$$

Now, we observe that

$$\begin{aligned} M(s) &= \int_0^s \int_{\Omega_1} \int_{I_{\omega_1}} |X_1(\omega_1)\xi(\omega_2)|\,d\mathbb{P}_2(\omega_2)\,d\mathbb{P}_1(\omega_1)\,dt \\ &= \int_0^s \int_{\Omega_1} |X_1(\omega_1)| \int_{I_{\omega_1}} |\xi(\omega_2)|\,d\mathbb{P}_2(\omega_2)\,d\mathbb{P}_1(\omega_1)\,dt. \end{aligned}$$

Let us take a closer look at the inner integral. Fix $t > 0$ and $\omega_1 \in \Omega_1$ and recall that the density of ξ is

$$f_\xi(x) = px^{-p-1}\mathbb{1}_{[1,\infty)}(x).$$

Therefore, if $t|X_1(\omega_1)| \le 1$,

$$\int_{I_{\omega_1}} |\xi(\omega_2)|\,d\mathbb{P}_2(\omega_2) = p \int_{\{z:\, zt|X_1(\omega_1)|\ge 1\}} z^{-p}\,dz = \frac{p}{p-1}(t|X_1|)^{p-1}.$$

Now assume that $t|X_1(\omega_1)| \ge 1$. Then we get

$$\int_{I_{\omega_1}} |\xi(\omega_2)|\,d\mathbb{P}_2(\omega_2) = \mathbb{E}|\xi| = \frac{p}{p-1}.$$

Hence, by splitting the integral over Ω_1, for fixed t we have

$$\begin{aligned} &\int_{\Omega_1}\int_{I_{\omega_1}} |X_1(\omega_1)\xi(\omega_2)|\,d\mathbb{P}_2(\omega_2)\,d\mathbb{P}_1(\omega_1) \\ &\qquad = \frac{p}{p-1}\int_{|X_1|\le 1/t} t^{p-1}|X_1|^p\,d\mathbb{P}_1(\omega_1) + \frac{p}{p-1}\int_{|X_1|>1/t} |X_1|\,d\mathbb{P}_1(\omega_1). \end{aligned}$$

This implies the result. ∎

Note that by Fubini's theorem,

$$\begin{aligned}\int_0^s\int_0^{1/t} t^{p-1}|x|^p\,d\mathbb{P}_{X_1}(x)\,dt &= \int_0^\infty |x|^p \int_0^{s\wedge|x|^{-1}} t^{p-1}\,dt\,d\mathbb{P}_{X_1}(x)\\ &= \frac{1}{p}\Big(s^p\int_0^{1/s} x^p\,d\mathbb{P}_X(x)+\mathbb{P}(|X_1|\ge s^{-1})\Big)\\ &\le \frac{1}{p}\big(\mathbb{P}(|X|\le s^{-1})+\mathbb{P}(|X|\ge s^{-1})\big)=\frac{1}{p}.\end{aligned}$$

Hence, the limit case in Theorem 3.1 for $p\to\infty$ coincides with Theorem 2.1.

Observe also that Theorem 3.1 provides a natural candidate for the probability density that appears in Theorem 1.1:

If the random variables $|X_1|,\dots,|X_n|$ have a density f_X, then

$$M''(s)=ps^{p-2}\int_0^{s^{-1}} x^p f_X(x)\,dx,$$

that is,

$$\int_0^{s^{-1}} x^p f_X(x)\,dx=\frac{1}{p}s^{2-p}M''(s).$$

Therefore, differentiating once again,

$$f_X(s^{-1})=\Big(1-\frac{2}{p}\Big)s^3M''(s)-\frac{1}{p}s^4M'''(s).$$

In the following section we will prove Theorem 1.1 in the case $p=\infty$. We then reduce the case of general p to the case $p=\infty$ in Section 5.

