Classes of measures closed under mixing and convolution. Weak stability

by

J. K. Misiewicz (Zielona Góra), K. Oleszkiewicz (Warszawa) and K. Urbanik (Wrocław)

Abstract. For a random vector $X$ with a fixed distribution $\mu$ we construct a class of distributions $\mathcal{M}(\mu) = \{\mu \circ \lambda : \lambda \in \mathcal{P}\}$, which is the class of all distributions of random vectors $X \Theta$, where $\Theta$ is independent of $X$ and has distribution $\lambda$. The problem is to characterize the distributions $\mu$ for which $\mathcal{M}(\mu)$ is closed under convolution. This is equivalent to the characterization of the random vectors $X$ such that for all random variables $\Theta_1, \Theta_2$ independent of $X, X'$ there exists a random variable $\Theta$ independent of $X$ such that

$$X \Theta_1 + X' \Theta_2 \overset{d}{=} X \Theta.$$ 

We show that for every $X$ this property is equivalent to the following condition:

$$\forall a, b \in \mathbb{R} \exists \Theta \text{ independent of } X, \quad aX + bX' \overset{d}{=} X \Theta.$$ 

This condition reminds the characterizing condition for symmetric stable random vectors, except that $\Theta$ is here a random variable, instead of a constant.

The above problem has a direct connection with the concept of generalized convolutions and with the characterization of the extreme points for the set of pseudo-isotropic distributions.

1. Introduction. Let $\mathbb{E}$ be a separable real Banach space. By $\mathcal{P}(\mathbb{E})$ we denote the set of all Borel probability measures on $\mathbb{E}$. For $\mathbb{E} = \mathbb{R}$ we will use the simplified notation $\mathcal{P}(\mathbb{R}) = \mathcal{P}$, and the set of all probability measures on $[0, \infty)$ will be denoted by $\mathcal{P}_+$. For every $a \in \mathbb{R}$ and every probability measure $\mu$, we define the rescaling operator $\mu \mapsto T_a \mu$ by the formula $(T_a \mu)(A) = \mu(A/a)$ when $a \neq 0$, and $T_0(\mu) = \delta_0$. This means that $T_a \mu$ is the distribution of the random vector $aX$ if $\mu$ is the distribution of the vector $X$. For every $\mu \in \mathcal{P}(\mathbb{E})$ and $\lambda \in \mathcal{P}$ we define a scale mixture $\mu \circ \lambda$

2000 Mathematics Subject Classification: Primary 60E05.

Key words and phrases: convolution, generalized convolution, pseudo-isotropic distributions, elliptically contoured distributions, weakly stable measures.

The research of the second named author was partially supported by the Polish KBN Grant 2 P03A 027 22.
of the measure $\mu$ with respect to the measure $\lambda$ by the formula
\[
(\mu \circ \lambda)(A) = \int_{\mathbb{R}} (T_a \mu)(A) \, \lambda(da).
\]
It is easy to see that $\mu \circ \lambda$ is the distribution of the random vector $X\Theta$ if $X$ and $\Theta$ are independent, $X$ has distribution $\mu$, and $\Theta$ has distribution $\lambda$.

We consider the set of all mixtures of the measure $\mu$, i.e.
\[
\mathcal{M}(\mu) = \{\mu \circ \lambda : \lambda \in \mathcal{P}\} = \mathcal{P} \circ \mu.
\]
When it is more convenient we will write $\mathcal{M}(\hat{\mu})$ instead of $\mathcal{M}(\mu)$. The corresponding set of characteristic functions is denoted by
\[
\Phi(\mu) = \{\varphi : \varphi = \int \hat{\mu}(\xi) \, \lambda(dt), \lambda \in \mathcal{P}, \xi \in \mathbb{E}^*\}.
\]

The problem discussed here has a very elementary formulation: characterize those probability measures $\mu$ on $\mathbb{E}$ for which the set $\mathcal{M}(\mu)$ is closed under convolution, i.e.
\[
(A) \quad \forall \nu_1, \nu_2 \in \mathcal{M}(\mu), \quad \nu_1 * \nu_2 \in \mathcal{M}(\mu).
\]
In the language of random vectors, this condition looks even simpler: Let $X$, $X'$, $\Theta_1$, $\Theta_2$ be independent, where $X$ and $X'$ have distribution $\mu$. If condition (A) holds, then there exists a random variable $\Theta$ independent of $X$ such that
\[
X\Theta_1 + X'\Theta_2 \overset{d}{=} X\Theta.
\]
In particular, under the previous assumptions,
\[
\forall a, b \in \mathbb{R} \exists \Theta = \Theta(a, b), \quad X \text{ and } \Theta \text{ independent and } aX + bX' \overset{d}{=} X\Theta.
\]
The main result of this paper states that condition (A) is equivalent to
\[
(B) \quad \forall a, b \in \mathbb{R}, \quad T_a \mu * T_b \mu \in \mathcal{M}(\mu).
\]

Example 1. The class of symmetric distributions on $\mathbb{R}$ is closed under mixing and under convolution. It is easy to see that this class can be written as $\mathcal{M}(\tau)$ for $\tau = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1}$. Checking (B) in this case is especially simple. In the language of characteristic functions we have
\[
\widehat{\tau}(at)\widehat{\tau}(bt) = \cos(at)\cos(bt) = \frac{1}{2} \cos((a + b)t) + \frac{1}{2} \cos((a - b)t)
\]
\[
= \int_{\mathbb{R}} \cos(ts) \left(\frac{1}{2} \delta_{a+b} + \frac{1}{2} \delta_{a-b}\right)(ds),
\]
which means that we can take $\frac{1}{2} \delta_{a+b} + \frac{1}{2} \delta_{a-b}$ for $\lambda$. But there are many other possibilities, since if $X$ is a symmetric random vector, and $X$ and $\Theta$ are independent, then $X\Theta \overset{d}{=} X|\Theta|$. Thus the measure $\lambda$ is not uniquely
determined and condition (B) holds for every \( \lambda_{pq} \), \( p, q \in [0, 1/2] \), where
\[
\lambda_{pq} := p\delta_{a+b} + \left( \frac{1}{2} - p \right)\delta_{-a-b} + q\delta_{a-b} + \left( \frac{1}{2} - q \right)\delta_{b-a}.
\]
It is easy to see that the set \( K(\delta_a, \delta_b) = \{ \lambda_{pq} : p, q \in [0, 1/2] \} \) is closed and convex. This property turns out to be general.

In [3, 10–13] Kucharczak, Urbanik and Vol’kovich considered a very similar problem. They studied the properties of weakly stable random variables and measures, where a random variable \( X \geq 0 \) with distribution \( \mu \) on \([0, \infty)\) is said to be weakly stable if for any \( a, b \in \mathbb{R}_+ \) there exists a nonnegative random variable \( Q \) with distribution \( \lambda \) such that
\[
(C) \quad T_a\mu \ast T_b\mu = \mu \circ \lambda.
\]
From now on we will say that a distribution \( \mu \) for which (C) holds is \( \mathbb{R}_+ \)-weakly stable, and that \( \mu \) is weakly stable when (B) is satisfied. The next example shows that these two conditions are not equivalent.

**Example 2.** Assume that a random vector \( X \) has a symmetric \( \alpha \)-stable distribution \( \mu \) with \( \alpha \in (0, 2] \). This means that for every \( a, b \in \mathbb{R} \) we have
\[
aX + bX' \overset{d}{=} cX,
\]
where \( c^\alpha = |a|^\alpha + |b|^\alpha \), so condition (B) holds for \( \lambda = \delta_c \).
It is easy to see that the opposite implication also holds, i.e. if for every \( a, b \in \mathbb{R} \) there exists a Dirac measure satisfying condition (B), then \( \mu \) is symmetric stable. This is a little different from the usual condition, where the assumption
\[
(D) \quad \forall a, b > 0 \ \exists c > 0, \quad aX + bX' \overset{d}{=} cX.
\]
is equivalent to \( X \) having a strictly stable distribution. Thus, a strictly stable distribution is \( \mathbb{R}_+ \)-weakly stable, but it may not be weakly stable. A symmetric stable distribution is both \( \mathbb{R}_+ \)-weakly stable and weakly stable.

**Example 3.** Consider the random vector \( X_{k,n} = (U_1, \ldots, U_k) \) for \( k \leq n \) which is the \( k \)-dimensional projection of \( U^n = (U_1, \ldots, U_n) \) with the uniform distribution on the unit sphere \( S_{n-1} \subset \mathbb{R}^n \). The distribution \( \mu_{k,n} \) of \( X_{k,n} \) for \( k < n \) is absolutely continuous with respect to the Lebesgue measure with density
\[
f(x_1, \ldots, x_k) = c(n, k) \left( 1 - \sum_{i=1}^k x_i^2 \right)^{(n-k)/2-1},
\]
where \( c(n, k) \) is a normalizing constant. The set \( \mathcal{M}(\mu_{n,n}) \) is well known, being the set of all rotationally invariant distributions on \( \mathbb{R}^n \). The set \( \mathcal{M}(\mu_{k,n}) \) is a convex and closed subset of \( \mathcal{M}(\mu_{k,k}) \). If \( n = k + 2 \), then \( \mu_{k,n} \) is the uniform distribution on the unit ball \( B_k \subset \mathbb{R}^k \). In particular, \( \mathcal{M}(\mu_{1,3}) \) is the set of symmetric unimodal probability measures on \( \mathbb{R} \).
In order to show that all these classes are also closed under convolution, we need to use the following characterization:

\( \mu \) is rotationally invariant on \( \mathbb{R}^k \)

\[ \Leftrightarrow \tilde{\mu}(\xi) \text{ depends only on } \|\xi\|_2 = (|\xi_1|^2 + \cdots + |\xi_k|^2)^{1/2}, \text{ i.e. } \tilde{\mu}(\xi) = \varphi(\|\xi\|_2) \]

for some function \( \varphi \)

\[ \Leftrightarrow \mu \text{ is the distribution of } U^n\Theta, \text{ where } \Theta \geq 0 \text{ is independent of } U^n. \]

Now, let \( \nu_1, \nu_2 \in \mathcal{M}(\mu_{k,n}) \). This means that there exist independent rotationally invariant random vectors \( X_1 \) and \( X_2 \) on \( \mathbb{R}^n \) such that \( \nu_1 \) and \( \nu_2 \) are the distributions of the \( k \)-dimensional projections of \( X_1 \) and \( X_2 \). For every \( a, b \in \mathbb{R} \), the random vector \( aX_1 + bX_2 \) is also rotationally invariant on \( \mathbb{R}^n \) since

\[ E \exp\{i\langle aX_1, \xi \rangle + i\langle bX_2, \xi \rangle\} = E \exp\{i\langle aX_1, \xi \rangle\}E \exp\{i\langle bX_2, \xi \rangle\} = f_1(|a||\xi||_2) f_2(|b||\xi||_2), \]

so the right hand side is a function depending only on \( |\xi||_2 \) (\( a, b \) are just some parameters here). This means that there exists a random variable \( Q = Q_{a,b} \) such that \( aX_1 + bX_2 \) \( d \sim U^nQ \). It is easy to see now that \( T_a\nu_1 + T_b\nu_2 \) is the distribution of a \( k \)-dimensional projection of \( U^nQ \), which was to be shown. It is interesting that the variable \( Q_{a,b} \) for the measure \( \mu_{k,n} \) does not depend on \( k \); in fact \( Q_{a,b} \) has the same distribution as \( \|aX_1 + bX_2\|_2 \).

### 2. Conditions (A) and (B) are equivalent

**Lemma 1.** Assume that a measure \( \mu \) has property (B). Then, for any discrete measures \( \nu_1 = \sum_i p_i \delta_{a_i} \) and \( \nu_2 = \sum_i q_i \delta_{b_i} \), the measure \( (\mu \circ \nu_1) \ast (\mu \circ \nu_2) \) belongs to \( \mathcal{M}(\mu) \).

**Proof.** Let \( \lambda_{ij} \) be such that \( T_{a_i}\mu \ast T_{b_j}\mu = \mu \circ \lambda_{ij} \). Then

\[
(\mu \circ \nu_1) \ast (\mu \circ \nu_2) = \sum_{i,j} p_iq_j T_{a_i}\mu \ast T_{b_j}\mu = \sum_{i,j} p_iq_j \mu \circ \lambda_{ij} = \mu \circ \left( \sum_{i,j} p_iq_j \lambda_{ij} \right). \]

**Lemma 2.** Let \( \mu \neq \delta_0 \) be a probability measure on a separable Banach space \( \mathcal{E} \) and let \( \mathcal{A} \subset \mathcal{P} \). If the set \( \mathcal{B} = \{ \mu \circ \lambda : \lambda \in \mathcal{A} \} \) is tight, then so is \( \mathcal{A} \).

**Proof.** Let \( \mu = \mathcal{L}(X) \) and \( \lambda = \mathcal{L}(Q\lambda) \) for \( X \) and \( Q\lambda \) independent, \( \lambda \in \mathcal{A} \). Let \( \varepsilon > 0 \). Since \( \mathcal{B} \) is tight there exists a compact set \( \mathcal{L} \subset \mathcal{E} \) such that

\[ P(Q\lambda X \in \mathcal{L}) \geq 1 - \varepsilon P(X \neq 0). \]

Put \( L_n = [-1/n, 1/n] \cdot \mathcal{L} = \{ sx : s \in [-1/n, 1/n], x \in \mathcal{L} \} \). Since \( \mathcal{L} \) is bounded we have

\[ \lim_{n \to \infty} P(X \not\in L_n) \geq P(X \neq 0). \]
Choose $n$ such that $P(X \notin L_n) \geq P(X \neq 0)/2$. Then

$$
p_{\varepsilon}P(X \neq 0) \geq P(Q \lambda X \notin L) \geq P(|Q \lambda| > n, X \notin L_n)
$$

$$
= P(|Q \lambda| > n)P(X \notin L_n) \geq P(|Q \lambda| > n)P(X \neq 0)/2,
$$

so that $P(|Q \lambda| > n) \leq 2\varepsilon$ for all $\lambda \in \mathcal{A}$. This implies tightness of $\mathcal{A}$. 

**Lemma 3.** The set $\mathcal{M}(\mu)$ is closed in the topology of weak convergence and the set of extreme points of $\mathcal{M}(\mu)$ is $\{T_a\mu : a \in \mathbb{R}\}$.

**Proof.** If $\mu = \delta_0$ then the assertion follows immediately, so we assume that $\mu \neq \delta_0$. Assume that $\mu \circ \lambda_n \Rightarrow \nu$. Then the set $\{\mu \circ \lambda_n : n \in \mathbb{N}\}$ is tight, and, by Lemma 2 the set $\{\lambda_n : n \in \mathbb{N}\}$ is also tight. Thus it contains a subsequence $\lambda_{nk}$ converging weakly to a probability measure $\lambda$ on $\mathbb{R}$. Since the function $\hat{\mu}(t)$ is bounded and continuous, we obtain

$$
\int \hat{\mu}(ts) \lambda_{nk}(ds) \to \int \hat{\mu}(ts) \lambda(ds).
$$

On the other hand, we have

$$
\int \hat{\mu}(ts) \lambda_n(ds) \to \hat{\nu}(t).
$$

This means that $\nu = \mu \circ \lambda$ and consequently $\nu \in \mathcal{M}(\mu)$.