4. The case of the ℓ_∞-norm. To obtain the case of ℓ_p-norms it is enough to settle the question for the ℓ_∞-norm. We will give a short explanation of that fact:

Assume that N is an arbitrary Orlicz function and we know how to choose a distribution (depending on N) so that, if $\xi_1,\dots,\xi_n$ are independent random variables distributed according to that law, then, for all $x=(x_i)_{i=1}^n\in\mathbb{R}^n$,

$$\mathbb{E}\max_{1\le i\le n}|x_i\xi_i|\sim\|x\|_N.$$

Now, let M be the normalized Orlicz function given in Theorem 1.1. We want to find a distribution and independent random variables $X_1,\dots,X_n$ defined on a measure spaces $(\Omega_1,\mathbb{P}_1)$ distributed according to this such that

$$\mathbb{E}_{\Omega_1}\|(x_iX_i)_{i=1}^n\|_p\sim_p\|x\|_M. \tag{4.1}$$

Of course, we can find a distribution and accordingly distributed independent random variables $Z_1, \ldots, Z_n$ so that

$$\mathbb{E} \max_{1 \le i \le n} |x_i Z_i| \sim \|x\|_M,$$

since we can just take $N = M$. On the other hand, observe that

$$\mathbb{E}_{\Omega_1} \|(x_i X_i)_{i=1}^n\|_p \sim_p \mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{1 \le i \le n} |x_i X_i Y_i|,$$

where the distribution of the independent random variables $Y_1, \ldots, Y_n$, say on $(\Omega_2, \mathbb{P}_2)$, is obtained by choosing $N(t) = t^p$. So, for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$,

$$\mathbb{E} \max_{1 \le i \le n} |x_i Z_i| \sim_p \|x\|_M \sim \mathbb{E}_{\Omega_1} \mathbb{E}_{\Omega_2} \max_{1 \le i \le n} |x_i X_i Y_i|.$$

Therefore, in order to get (4.1), we just have to choose the distribution of $X_1, \ldots, X_n$ so that $X_1 Y_1 \overset{\mathcal{D}}{=} Z_1$. Of course, here the distribution of Z and Y is known.

Before we continue, we observe that the transformation formula for integrals yields the following substitution rule for Stieltjes integrals:

$$\int_a^b f \circ u \, d(F \circ u) = \int_{u(a)}^{u(b)} f \, dF, \tag{4.2}$$

where f is an arbitrary measurable function, F is a non-decreasing function and u is monotone on the interval $[a, b]$.

The following result is the converse to Theorem 2.1:

Proposition 4.1. *Let M be a normalized Orlicz function with $M'(0) = 0$. Let $X_1, \ldots, X_n$ be independent copies of a random variable X with distribution*

$$\mathbb{P}(X \le t) = \int_{[1/t, \infty)} s \, dM'(s), \quad t > 0. \tag{4.3}$$

Then, for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$,

$$c_1 \|x\|_M \le \mathbb{E} \max_{1 \le i \le n} |x_i X_i| \le c_2 \|x\|_M,$$

where c_1, c_2 are constants independent of the Orlicz function M.

Proof. We first observe that for an arbitrary random variable X which is ≥ 0 a.s., by (4.2) we have

$$F_X(t) := \mathbb{P}(X \le t) = \int_{(0,t]} dF_X(s) = -\int_{[1/t, \infty)} d(F_X \circ u)(s),$$

where $u(s) = 1/s$. If the distribution of X is given by (4.3), we obtain

$$d(F_X \circ u)(s) = -s \, dM'(s).$$

Now we obtain, again by (4.2) and this identity,

$$\int_0^s \int_{[1/t,\infty)} x\,dF_X(x)\,dt = -\int_0^s \int_{(0,t]} \frac{1}{x}\,d(F_X \circ u)(x)\,dt$$
$$= \int_0^s \int_{(0,t]} dM'(x)\,dt = M(s).$$