If $a = 0$, then $T_a\mu = \delta_0$ and it is easy to check that $\delta_0$ is an extreme point in $\mathcal{M}(\mu)$. Assume that for some $a \in \mathbb{R}$, $a \neq 0$, there exist $\lambda_1, \lambda_2 \in \mathcal{P}$ and $p \in (0,1)$ such that

$$
T_a\mu = p\mu \circ \lambda_1 + (1-p)\mu \circ \lambda_2 = \mu \circ (p\lambda_1 + (1-p)\lambda_2).
$$

This means that $aX \overset{d}{=} X\Theta$ for some random variable $\Theta$ independent of $X$ with distribution $p\lambda_1 + (1-p)\lambda_2$. The result of Mazurkiewicz (see [5]) implies that $P\{\Theta = a\} = 1$ if the distribution of $X$ is not symmetric, and $P\{|\Theta| = |a|\} = 1$ otherwise. In the first situation we would have

$$
\delta_a = p\lambda_1 + (1-p)\lambda_2,
$$

so $\lambda_1 = \lambda_2 = \delta_a$ since $\delta_a$ is an extreme point in $\mathcal{P}$. If $X$ has symmetric distribution we obtain

$$
\delta_{||a||}(A) = p\lambda_1(A) + (1-p)\lambda_2(A) + p\lambda_1(-A) + (1-p)\lambda_2(-A) =: p|\lambda_1|(A) + (1-p)|\lambda_2|(A)
$$

for every Borel set $A \subset (0,\infty)$. Since $\delta_{||a||}$ is an extreme point in $\mathcal{P}_+$, we have $\delta_{||a||} = |\lambda_1| = |\lambda_2|$. Now, it is enough to notice that for a symmetric distribution $\mu$, the equality $\mu \circ \lambda = \mu \circ |\lambda|$ holds for every probability measure $\lambda$. Consequently, we obtain

$$
T_a\mu = \mu \circ |\lambda_1| = \mu \circ \lambda_1 = \mu \circ |\lambda_2| = \mu \circ \lambda_2.
$$

The above reasoning works for $\mu \in \mathcal{P}$. For $\mu \in \mathcal{P}(\mathbb{E})$ the following two situations are possible. If $\mu$ is nonsymmetric then one can choose $\xi \in \mathbb{E}^*$ such that $\xi(X)$ is nonsymmetric and use the result of Mazurkiewicz as before. If
\(\mu\) is symmetric then there exists \(\xi \in \mathbb{E}^*\) such that \(\xi(X) \neq 0\) since \(\mu \neq \delta_0\), so that \(\delta_{|\mu|} = |\lambda_1| = |\lambda_2|\), as before. The rest of the reasoning does not need any change.

Assume now that the probability measure \(\nu\) is an extreme point of \(\mathcal{M}(\mu)\). Then there exists a probability measure \(\lambda\) such that \(\nu = \mu \circ \lambda\). If \(\lambda \neq \delta_0\) for any \(a \in \mathbb{R}\) then we could divide \(\mathbb{R}\) into two Borel sets \(A\) and \(A' = \mathbb{R} \setminus A\) such that \(\lambda(A) = \alpha \in (0, 1)\). Then
\[
\mu = \alpha \mu \circ (\alpha^{-1}\lambda|_A) + (1 - \alpha) \mu \circ ((1 - \alpha)^{-1}\lambda|_{A'}),
\]
in contradiction with the assumption that \(\nu\) is extremal. ■

**Lemma 4.** Assume that for a probability measure \(\mu \neq \delta_0\) and some \(\nu_1, \nu_2 \in \mathcal{P}\) the set
\[
K_\mu(\nu_1, \nu_2) := \{\lambda : (\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda\}
\]
is not empty. Then it is convex and weakly compact.

**Proof.** Notice that
\[
\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\} = \{\mu \circ \lambda : \lambda \in K_\mu(\nu_1, \nu_2)\},
\]
and the set \(\{(\mu \circ \nu_1) * (\mu \circ \nu_2)\}\) contains only one point. Then the weak compactness of \(K_\mu(\nu_1, \nu_2)\) follows from Lemma 2. The convexity is trivial. ■

**Lemma 5.** Assume that \(\mu \neq \delta_0\) is a probability measure and \(K_\mu(\nu_{n1}, \nu_{n2}) \neq \emptyset\) for every \(n \in \mathbb{N}\), where \(\nu_{n1} \to \nu_1\) weakly, \(\nu_{n2} \to \nu_2\) weakly, and \(\nu_{n1}, \nu_{n2} \in \mathcal{P}\). Then \(K_\mu(\nu_1, \nu_2) \neq \emptyset\).

**Proof.** Let \(A = \bigcup_{n=1}^{\infty} K_\mu(\nu_{n1}, \nu_{n2})\) and
\[
B = \{\mu \circ \lambda : \lambda \in A\} = \{((\mu \circ \nu_{n1}) * (\mu \circ \nu_{n2}) : n \in \mathbb{N}\}.
\]
Since \(B\) is tight, so is \(A\) by Lemma 2. Choosing now \(\lambda_n \in K_\mu(\nu_{n1}, \nu_{n2})\) for every \(n \in \mathbb{N}\), we can find a subsequence \(\lambda_{nk}\) converging weakly to a probability measure \(\lambda\). Since
\[
(\mu \circ \nu_{nk}) * (\mu \circ \nu_{nk}) = \mu \circ \lambda_{nk},
\]
we also have
\[
(\mu \circ \nu_{nk}) * (\mu \circ \nu_{nk}) = \mu \circ \lambda,
\]
and consequently \(\lambda \in K_\mu(\nu_1, \nu_2) \neq \emptyset\). ■

**Theorem 1.** For every probability distribution \(\mu\) properties (A) and (B) are equivalent.

**Proof.** The implication (A)⇒(B) is trivial. Assume that \(\mu \neq \delta_0\) and (B) holds. This means that \(K_\mu(\delta_a, \delta_b) \neq \emptyset\) for any \(a, b \in \mathbb{R}\). It follows from Lemma 1 that \(K_\mu(\nu_1, \nu_2) \neq \emptyset\) for any discrete measures \(\nu_1, \nu_2\). Let now \(\lambda_1, \lambda_2 \in \mathcal{P}\). We can find two sequences of discrete measures \(\nu_{1n}\) and \(\nu_{2n}\) converging weakly to \(\lambda_1\) and \(\lambda_2\) respectively. Since \(K_\mu(\nu_{1n}, \nu_{2n}) \neq \emptyset\) for every \(n \in \mathbb{N}\), Lemma 5 shows that also \(K_\mu(\lambda_1, \lambda_2) \neq \emptyset\), which implies (A). ■
Proposition 1. Let \( X = (X_1, \ldots, X_n) \) be a symmetric \( \alpha \)-stable random vector, and let \( \Theta \) be a random variable independent of \( X \). Then \( Y = X\Theta \) is weakly stable iff \( |\Theta|^{\alpha} \) is \( \mathbb{R}_+ \)-weakly stable.

Proof. Notice that
\[
aX\Theta + bX'\Theta' \overset{d}{=} (|a\Theta|^{\alpha} + |b\Theta'|^{\alpha})^{1/\alpha} X,
\]
where \( X', \Theta' \) are independent copies of \( X, \Theta \) such that \( X, X', \Theta, \Theta' \) are independent. Assume that \( Y \) is weakly stable. Since \( X\Theta \overset{d}{=} X|\Theta| \) we obtain
\[
(|a\Theta|^{\alpha} + |b\Theta'|^{\alpha})^{1/\alpha} X \overset{d}{=} X \cdot |\Theta| \cdot Q
\]
for some random variable \( Q \). Without loss of generality we can assume that \( Q \geq 0 \). A symmetric stable distribution is cancellable (see [3, Prop. 1.1]), thus we obtain
\[
|a|^{\alpha}|\Theta|^{\alpha} + |b|^{\alpha}|\Theta'|^{\alpha} \overset{d}{=} |\Theta|^{\alpha}Q^{\alpha}.
\]
This implies that \( |\Theta|^{\alpha} \) is \( \mathbb{R}_+ \)-weakly stable. The converse is trivial. \( \blacksquare \)

3. Symmetrizations of mixing measures are uniquely determined. Assume that a measure \( \mu \neq \delta_0 \) on \( \mathbb{R} \) is weakly stable. We have seen before that \( K_{\mu}(\nu_1, \nu_2) \) is a nonempty convex and weakly compact set in \( \mathcal{P} \) for all \( \nu_1, \nu_2 \in \mathcal{P} \). In this section we discuss further properties of \( K_{\mu}(\nu_1, \nu_2) \).

For a weakly stable measure \( \mu \) we define
\[
\Phi(\mu) = \{ \hat{\nu} : \nu = \mu \circ \lambda, \lambda \in \mathcal{P} \},
\]
and let \( L(\mu) \) denote the complex linear space generated by \( \Phi(\mu) \). Weak stability of \( \mu \) implies that for any \( f, g \in L(\mu) \) we have \( fg, f^2 \in L(\mu) \). Since \( \mu \circ \delta_0 = \delta_0 \) the space \( L(\mu) \) contains the constants.

We denote by \( \overline{\mathbb{R}} = \mathbb{R} \cup \{ \Delta \} \) the one-point compactification of the real line, and by \( \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{ \infty \} \) the one-point compactification of the nonnegative half-line. Let \( C(Y) \) denote the space of continuous real functions on the topological space \( Y \). Then \( C(\overline{\mathbb{R}}_+) \) can be identified with the set of even (symmetric) functions from \( C(\mathbb{R}) \).

Now, for a probability measure \( \mu \), we define
\[
A(\mu) = \{ f \in L(\mu) : f = \tilde{f}, \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) \}.
\]
If \( \mu \) is weakly stable then \( A(\mu) \) is an algebra (over the reals).

Lemma 6. If a probability measure \( \mu \) on \( \mathbb{R} \) is not symmetric, then the set \( A(\mu) \) separates points of \( \overline{\mathbb{R}} \).

Proof. Let \( \gamma \) be a symmetric Cauchy distribution with Fourier transform \( \hat{\gamma}(t) = e^{-|t|} \). For every \( c \in \mathbb{R} \), we define
\[
h_c(t) = (\mu \circ (\gamma * \delta_c))^\wedge(t) \in \Phi(\mu).
\]
First we show that there exists $a \in \mathbb{R}$ such that $\Im m(h_a) \neq 0$. Assume the opposite, i.e. $\Im m(h_c) \equiv 0$ for every $c \in \mathbb{R}$. This means that

$$\Im m(h_c(t)) = \int_{-\infty}^{\infty} e^{-|tx|} \sin(ctx) \mu(dx) = 0$$

for all $c, t \in \mathbb{R}$. Substituting $u = ct$, we obtain

$$\int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux) \mu(dx) = 0$$

for $u \in \mathbb{R}$ and $c \neq 0$. This implies that

$$\lim_{c \to \infty} \int_{-\infty}^{\infty} e^{-|ux|/|c|} \sin(ux) \mu(dx) = \int_{-\infty}^{\infty} \sin(ux) \mu(dx) = 0,$$

which means that the characteristic function $\hat{\mu}$ is real, which contradicts our assumption.

Now let $a, t_0 \in \mathbb{R}$ be such that $\Im m(h_a(t_0)) \neq 0$. For every $s \neq 0$, we define

$$g_s(t) = \Im m \left( h_a \left( \frac{t \cdot t_0}{s} \right) \right).$$

It is easy to see that $g_s(t) \in A(\mu)$, and $g_s(t) = -g_s(-t)$. We can now see that for every $r \in \mathbb{R}$, $r \neq 0$, the function $g_r(t)$ separates the points $r$ and $-r$ since $g_r(r) = h_a(t_0) \neq g_r(-r)$. To finish the proof, it is enough to notice that the function

$$h_0(t) = \int_{-\infty}^{\infty} e^{-|tx|} \mu(dx)$$

separates points $t_1, t_2 \in \mathbb{R}$ if $|t_1| \neq |t_2|$, including the case $t_i = \Delta$. ■

**Lemma 7.** If a probability measure $\mu$ on $\mathbb{R}$ is symmetric and $\mu \neq \delta_0$, then $A(\mu)$ separates points of $\mathbb{R}_+$. ■

**Theorem 2.** If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is not symmetric, then for any $\nu_1, \nu_2 \in \mathcal{P}$ the set $K_\mu(\nu_1, \nu_2)$ contains only one measure.

**Proof.** Assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This means that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently, for every $\lambda \in \mathcal{P}$,

$$(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2.$$ 

Hence, for every $\lambda \in \mathcal{P}$,

$$\int_{-\infty}^{\infty} (\mu \circ \lambda)^\wedge(tx) \lambda_1(dx) = \int_{-\infty}^{\infty} (\mu \circ \lambda)^\wedge(tx) \lambda_2(dx).$$
This implies that for every $f \in A(\mu)$,

$$\int_{-\infty}^{\infty} f(x) \lambda_1(dx) = \int_{-\infty}^{\infty} f(x) \lambda_2(dx).$$

From Lemma 6 we know that the algebra $A(\mu)$ separates points of $\mathbb{R}$, so by the Stone–Weierstrass Theorem (see Theorem 4E in [4]), it is dense in $C(\mathbb{R})$ in the topology of uniform convergence. This means that (*) holds for every $f \in C(\mathbb{R})$, and consequently $\lambda_1 = \lambda_2$.

Let $\tau = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1}$. The symmetrization of a measure $\lambda \in \mathcal{P}$ is defined to be the measure $\lambda \circ \tau$. Notice that $\lambda$ is symmetric if and only if $\lambda = \lambda \circ \tau$.

**Theorem 3.** If a weakly stable measure $\mu \neq \delta_0$ on $\mathbb{R}$ is symmetric and $\nu_1, \nu_2 \in \mathcal{P}$, then

$$\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2) \Rightarrow \lambda_1 \circ \tau = \lambda_2 \circ \tau.$$

If $\lambda_1 \circ \tau = \lambda_2 \circ \tau$ and $\lambda_1 \in K_\mu(\nu_1, \nu_2)$ then $\lambda_2 \in K_\mu(\nu_1, \nu_2)$.

**Proof.** The second implication is trivial because for every symmetric measure $\mu$ we have $\mu \circ \lambda = \mu \circ (\lambda \circ \tau)$. To prove the first implication assume that $\lambda_1, \lambda_2 \in K_\mu(\nu_1, \nu_2)$. This implies that $\mu \circ \lambda_1 = \mu \circ \lambda_2$, and consequently $(\mu \circ \lambda) \circ \lambda_1 = (\mu \circ \lambda) \circ \lambda_2$ for every $\lambda \in \mathcal{P}$. This means that for every even function $f \in A(\mu)$ the following equality holds:

$$\int_{0}^{\infty} f(x)(\tau \circ \lambda_1)(dx) = \int_{0}^{\infty} f(x)(\tau \circ \lambda_2)(dx).$$

It follows from the proof of Lemma 7 that the even functions from $A(\mu)$ separate points in $\mathbb{R}_+$. Applying the Stone–Weierstrass Theorem again we conclude that the set of even functions from $A(\mu)$ is dense in $C(\mathbb{R}_+)$ in the topology of uniform convergence. This means that (**) holds for every $f \in C(\mathbb{R}_+)$, so the measures $\tau \circ \lambda_1$ and $\tau \circ \lambda_2$ coincide on $\mathbb{R}_+$, and, by symmetry, also on $\mathbb{R}$. ■

**Remark 1.** Notice that it follows from the proofs of Theorems 2 and 3 that weakly stable distributions are reducible in the sense that:

- If $X, Y, Z$ are independent real random variables and $X$ is nonsymmetric and weakly stable then the equality $XY \overset{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$.

- If $X, Y, Z$ are independent, $Y, Z$ are real, and $X$ is a nonsymmetric weakly stable random vector taking values in a separable Banach space $E$, then $XY \overset{d}{=} XZ$ implies $\mathcal{L}(Y) = \mathcal{L}(Z)$. To see this, apply the previous remark to the random variable $\xi(X)$, where $\xi \in E^*$ is such that $\xi(X)$ is not symmetric.