The assertion is now a consequence of Theorem 2.1. ∎

Remark 4.2. The assumption that M is normalized, i.e., $\int_0^\infty x\,dM'(x) = 1$, ensures that the constants do not depend on M. Note also that, as an immediate consequence of Proposition 4.1, by the integration by parts rule for Stieltjes integrals we obtain

$$\mathbb{P}(X > t) = \int_0^{1/t} s\,dM'(s) = \frac{1}{t}M'\Big(\frac{1}{t}\Big) - M\Big(\frac{1}{t}\Big) \tag{4.4}$$

for any $t > 0$. If M is "sufficiently smooth", we deduce that the density f_X of X is given by

$$f_X(t) = t^{-3}M''(t^{-1}).$$

Remark 4.3. To generate an ℓ_p-norm in Proposition 4.1, i.e., to consider the case $M(t) = t^p$, one needs to pass to an equivalent Orlicz function so that the normalization condition is satisfied. The function $\widetilde{M}$ with $\widetilde{M}(t) = t^p$ on $[0,(p-1)^{-1/p}]$ which is then linearly extended does the trick.

5. The case of ℓ_p-norms. We will now prove the result which will then imply the main result, Theorem 1.1. Of course, in the proposition we could also assume $M \in \mathcal{C}^3$, but $M \in \mathcal{C}^2$ with M'' absolutely continuous on each compact subinterval of $(0,\infty)$ is sufficient.

Proposition 5.1. *Let $M \in \mathcal{C}^2(0,\infty)$ be a normalized Orlicz function and M'' be absolutely continuous on each compact subinterval of $(0,\infty)$. Assume that $M'(0) = 0 = M''(T)$ for $T = M^{-1}(1)$ and that $M|_{[T,\infty)}$ is linear. Let $1 < p < \infty$ and X, Y be two independent random variables distributed according to the laws*

$$\mathbb{P}(Y \ge y) = \min(1, y^{-p}) \quad \textit{and}$$
$$\mathbb{P}(X \ge x) = -M\Big(\frac{1}{x}\Big) + \frac{1}{x}M'\Big(\frac{1}{x}\Big) - \frac{1}{px^2}M''\Big(\frac{1}{x}\Big).$$

Then the tail distribution function of XY is

$$\mathbb{P}(XY \ge z) = \frac{1}{z}M'\Big(\frac{1}{z}\Big) - M\Big(\frac{1}{z}\Big), \quad z > 0. \tag{5.1}$$

Proof. First note that the density function of X is given by

$$f_X(x) = \left(1 - \frac{2}{p}\right)\frac{1}{x^3}M''\left(\frac{1}{x}\right) - \frac{1}{px^4}M'''\left(\frac{1}{x}\right), \tag{5.2}$$

Inserting the expression for $\mathbb{P}(Y \geq y)$, we obtain

$$\begin{aligned} \mathbb{P}(XY \geq z) &= \int \mathbb{1}_{\{XY\geq z\}}\, d\mathbb{P} = \int_0^\infty \mathbb{P}(Y \geq z/x) f_X(x)\, dx \\ &= \int_0^\infty \min(1, x^p/z^p) f_X(x)\, dx \\ &= \mathbb{P}(X \geq z) + z^{-p}\int_0^z x^p f_X(x)\, dx. \end{aligned} \tag{5.3}$$

Observe that, under the above assumptions and for $z \leq T^{-1}$, we have $\mathbb{P}(X \geq z) = 1 = z^{-1}M'(z^{-1}) - M(z^{-1})$ and $f_X(z) = 0$, since $\int_0^\infty x\, dM'(x) = TM'(T) - M(T) = 1$. This yields (5.1) for $z \leq 1/T$. Thus we now assume $z > 1/T$ and continue with calculating the integral $\int_0^z x^p f_X(x)\, dx$. We substitute $u = 1/x$ and obtain

$$\begin{aligned} \int_0^z x^p f_X(x)\, dx &= \int_{z^{-1}}^\infty u^{-p-2} f_X(u^{-1})\, du \\ &= \int_{z^{-1}}^T \left(\left(1 - \frac{2}{p}\right)u^{1-p}M''(u) - \frac{u^{2-p}}{p}M'''(u)\right) du. \end{aligned}$$