- If $X, Y, Z$ are independent, $Y, Z$ take values in $E$, and $X$ is a nonsymmetric weakly stable real random variable, then $XY \overset{d}{=} XZ$ implies
\[ \mathcal{L}(Y) = \mathcal{L}(Z). \] To see this, note that it suffices to prove \( \xi(Y) \overset{d}{=} \xi(Z) \) for all \( \xi \in \mathbb{E}^*. \)

- If \( X, Y, Z \) are independent and \( X \neq 0 \) is symmetric weakly stable, then \( XY \overset{d}{=} XZ \) implies \( \mathcal{L}(Y) \circ \tau = \mathcal{L}(Z) \circ \tau. \)

**Remark 2.** Notice that if \( \mu \) is weakly stable then so is \( \mu \circ \tau. \) Indeed, if \( T_a \mu * T_b \mu = \nu_1 \circ \mu \) and \( T_a \mu * T_{-b} \mu = \nu_2 \circ \mu \) then

\[
T_a(\mu \circ \tau) * T_b(\mu \circ \tau) = \left( \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \right) \circ (\mu \circ \tau).
\]

### 4. Some general properties of weakly stable distributions

**Lemma 8.** If a measure \( \mu \) on \( \mathbb{R} \) is weakly stable then \( \mu(\{0\}) = 0 \) or 1.

**Proof.** Let \( X \) be a weakly stable variable such that \( \mathcal{L}(X) = \mu, \) \( \mathbb{P}\{X = 0\} = p < 1, \) and let \( X' \) be its independent copy. We define the random variable \( Y \) with distribution \( \mathcal{L}(X \mid X \neq 0) \) and \( Y' \) its independent copy. The random variable \( Y/Y' \) has at most countably many atoms, so there exists \( a \in \mathbb{R}, \) \( a \neq 0, \) such that \( \mathbb{P}\{Y = aY'\} = 0. \) Now let \( \Theta \) be the random variable independent of \( X \) such that

\[ X - aX' \overset{d}{=} X \Theta. \]

Then we have

\[
p \leq \mathbb{P}\{X \Theta = 0\} = \mathbb{P}\{X - aX' = 0\} = p^2 + (1 - p)^2 \mathbb{P}\{Y - aY' = 0\} = p^2.
\]

This holds only if \( p = 0, \) which ends the proof. ■

**Lemma 9.** Assume that a weakly stable probability measure \( \mu \neq \delta_0 \) on \( \mathbb{R} \) has at least one atom. Then the discrete part of \( \mu \) (normalized to be a probability measure) is also weakly stable.

**Proof.** Let \( \mu = \alpha \mu_1 + (1 - \alpha) \mu_2, \) \( \alpha \in (0, 1), \) where \( \alpha \mu_1 \) is the discrete part of \( \mu, \) \( \mu_1(\mathbb{R}) = 1, \) and \( \mu_2 \) is such that \( \mu_2(\mathbb{R}) = 1 \) and \( \mu_2(\{x\}) = 0 \) for every \( x \in \mathbb{R}. \) If \( \mu \) is weakly stable, then for every \( a \in \mathbb{R} \) there exists a probability measure \( \lambda \) such that \( \mu * T_a \mu = \mu \circ \lambda. \) Now we have

\[
\mu * T_a \mu = \alpha^2 \mu_1 * T_a \mu_1 + \alpha(1 - \alpha) \mu_1 * T_a \mu_2 + \alpha(1 - \alpha) \mu_2 * T_a \mu_1 + (1 - \alpha)^2 \mu_2 * T_a \mu_2.
\]

Clearly for \( a \neq 0 \) the discrete part of \( \mu * T_a \mu \) is equal to \( \alpha^2 \mu_1 * T_a \mu_1. \) On the other hand, we have

\[
\mu \circ \lambda = (1 - \beta) \mu \circ \lambda_2 + \alpha \beta \mu_1 \circ \lambda_1 + (1 - \alpha) \beta \mu_2 \circ \lambda_1,
\]

where \( \lambda_1(\mathbb{R}) = \lambda_2(\mathbb{R}) = 1, \) \( \lambda_1 \) is a discrete measure, \( \lambda_2(\{x\}) = 0 \) for every \( x \in \mathbb{R} \) and \( \lambda = \beta \lambda_1 + (1 - \beta) \lambda_2. \)
Let $S = \{ a \in \mathbb{R} : \mu \ast T_a \mu(\{0\}) = 0 \}$. If $a \in S$, $a \neq 0$, then $\lambda(\{0\}) = 0$ and $\mu \circ \lambda_2(\{x\}) = \beta \mu_2 \circ \lambda_1(\{x\}) = 0$ for every $x \in \mathbb{R}$, so

$$\alpha^2 \mu_1 \ast T_a \mu_1 = \alpha \beta \mu_1 \circ \lambda_1.$$  

This means that $\alpha = \beta$ and $\mu_1 \ast T_a \mu_1 = \mu_1 \circ \lambda_1$.

If $a \notin S$ then there exists a sequence $a_n \in S \setminus \{0\}$, $n \in \mathbb{N}$, such that $\lim_n a_n = a$. Then $\mu \ast T_{a_n} \mu \Rightarrow \mu \ast T_a \mu$ and $\mu_1 \ast T_{a_n} \mu_1 \Rightarrow \mu_1 \ast T_a \mu_1$. For every $n \in \mathbb{N}$ there exists $\lambda_n$ such that $\mu_1 \ast T_{a_n} \mu_1 = \mu_1 \circ \lambda_n$, i.e. $\lambda_n \in K_{\mu_1}(\delta_1, \delta_{a_n})$. In view of Lemma 5 there exists $\lambda \in K_{\mu_1}(\delta_1, \delta_a)$, which ends the proof. ■

**Theorem 4.** Assume that a random vector $X$ taking values in a separable Banach space $\mathbb{E}$ and having distribution $\mu$ is such that $\mathbb{E}\|X\| < \infty$ and $\mathbb{E}X = a \neq 0$. Then $\mu$ is weakly stable if and only if $\mu = \delta_a$.

**Proof.** Assume first that $\mathbb{E} = \mathbb{R}$. If $\mu = \delta_a$ for some $a \neq 0$, then it is weakly stable. Conversely, let $\mu$ be weakly stable and $\mathbb{E}X = a \neq 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with distribution $\mu$. The Weak Law of Large Numbers implies that

$$\frac{1}{n} \sum_{k=1}^n X_k \to a$$

weakly as $n \to \infty$. The measure $\mu$ is weakly stable, thus for every $n \in \mathbb{N}$ there exists a measure $\nu_n$ such that

$$\mu_n = L \left( \frac{1}{n} \sum_{k=1}^n X_k \right) = (T_{1/n} \mu)^*n = \mu \circ \nu_n.$$  

Since $\mu_n \Rightarrow \delta_a$, it follows from Lemma 2 that the family $\{\nu_n\}$ is tight and it contains a sequence $\nu_{n_k}$ such that $\nu_{n_k} \Rightarrow \nu$ for some probability measure $\nu$. Now, we obtain

$$\delta_a = \lim_{n \to \infty} \mu_n = \lim_{k \to \infty} \mu \circ \nu_{n_k} = \mu \circ \nu.$$  

Since $a \neq 0$ the last equality is possible only if $\mu = \delta_x$ and $\nu = \delta_y$ for some $x, y \in \mathbb{R}$ with $xy = a$. Since $\mathbb{E}X = a$, we conclude that $\mu = \delta_a$.

If $X$ is a random vector in a separable Banach space $\mathbb{E}$ with $\mathbb{E}X = a \neq 0$ then the previous considerations yield $\mathbb{P}\{\xi(X) = \xi(a)\} = 1$ for each $\xi \in \mathbb{E}^*$ with $\xi(a) \neq 0$. Such $\xi$’s form a dense subset in $\mathbb{E}^*$. Consequently, $\mathbb{P}\{X = a\} = 1$. ■

**Theorem 5.** Assume that for a weakly stable measure $\mu \neq \delta_0$ on a separable Banach space $\mathbb{E}$ there exists $\varepsilon \in (0, 1]$ such that for every $\xi \in \mathbb{E}^*$ and every $p \in (0, \varepsilon)$,

$$\int_{\mathbb{E}} |\xi(x)|^p \mu(dx) < \infty.$$
Then there exists $\alpha_0 \in [\varepsilon, 2]$ such that $\mathcal{M}(\mu)$ contains a strictly $\alpha$-stable measure for every $\alpha \in (0, \alpha_0)$.

Proof. Let $p \in (0, \varepsilon)$. Since $\mathcal{M}(\mu)$ is closed under scale mixing, $\mu \circ m_n \in \mathcal{M}(\mu)$ for every $n \in \mathbb{N}$, where

$$m_n(dx) = c(n)x^{-p-1}\mathbf{1}_{(1/n, \infty)}dx, \quad c(n) = pn^{-p}.$$  

As $\mathcal{M}(\mu)$ is also closed under convolution and under taking convex linear combinations, and weakly closed, for every $n \in \mathbb{N}$ we have

$$\nu_n = \exp\{c(n)^{-1}(\mu \circ m_n)\} \in \mathcal{M}(\mu),$$  

where $\exp(\kappa) := e^{-\kappa(E)} \sum_{k=0}^{\infty} \kappa^k/k!$ for every finite measure $\kappa$ on $E$. Notice that for every $\xi \in \mathbb{E}^*$ we have

$$\hat{\nu}_n(\xi) = \exp\left\{-\int_{E} \int_{1/n}^{\infty} (1 - e^{i\xi(sx)})s^{-p-1}ds \mu(dx) \right\}$$  

$$= \exp\left\{-\int_{E} |\xi(x)|^p \int_{|\xi(x)|/n}^{\infty} (1 - e^{iu\text{sgn}(\xi(x))})u^{-p-1}du \mu(dx) \right\}.$$  

Let $h(u) = (1 - e^{iu\text{sgn}(\xi(x))})u^{-p-1}$. Then $h$ is integrable on $[0, \infty)$ since $p \in (0, 1)$, and $|h(u)| = 2|\sin(u/2)|u^{-p-1}$, thus $|h(u)| \leq u^{-p}$ for $u < 1$ and $|h(u)| \leq 2u^{-p-1}$ for $u \geq 1$. This implies that the function

$$H_p(\xi(x)) = \int_{0}^{\infty} (1 - e^{iu\text{sgn}(\xi(x))})u^{-p-1}du$$  

is well defined and bounded on $E$, thus

$$\hat{\nu}_n(\xi) \to \exp\left\{-\int_{E} |\xi(x)|^pH_p(\xi(x)) \mu(dx) \right\} =: \hat{\gamma}_p(\xi).$$  

It is easy to see that $\hat{\gamma}_p$ is the characteristic function of a strictly $p$-stable random variable and the corresponding measure $\gamma_p$ belongs to $\mathcal{M}(\mu)$ since this class is weakly closed. Now we define

$$\alpha_0 = \sup\{\alpha \in (0, 2] : \mathcal{M}(\mu) \text{ contains a strictly } \alpha \text{-stable measure}\}.$$  

To end the proof it is enough to recall that for every $0 < \beta < \alpha \leq 2$ and every strictly $\alpha$-stable measure $\gamma_{\alpha}$ the measure $\gamma_{\alpha} \circ \lambda_{\beta/\alpha}$ is strictly $\beta$-stable, where $\lambda_{\beta/\alpha}$ is the distribution of the random variable $\Theta_{\beta/\alpha}^{1/\alpha}$, and $\Theta_{\beta/\alpha} \geq 0$ is such that $\mathbb{E}\exp\{-t\Theta_{\beta/\alpha}\} = \exp\{-t^{\beta/\alpha}\}$. □

Remark 3. Notice that if a weakly stable measure $\mu \neq \delta_0$ on $E$ is such that $\int |\xi(x)|^p \mu(dx) < \infty$ for every $\xi \in \mathbb{E}^*$ and $p \in (0, \varepsilon)$ for some $\varepsilon \in (0, 2]$ then $\mathcal{M}(\mu)$ contains a symmetric $p$-stable measure for every $p \in (0, \varepsilon)$. 

206 J. K. Misiewicz et al.
To see this it is enough to notice that if \( \mu \) is symmetric, then so is the measure \( \nu_n \) constructed in the proof of Theorem 5. Consequently,

\[
\hat{\nu}_n(\xi) = \exp \left\{ - \int \frac{|\xi(x)|^p}{\mathbb{E}} \left( 1 - \cos u \right) u^{-p-1} du \mu(dx) \right\}.
\]

Let \( h(u) = (1 - \cos u)u^{-p-1} \). Then \(|h(u)| < u^{1-p}\) for \( u < 1 \), and \(|h(u)| < 2u^{-p-1}\) for \( u > 1 \), so \( h \) is integrable on \([0, \infty)\) for every \( p \in (0, 2) \). For the constants

\[
H_p = \int_0^\infty (1 - \cos u) u^{-p-1} du
\]

we obtain

\[
\hat{\nu}_n(\xi) \to \exp \left\{ -H_p \int \frac{|\xi(x)|^p}{\mathbb{E}} \mu(dx) \right\},
\]

which is the characteristic function of a symmetric \( p \)-stable random vector.

If \( \mu \) is not symmetric then we replace \( \mu \) by \( \mu \circ \tau \) in this construction. This is possible since \( \mu \circ \tau \) is symmetric, belongs to \( \mathcal{M}(\mu) \), and has the same moments as \( \mu \).

**Remark 4.** In the situation described in Remark 3, if \( \mathbb{E} = \mathbb{R} \) then \( \mathcal{M}(\mu) \) also contains a symmetric \( \varepsilon \)-stable random variable. Indeed, it follows from Remark 3 that

\[
\exp \left\{ -H_p \int \frac{|t|^p}{\mathbb{R}} \mu(dx) \right\} = \exp \left\{ -|t|^p H_p \int \frac{|x|^p}{\mathbb{R}} \mu(dx) \right\}
\]

is the characteristic function of some measure from \( \mathcal{M}(\mu) \). Since rescaling is admissible, \( \exp\{|-t|^p\} \) is also the characteristic function of some measure from \( \mathcal{M}(\mu) \). Now it is enough to notice that

\[
\lim_{p/\varepsilon} \exp\{-|t|^p\} = \exp\{-|t|^\varepsilon\},
\]

and use Lemma 3.

**Remark 5.** There exist measures \( \mu \) such that \( \mu \circ \nu \) is symmetric \( \alpha \)-stable for some probability measure \( \nu \), but \( \mu \) is not weakly stable. Any measure of the form \( \mu = q\delta_{-1} + (1 - q)\delta_1 \) for \( q \in (0, 1) \setminus \{1/2\} \) can serve as an example.

**Lemma 10.** Let \( X \) be a real random variable with distribution \( \mu \). If \( \mu \) is weakly stable and supported on a finite set then either there exists \( a \in \mathbb{R} \) such that \( \mu = \delta_a \) or there exists \( a \neq 0 \) such that \( \mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a \).

**Proof.** Let \( X' \) be an independent copy of \( X \). Assume that \( \mu \neq \delta_a \) for all \( a \in \mathbb{R} \). Theorem 4 implies that \( X \) must take on both negative and positive values with positive probability. Let \( V = \{ x \in \mathbb{R} : \mu(\{x\}) > 0 \} \). By Lemma 8 we have \( 0 \not\in V \). Let \( b \) be the greatest and \( -a \) the least element of \( V \). Clearly, \( a, b > 0 \). We will prove first that \( a = b \).
Assume that \( b > a \). For \( \lambda \in \mathbb{R} \) consider the set of values taken on by \( X - \lambda X' \) with positive probability: \( V_\lambda = \{v - \lambda w : v, w \in V\} \). Clearly, for \( \lambda \in (0, 1) \) the greatest element of \( V_\lambda \) is \( b + \lambda a \), and the least is \(- (a + \lambda b)\). Moreover \( a + \lambda b < b + \lambda a \) (hence \( b + \lambda a \) has strictly the greatest absolute value among all elements of \( V_\lambda \)). Since \( \mu \) is weakly stable there exists a real random variable \( Y_\lambda \) independent of \( X \) and such that \( Y_\lambda X \overset{d}{=} X - \lambda X' \). One can easily see that \( Y_\lambda \) is also finitely supported. We have \( b + \lambda a \in V_\lambda \) so that there exist \( c, d \neq 0 \) such that \( \mathbb{P}\{Y_\lambda = c\} > 0 \), \( d \in V \) and \( cd = b + \lambda a \). Also for any \( d' \in V \) we have \( cd' \in V_\lambda \), so that \(|d'| > |d| \) would imply \(|cd' | > b + \lambda a\), contrary to the fact that \( b + \lambda a \) has the maximal absolute value among all elements of \( V_\lambda \). Hence \( d \) must have maximal absolute value among all elements of \( V \) and therefore \( d = b \) so that \( c = 1 + \lambda a/b \).