Partial integration further yields

$$\int_0^z x^p f_X(x)\, dx = -\frac{u^{2-p}}{p}M''(u)\bigg|_{z^{-1}}^T = \frac{1}{p}z^{p-2}M''(z^{-1}),$$

since $M''(T) = 0$. Combining equation (5.3) with this result and the expression for the distribution of X, we obtain (5.1) for $z > 1/T$. ∎

Now we can finally prove our main theorem:

Proof of Theorem 1.1. Let M be the given Orlicz function and $(X_i)_{i=1}^n$ the given random variables on a measure space $(\Omega_1, \mathbb{P}_1)$. First note that by Proposition 4.1 and Remark 4.2 we get

$$\|x\|_M \sim \mathbb{E} \max_{1\leq i\leq n} |x_i Z_i|, \tag{5.4}$$

where $\mathbb{P}(Z \geq z) = z^{-1}M'(z^{-1}) - M(z^{-1})$. Secondly, by Theorem 2.5,

(5.5)

$$\mathbb{E}_{\Omega_1}\|(x_i X_i)_{i=1}^n\|_p \lesssim \mathbb{E}_{\Omega_1}\mathbb{E}_{\Omega_2} \max_{1\leq i\leq n} |x_i X_i Y_i| \lesssim (p-1)^{-1/p}\mathbb{E}_{\Omega_1}\|(x_i X_i)_{i=1}^n\|_p$$

where the random variables $(Y_i)_{i=1}^n$, defined on $(\Omega_2, \mathbb{P}_2)$, are independent and $\log \gamma_{1,p}$ distributed. Since, by Proposition 5.1, $X_1 Y_1 \overset{\mathcal{D}}{=} Z_1$, we combine (5.4) and (5.5) to obtain the assertion of the theorem. ■

In case $p = 2$, we obtain the following corollary:

COROLLARY 5.2. *Let $M \in \mathcal{C}^3(0,\infty)$ be a normalized Orlicz function with $M'(0) = 0$ and $M'''(x) \le 0$ for all $x \ge 0$ and assume that $M''(M^{-1}(1)) = 0$. Then*

$$f_X(x) = -\frac{1}{2x^4} M'''\left(\frac{1}{x}\right) \tag{5.6}$$

is a probability density and for all $x \in \mathbb{R}^n$,

$$c_1 \|x\|_M \le \mathbb{E}\|(x_i X_i)_{i=1}^n\|_2 \le c_2 \|x\|_M,$$

where c_1, c_2 are positive absolute constants and $X_1, \dots, X_n$ are iid with density f_X.

Again, the normalization condition $\int_0^\infty y\, dM'(y) = 1$ ensures that constants do not depend on M and, in fact, is of the same form as the normalization condition in Theorem 2 from [25]. Note also that in the proof of Theorem 5.1 and its corollaries we need that $M''(T) = 0$ for $T = M^{-1}(1)$. This, indeed, is no restriction, since Lemma 8.2 in Section 8 shows that for any 2-concave Orlicz function we can assume that $M''(T) = 0$, otherwise we pass to an equivalent Orlicz function which has this property. Recall also that every Orlicz function which satisfies $M''' \le 0$ is already 2-concave. The authors do not know whether for an Orlicz function M, being 2-concave is equivalent (up to equivalent Orlicz functions) to having non-positive third derivative.

REMARK 5.3. Note that another proof of Corollary 5.2 via a Choquet-type representation theorem in the spirit of Lemma 7 in [25] also yields the condition that the function $z \mapsto zM'(z) - M(z)$ has to be 2-concave (or equivalently $M''' \le 0$).