We deduce that

\[-a/b (b + \lambda a) = c \cdot (-a) \in V_\lambda\]

and therefore there exist \( v, w \in V \) such that \(- (a/b)(b + \lambda a) = v - \lambda w \), and consequently \( \lambda (w - a^2/b) = v + a \). Assume that \( a^2/b \not\in V \). Then the last equation may be satisfied for finitely many values of the parameter \( \lambda \) only (because \( v \) and \( w \) can be chosen from a finite set only). It was proved for all \( \lambda \in (0, 1) \), however. Hence \( a^2/b \in V \). Therefore

\[a^2/b^2 (b + \lambda a) = c \cdot a^2/b \in V_\lambda\]

and again, there exist \( v, w \in V \) such that \((a^2/b^2)(b + \lambda a) = v - \lambda w \) so that \( \lambda (w + a^3/b^2) = v - a^2/b \). As before we infer that \(- a^3/b^2 \in V \). By iterating this reasoning we prove that \((-1)^{k+1}a^{k+1}/b^k \in V \) for every \( k \in \mathbb{N} \). Since \( 0 < a/b < 1 \) this implies that \( V \) contains an infinite subset, contradicting our assumptions. The case \( a > b \) is excluded in a similar way. Hence \( a = b \).

Now, let \(-\alpha \) be the greatest negative element and \( \beta \) the least positive element of \( V \). Consider \( X - \lambda X' \) for \( 0 < \lambda < \min(\alpha, \beta)/a \). Clearly, the least positive element of \( V_\lambda \) is \( \beta - \lambda a \), whereas \(- (\alpha - \lambda a) \) is the greatest negative one. Assume without loss of generality that \( \beta \leq \alpha \) so that \( \beta - \lambda a \) has the least absolute value among all elements of \( V_\lambda \) (otherwise consider \(-X \) instead of \( X \)). Again, we choose \( Y_\lambda \) and parameters \( c, d \neq 0 \) such that \( \mathbb{P}\{Y_\lambda = c\} > 0 \), \( d \in V \) and \( cd = \beta - \lambda a \). We obtain \( d \in \{-\alpha, \beta\} \) by a similar reasoning—no element can be both at the same side of zero as \( d \) and closer to zero than \( d \) because multiplying by \( c \) we would get a positive element of \( V_\lambda \) less than \( \beta - \lambda a \). Hence \( c \in \{(\beta - \lambda a)/\beta, - (\beta - \lambda a)/\alpha\} \). However, \( ca \in V_\lambda \) so that there exist \( v, w \in V \) such that \( ca = v - \lambda w \), which means that \( \lambda (w - a^2/\beta) = v - a \) or \( \lambda (w + a^2/\alpha) = v + a\beta/\alpha \). Since we proved this alternative for infinitely many \( \lambda \)’s and we know that \( v \) and \( w \) can have only
finitely many values we infer that $a^2/\beta \in V$ (if $d = \beta$) or $-a^2/\alpha \in V$ (if $d = -\alpha$).

We have proved that $V \subset [-a, a]$, so that $\beta = a$ if $d = \beta$, or $\alpha = a$ if $d = -\alpha$. Anyway, $|d| = a$ so that $|c| = \beta/a - \lambda$. Since $[-a, a] \subset V$ we have $\{-ca, ca\} \subset V_\lambda$ and therefore also $-(\beta - \lambda a) \in V_\lambda$. We have assumed though that $\beta - \lambda a$ has the least absolute value among all elements of $V_\lambda$, so in particular $-(\alpha - \lambda a) \leq -(\beta - \lambda a)$. Since $-(\alpha - \lambda a)$ is the greatest negative element of $V_\lambda$ we also have $-(\alpha - \lambda a) \geq -(\beta - \lambda a)$. Hence $\alpha = \beta$.

We have proved earlier that $\alpha = a$ or $\beta = a$, so finally $\alpha = \beta = a$ and the support of $\mu$ is $\{-a, a\}$. Theorem 4 implies that $\mu$ is symmetric. ■

Lemma 11. Let $X$ be a real random variable with distribution $\mu \neq \delta_0$ and let $X'$ be its independent copy. Assume that $\mu$ is weakly stable, so that for any $\lambda \in \mathbb{R}$ there exists a real random variable $Y_\lambda$ independent of $X$ such that $X - \lambda X' \overset{d}{=} Y_\lambda X$. If $X$ is symmetric, assume additionally that $Y_\lambda \geq 0$ a.s. Then the map

$$\lambda \mapsto \mathcal{L}(Y_\lambda)$$

is well defined and continuous on $\mathbb{R}$.

Proof. The existence and uniqueness of distribution of $Y_\lambda$ follows from Theorems 2 and 3. We only need to prove that $\lambda_n \to \lambda$ implies that $Y_{\lambda_n} \overset{d}{\to} Y_\lambda$ as $n \to \infty$. Suppose not. Then we can find $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that for any $k$ the law of $Y_{\lambda_{n_k}}$ is $\varepsilon$-separated from the law of $Y_\lambda$ in Lévy’s metric. Since

$$Y_{\lambda_{n_k}} X \overset{d}{=} X - \lambda_{n_k} X' \overset{d}{=} X - \lambda X' \overset{d}{=} Y_\lambda X,$$

by Lemma 2 we can choose a subsequence $\{n_{k_l}\} \subset \{n_k\}$ such that $Y_{\lambda_{n_{k_l}}} \overset{d}{\to} Z$ for some real random variable $Z$ as $l \to \infty$. Hence $\mathcal{L}(Z) \neq \mathcal{L}(Y_\lambda)$. Moreover $Z \geq 0$ a.s. if $X$ is symmetric because then all $Y_{\lambda_n}$’s are nonnegative a.s.

On the other hand, $Z' X \overset{d}{=} Y_\lambda X$, where $Z'$ is a copy of $Z$ independent of $X$ since the map $\lambda \mapsto \mathcal{L}(X - \lambda X')$ is continuous. Therefore $Y_\lambda \overset{d}{=} Z' \overset{d}{=} Z$, by Theorems 2 and 3 (or by Remark 1). The contradiction obtained ends the proof. ■

Remark 6. Let $\alpha \in [1, 2]$. Note that if $X$ is a random variable with a weakly stable distribution $\mu$ and $\mathbb{E}|X|^p < \infty$ for all $p \in (0, \alpha)$ then

$$1 + |\lambda| \geq \begin{cases} |Y_\lambda| \text{ a.s.} & \text{if } \alpha = 2, \\ \|Y_\lambda\|_\alpha & \text{if } \alpha < 2. \end{cases}$$

Indeed, by Theorem 5 there exists $\Theta$ independent of $X$ such that $X\Theta$ is strictly $\alpha$-stable. If $\alpha < 2$ then $\mathbb{E}|X\Theta|^\beta < \infty$ for every $\beta < \alpha$, thus $\mathbb{E}|X|^\beta < \infty$ for every $\beta < \alpha$. If $\alpha = 2$ then $X\Theta$ is Gaussian so $\mathbb{E}|X|^\beta < \infty$
for every $\beta > 0$. Now it is enough to notice that for $\beta \geq 1$ we have
\[
\|Y_\lambda\|_\beta \|X\|_\beta = \|Y_\lambda X\|_\beta = \|X - \lambda X'\|_\beta \leq \|X\|_\beta + |\lambda| \|X'\|_\beta = (1 + |\lambda|) \|X\|_\beta.
\]
The case $\beta = \alpha$ can be obtained by observing that $\|Y_\lambda\|_\alpha = \lim_{\beta \to \alpha^-} \|Y_\lambda\|_\beta$. If $\alpha = 2$ the inequality holds for all $\beta \geq 1$, which implies that $\|Y_\lambda\|_\infty \leq 1 + |\lambda|$.

**Lemma 12.** Let $X$ be a real random variable with distribution $\mu$. If $\mu$ is weakly stable and supported on a countable set then there exists $a \in \mathbb{R}$ such that $\mu = \delta_a$ or there exists $a \neq 0$ such that $\mu = \frac{1}{2} \delta_{-a} + \frac{1}{2} \delta_a$.

**Proof.** Assume that the support of $\mu$ is an infinite countable set. For $\lambda \in (0, 1)$ we have $X - \lambda X' \overset{d}{=} Y_\lambda X$, where $X'$ and $Y_\lambda$ are defined as in the preceding lemma (so that if $X$ is symmetric then $Y_\lambda \geq 0$ a.s.). By Lemma 11, $Y_\lambda \overset{d}{\to} Y_0 = 1$ as $\lambda \to 0$. Let
\[
\mu = \sum_{n=1}^\infty p_n \delta_{x_n},
\]
where $x_n$’s are nonzero (by Lemma 8) and pairwise different, and $(p_n)_{n=1}^\infty$ is a nonincreasing sequence of positive numbers. Let
\[
M = \left\{ \frac{x_i - x_j}{x_k - x_l} : k \neq l \right\}.
\]
Clearly, $M$ is a countable set. We see that for $\lambda \notin M$ the equality $x_k - \lambda x_i = x_l - \lambda x_j$ implies $i = j$ and $k = l$. Finally, let $N \in \mathbb{N}$ be such that $\sum_{n>N} p_n \leq p_1^2/2$. Then for $\lambda \in (0, 1) \setminus M$ we have
\[
p_1^2 = P\{X = x_1, X' = x_1\} = P\{X - \lambda X' = x_1 - \lambda x_1\}
= P\{Y_\lambda X = x_1(1 - \lambda)\} = \sum_{n=1}^\infty P\left\{ Y_\lambda = \frac{x_1}{x_n} (1 - \lambda) \right\} \cdot p_n
\leq P\{Y_\lambda = 1 - \lambda\} \cdot p_1 + \frac{p_1^2}{2} + \sum_{n=2}^N P\left\{ Y_\lambda = \frac{x_1}{x_n} \right\} \cdot p_n,
\]
and the summands for $2 \leq n \leq N$ tend to zero as $\lambda \to 0$ (since $\frac{Y_\lambda}{1 - \lambda} \overset{d}{\to} 1$) so that
\[
\liminf_{\lambda \to 0, \lambda \in (0, 1) \setminus M} P\{Y_\lambda = 1 - \lambda\} \geq p_1/2.
\]
On the other hand, for $\lambda \in (0, 1) \setminus M$ and $k \in \mathbb{N}$ we have
\[
p_k^2 = P\{X = x_k, X' = x_k\} = P\{X - \lambda X' = x_k(1 - \lambda)\}
= P\{Y_\lambda X = x_k(1 - \lambda)\} \geq P\{Y_\lambda = 1 - \lambda\} \cdot p_k,
\]
so that
\[ \limsup_{\lambda \to 0, \lambda \in (0,1) \setminus M} \mathbf{P}\{y_\lambda = 1 - \lambda\} \leq p_k. \]

Hence \( p_k \geq p_1/2 \) for any \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} p_k = \infty \), which is clearly not possible. This proves that \( \mu \) has finite support and the assertion follows from Lemma 10. 

**Theorem 6.** Let \( \mu \) be a weakly stable probability measure on a separable Banach space \( \mathbb{E} \). Then either there exists \( a \in \mathbb{E} \) such that \( \mu = \delta_a \), or there exists \( a \in \mathbb{E} \setminus \{0\} \) such that \( \mu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a \), or \( \mu(\{a\}) = 0 \) for all \( a \in \mathbb{E} \).

**Proof.** Assume first that \( \mathbb{E} = \mathbb{R} \). One can express \( \mu \) as \( p\mu_1 + (1 - p)\mu_2 \), where \( p \in [0,1] \), \( \mu_1 \) is a discrete probability measure and \( \mu_2(\{x\}) = 0 \) for any \( x \in \mathbb{R} \). The case \( p = 0 \) is trivial, so assume that \( p > 0 \). Lemma 9 implies that \( \mu_1 \) is weakly stable, and therefore by Lemma 12, \( \mu_1 = \delta_a \) for some \( a \in \mathbb{R} \) or \( \mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a \) for some \( a \neq 0 \).

**Case 1:** \( \mu_1 = \delta_a \). If \( a = 0 \) then by Lemma 8 we have \( p = 1 \) and the proof is finished. If \( a \neq 0 \), note that for \( \lambda \in (0,1) \) the random variable \( X - \lambda X' \overset{d}{=} Y_\lambda X \) has exactly one atom with mass \( p^2 \) at \( (1 - \lambda)a \). Hence \( Y_\lambda \) has an atom with mass \( p \) at \( 1 - \lambda \). Since \( Y_\lambda \overset{d}{\to} Y_1 \) as \( \lambda \to 1 \) we have \( \mathbf{P}\{Y_1 = 0\} \geq p \), and therefore \( \mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \geq p \). On the other hand, \( \mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X'\} = p^2 \) because \( X \) has only one atom, at \( a \). Hence \( p^2 \geq p \) so that \( p = 1 \).

**Case 2:** \( \mu_1 = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a \) for some \( a \neq 0 \). For \( \lambda \in (0,1) \) the random variable \( X - \lambda X' \overset{d}{=} Y_\lambda X \) has exactly four atoms, with mass \( p^2/4 \) each, at \( \pm(1-\lambda)a \) and \( \pm(1+\lambda)a \). Hence \( Y_\lambda \) has atoms with total mass \( p/2 \) at \( \pm(1-\lambda) \) (and atoms with total mass \( p/2 \) at \( \pm(1+\lambda) \)). Since \( Y_\lambda \overset{d}{\to} Y_1 \) as \( \lambda \to 1 \) we have \( \mathbf{P}\{Y_1 = 0\} \geq p/2 \), and therefore \( \mathbf{P}\{X - X' = 0\} = \mathbf{P}\{Y_1 X = 0\} \geq p/2 \). On the other hand, \( \mathbf{P}\{X - X' = 0\} = \mathbf{P}\{X = X' = a\} + \mathbf{P}\{X = X' = -a\} = p^2/2 \) so that \( p^2/2 \geq p/2 \) and \( p = 1 \).

Let now \( \mathbb{E} \) be an arbitrary separable Banach space. By making use of the above result for real random variables \( \xi(X) \), where \( \xi \in \mathbb{E}^* \), we can easily finish the proof. 

**Remark 7.** Let \( X \) be a given random variable. It may happen that for some \( a, b \in \mathbb{R} \) there exists a random variable \( Q_{a,b} \) independent of \( X \) such that \( aX + bX' \overset{d}{=} XQ_{a,b} \), and for some other \( a, b \) such a random variable does not exist.

Consider, for example, \( X \) with exponential distribution and characteristic function \((1 - it)^{-1}\). It follows from Theorem 4 that the class \( \mathcal{M}((1 - it)^{-1}) \) is not closed under convolutions. However, if \( a, b \in \mathbb{R} \) are such that \( ab < 0 \).
then the characteristic function of $aX + bX'$ can be written as

$$
E \exp \{ i(aX + bX')t \} = \frac{1}{1 - iat} \frac{1}{1 - ibt} = \int \frac{1}{1 - ist} \lambda(ds),
$$

where $\lambda(\{a\}) = p = 1 - \lambda(\{b\})$, with $p = a/(a - b)$. This means that

$$
aX + bX' \overset{d}{=} XQ_{a,b},
$$

where $Q_{a,b}$ is independent of $X$ and has distribution $\lambda$.

Assume that for some $a, b > 0$ there exists $Q \geq 0$ such that $aX + bX' \overset{d}{=} XQ$. Then the density $g$ of $aX + bX'$ can be written as

$$
g(x) = \int_{0}^{\infty} e^{-x/s} s^{-1} \mathcal{L}(Q)(ds).
$$

On the other hand, we have

$$
g(x) = \int_{0}^{\infty} e^{-x/s} s^{-1} \lambda(ds).
$$

The uniqueness of the Laplace transform for signed $\sigma$-finite measures implies that $\mathcal{L}(Q) = \lambda$, which is impossible since $\mathcal{L}(Q)$ is a probability measure while $\lambda$ is a signed measure only. Similar arguments can be used for $a, b < 0$.

Finally, if $ab > 0$ then $aX + bX'$ cannot have the same distribution as $XQ$ for any random variable $Q$ independent of $X$.

References

Classes of measures closed under mixing


Department of Mathematics, Informatics and Econometry
University of Zielona Góra
Podgórna 50
65-246 Zielona Góra, Poland
E-mail: j.misiewicz@wmie.uz.zgora.pl

Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: urbanik@math.uni.wroc.pl

Received December 2, 2002

Revised version December 7, 2004