6. Orlicz spaces that are isomorphic to subspaces of L_1. As we will see, it is an easy consequence of Corollary 5.2 that the sequence of Orlicz spaces ℓ_M^n, $n \in \mathbb{N}$, where $t \mapsto tM'(t) - M(t)$ is 2-concave, embeds uniformly into L_1. Although we need $t \mapsto tM'(t) - M(t)$ to be a 2-concave function, which seems a bit stronger than to assume that M is 2-concave, the simplicity of the representation (5.6) of the density that we need in our embedding has a strong advantage over the representation in [25, Theorem 2], since it is much easier to handle.

We obtain the following result:

Corollary 6.1. *Let M be a normalized Orlicz function with $M'(0)=0$ and $M'''\le 0$. Then there exists a positive absolute constant C (independent of M) such that for all $n\in\mathbb{N}$ there is a subspace Y_n of L_1 with* $\dim(Y_n)=n$ *and*

$$d(\ell_M^n, Y_n)\le C,$$

i.e., $(\ell_M^n)_n$ embeds uniformly into L_1.

Proof. The proof is a simple consequence of Corollary 5.2, Khinchin's inequality and Theorem 2.6. Given $n\in\mathbb{N}$, we let $\mu_1=\cdots=\mu_n$ be the distribution of Rademacher functions, that is,

$$\mu_i(\{1\})=\mu_i(\{-1\})=1/2,\quad 1\le i\le n.$$

Additionally, we let $\mu_{n+1}=\cdots=\mu_{2n}$ be the distribution of X_i given in Corollary 5.2. Then we apply Theorem 2.6 to the finite sequence $(\mu_i)_{i=1}^{2n}$ of probability measures to get a sequence of independent random variables $r_1,\dots,r_n,X_1,\dots,X_n$ defined on the unit interval $[0,1]$ such that the distribution of r_i is μ_i and the distribution of X_i is μ_{n+i} for all $1\le i\le n$. Then the asserted isomorphism is given by

$$\Psi_n:\ell_M^n\to L_1[0,1],\quad a\mapsto\sum_{i=1}^n a_i r_i(\cdot)X_i(\cdot).$$

Thus, applying Khinchin's inequality, for any $a=(a_i)_{i=1}^n\in\mathbb{R}^n$,

$$\begin{aligned}
&\|\Psi_n(a)\|_{L_1}\\
&=\int_0^1\Big|\sum_{i=1}^n a_i r_i(t)X_i(t)\Big|\,dt\\
&=\int_{\mathbb{R}^n}\int_{\{-1,1\}^n}\Big|\sum_{i=1}^n a_i\varepsilon_i x_i\Big|\,d(\mu_1\otimes\cdots\otimes\mu_n)(\varepsilon)\,d(\mu_{n+1}\otimes\cdots\otimes\mu_{2n})(x)\\
&\sim\int_{\mathbb{R}^n}\Big(\sum_{i=1}^n|a_i x_i|^2\Big)^{1/2}d(\mu_{n+1}\otimes\cdots\otimes\mu_{2n})(x)\\
&=\int_{[0,1]}\Big(\sum_{i=1}^n|a_i X_i(t)|^2\Big)^{1/2}dt\sim\|a\|_M,
\end{aligned}$$

where we used Corollary 5.2 in the last step. ■

7. The general result. Following the ideas described in Section 4, we now generalize our results to find an inequality of the form

$$\frac{1}{C}\|x\|_M\le\mathbb{E}\|(x_iX_i)_{i=1}^n\|_N\le C\|x\|_M$$

for a general Orlicz function N. For each normalized Orlicz function L, we write

$$\overline{F}_L(t) = \int_0^{1/t} s\,dL'(s) = \frac{1}{t}L'\Big(\frac{1}{t}\Big) - L\Big(\frac{1}{t}\Big)$$

and call this function the *tail distribution function* associated to L, motivated by Proposition 4.1 and equation (4.4).

PROPOSITION 7.1. *Let M, N be normalized Orlicz functions with $M'(0) = N'(0) = 0$.*

(i) *If there exists a probability measure μ on $(0,\infty)$ such that*

$$\overline{F}_M(t) = \int_{(0,\infty)} \overline{F}_N(t/x)\,d\mu(x), \tag{7.1}$$

then, for all $x = (x_i)_{i=1}^n \in \mathbb{R}^n$,

$$c_1\|x\|_M \le \mathbb{E}\|(x_iX_i)_{i=1}^n\|_N \le c_2\|x\|_M,$$

where c_1, c_2 are positive absolute constants and $X_1,\dots,X_n$ are iid random variables with distribution μ.

(ii) *If there exist iid random variables $X_1,\dots,X_n$ with distribution μ on $(0,\infty)$ such that*

$$c_1\|x\|_M \le \mathbb{E}\|(x_iX_i)_{i=1}^n\|_N \le c_2\|x\|_M,$$

where c_1, c_2 are positive absolute constants, then there exists an Orlicz function $\widetilde{M}$ equivalent to M such that

$$\overline{F}_{\widetilde{M}}(t) = \int_{(0,\infty)} \overline{F}_N(t/x)\,d\mu(x).$$

Proof. (i) Note that condition (7.1) guarantees that we can follow the line of argument in the proof of Theorem 1.1. Indeed, we choose independent sequences $(Z_1,\dots,Z_n)$ of iid random variables defined on $(\Omega_1,\mathbb{P}_1)$ and $(Y_1,\dots,Y_n)$ defined on $(\Omega_2,\mathbb{P}_2)$ with tail distribution functions $\overline{F}_M$ and $\overline{F}_N$, respectively. By Proposition 4.1 we have

$$\|x\|_M \sim \mathbb{E}_{\Omega_1} \max_{1\le i\le n} |x_iZ_i| \quad \text{and} \quad \|x\|_N \sim \mathbb{E}_{\Omega_2} \max_{1\le i\le n} |x_iY_i|$$

for all $(x_i)_{i=1}^n \in \mathbb{R}^n$. By (7.1), $X_1Y_1 \overset{\mathcal{D}}{=} Z_1$, since for all $t>0$,

$$\begin{aligned}\mathbb{P}(Z_1 > t) = \overline{F}_M(t) &= \int_{(0,\infty)} \overline{F}_N(t/x)\,d\mu(x) \\ &= \int_{(0,\infty)} \mathbb{P}(xY_1 > t)\,d\mu(x) = \mathbb{P}(X_1Y_1 > t).\end{aligned} \tag{7.2}$$

Therefore,

$$\begin{aligned}\|x\|_M &\sim \mathbb{E}_{\Omega_1} \max_{1\le i\le n} |x_i Z_i| = \mathbb{E}_{\Omega}\mathbb{E}_{\Omega_2} \max_{1\le i\le n} |x_i X_i Y_i| \\ &= \int_{\Omega} \mathbb{E}_{\Omega_2} \max_{1\le i\le n} |x_i X_i(\omega) Y_i|\, d\mathbb{P}(\omega) \\ &\sim \int_{\Omega} \|(x_i X_i(\omega))_{i=1}^n\|_N \, d\mathbb{P}(\omega) = \mathbb{E}_{\Omega}\|(x_i X_i)_{i=1}^n\|_N.\end{aligned}$$

(ii) Assume that

$$\mathbb{E}\|(x_i X_i)_{i=1}^n\|_N \sim \|x\|_M$$

for iid random variables $X_1, \ldots, X_n$ with distribution μ. Define the tail distribution function $\overline{F}$ by

$$\overline{F}(t) = \int_{(0,\infty)} \overline{F}_N(t/x)\, d\mu(x),$$

and choose a sequence of iid random variables $(Z_1, \ldots, Z_n)$ defined on $(\Omega_1, \mathbb{P}_1)$ with tail distribution function $\overline{F}$, and a sequence $(Y_1, \ldots, Y_n)$ independent of $(X_1, \ldots, X_n)$ defined on $(\Omega_2, \mathbb{P}_2)$ with tail distribution function $\overline{F}_N$. By construction, Z_i has the same distribution as $X_i Y_i$, $i = 1, \ldots, n$. Now define an Orlicz function $\widetilde{M}$ by

$$\widetilde{M}(s) = \int_0^s \int_{1/t\le |Z_1|} |Z_1|\, d\mathbb{P}_1\, dt.$$

By Theorem 2.1, $\|x\|_{\widetilde{M}} \sim \mathbb{E}_{\Omega_1} \max_{1\le i\le n} |x_i Z_i|$ and, therefore, we obtain

$$\begin{aligned}\|x\|_M \sim \mathbb{E}_{\Omega}\|(x_i X_i)_{i=1}^n\|_N &= \int_{\Omega} \|(x_i X_i(\omega))_{i=1}^n\|_N \, d\mathbb{P}(\omega) \\ &\sim \int_{\Omega} \mathbb{E}_{\Omega_2} \max_{1\le i\le n} |x_i X_i(\omega) Y_i|\, d\mathbb{P}(\omega) = \mathbb{E}_{\Omega}\mathbb{E}_{\Omega_2} \max_{1\le i\le n} |x_i X_i Y_i| \\ &= \mathbb{E}_{\Omega_1} \max_{1\le i\le n} |x_i Z_i| \sim \|x\|_{\widetilde{M}}.\end{aligned}$$

Thus, M and $\widetilde{M}$ are equivalent [14, Proposition 4.a.5]. ∎

Condition (7.1) seems hard to check for general Orlicz functions M and N. However, in the special case of $N(t) = t^2$ on $[0, 1]$, extended linearly to the right of 1 as a C^1 function, condition (7.1) is equivalent to the positivity of the function f_X in (5.6). Indeed,

$$\overline{F}_M(t) = \int_{(0,\infty)} \overline{F}_N(t/x)\, d\mu(x) = \int_{(0,\infty)} \min(1, x^2/t^2)\, d\mu(x).$$

Note that

$$\int_{(0,\infty)} \min(1, x^2 z^2)\, d\mu(x) = \overline{F}_M(1/z) = zM'(z) - M(z)$$

is obviously a 2-concave function in z as an average over such functions, in correspondence with the discussion before. On the other hand, Corollary 5.2 can be restated in the following form that shows that the converse is also true: if $z \mapsto zM'(z) - M(z)$ is 2-concave under the conditions stated in Corollary 5.2, then the tail distribution function $\overline{F}_M$ has a representation of the form (7.1) and the distribution μ is explicitly given by the density

$$f(x) = -\frac{1}{2x^4} M'''\left(\frac{1}{x}\right).$$

8. Appendix. We provide some approximation results for Orlicz functions that we need in this paper and which might be interesting in further applications.

Lemma 8.1. *Let $M \in \mathcal{C}^2(0,\infty)$ be an Orlicz function with $M'(0) = 0$ and such that M'' is decreasing. Then M is 2-concave.*

Proof. Recall that M is 2-concave if and only if $xM''(x) \le M'(x)$. For all $\varepsilon \in (0,x)$, there exists $\xi_\varepsilon \in (\varepsilon, x)$ such that

$$M'(x) = M'(\varepsilon) + (x-\varepsilon)M''(\xi_\varepsilon).$$

Since M'' is decreasing, we get

$$M'(x) \ge M'(\varepsilon) + (x-\varepsilon)M''(x),$$

and so, for $\varepsilon \to 0$, $M'(x) \ge xM''(x)$, which means that M is 2-concave. ∎

Lemma 8.2. *Let $M \in \mathcal{C}^2(0, M^{-1}(1))$ be an Orlicz function that is linear to the right of $T := M^{-1}(1)$. Then, for all constants $c > 1$, there exists an Orlicz function N such that*

(1) $N''(T) = 0$,
(2) $N(t) \le M(t) \le cN(t)$ *for all* $t \in [0,\infty)$.

Additionally, if M'' is decreasing, we can choose N so that N'' is decreasing.

Proof. We let $\delta \in (0,1)$ and define N as follows: We set $N(t) = M(t)$ for all $t \le T(1-\delta)$ and we extend M to $[0,T]$ so that N'' is smooth, decreasing, $N''(t) \le M''(t)$ for $t \in [0,T)$ and $N''(T) = 0$. For $t > T$, we define N linearly with the same slope as M.

We have to show property (2). The inequality $N(t) \le M(t)$ follows from the construction for all $t \in [0,\infty)$. The second inequality is trivial for $t \le T(1-\delta)$ since $M(t) = N(t)$ for such t. Next, we explore the case

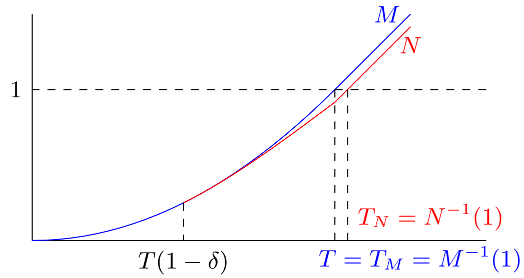

Fig. 1. Approximation of the Orlicz function M

$t \in [T(1-\delta), T]$. If we choose t in this interval, by the above definition of N,

$$\begin{aligned}0 \le M(t) - N(t) &= \int\limits_{T(1-\delta)}^{t} \int\limits_{T(1-\delta)}^{s} (M''(x) - N''(x))\,dx\,ds \\ &\le T\delta^2 \max_{x\in[T(1-\delta),T]} (M''(x) - N''(x)) \le T\delta^2 \max_{x\in[T(1-\delta),T]} M''(x).\end{aligned}$$

Now we choose δ such that $T\delta^2 \max_{x\in[T(1-\delta),T]} M''(x) \le (c-1)M(T(1-\delta))$. This is possible since $\max_{x\in[T(1-\delta),T]} M''(x)$ is an increasing function of δ and $M(T(1-\delta))$ is a decreasing function of δ. Then for $t \in [T(1-\delta), T]$ we obtain

$$\begin{aligned}M(t) &= N(t) + M(t) - N(t) \\ &\le N(t) + (c-1)M(T(1-\delta)) = N(t) + (c-1)N(T(1-\delta)) \le cN(t).\end{aligned}$$

This is property (2) for $t \in [T(1-\delta), T]$. Since, for $t \ge T$, the difference $M(t) - N(t)$ is constant by definition of N, and the two Orlicz functions M and N are both increasing, the inequality $M(t) \le cN(t)$ also holds for $t \ge T$ by the following simple calculation:

$$\begin{aligned}M(t) &= N(t) + M(t) - N(t) = N(t) + M(T) - N(T) \\ &\le N(t) + (c-1)N(T) \le cN(t). \ \blacksquare\end{aligned}$$

Figure 1 illustrates the choice of the equivalent Orlicz function in the proof of Lemma 8.2 which has the desired properties.

Remark 8.3. Let M and N be as in Lemma 8.2. In order to apply this lemma to Proposition 5.1, we have to pass once again to an equivalent Orlicz function $\widetilde{N}$, a multiple of the function N constructed in Lemma 8.2 (see Figure 1), to ensure $M^{-1}(1) = \widetilde{N}^{-1}(1)$ and hence that $\widetilde{N}$ is "smooth" up to the point $\widetilde{N}^{-1}(1)$.

Acknowledgments. The first author is partially supported by MICINN project MTM2010-16679, MICINN-FEDER project MTM2009-10418 and "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 04540/GERM/06. The third and fourth authors are supported by the Austrian Science Fund, FWF project P23987 "Projection Operators in Analysis and Geometry of Classical Banach Spaces".

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Received July 1, 2013
Revised version October 15, 2013 (7804)