A condition equivalent to uniform ergodicity

by

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Abstract. Let $T$ be a linear operator on a Banach space $X$ with $\sup_n \|T^n/n^w\| < \infty$ for some $0 \leq w < 1$. We show that the following conditions are equivalent:

(i) $n^{-1} \sum_{k=0}^{n-1} T^k$ converges uniformly; (ii) $\text{cl} (I-T)X = \{z \in X : \lim_n \sum_{k=1}^n T^kz/k \text{ exists} \}$.

1. Introduction. Let $X$ be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators from $X$ to itself. An operator $T \in \mathcal{L}(X)$ is called uniformly ergodic if the averages

$$A_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k$$

converge in the uniform operator topology.

M. Lin [5] showed that when $\lim_{n \to \infty} \|T^n/n\| = 0$, $T$ is uniformly ergodic if and only if $(I-T)X$ is closed. From this it is easy to see that a power-bounded $T$ (that is, $\sup_n \|T^n\| < \infty$) is uniformly ergodic if and only if $\{z \in X : \sup_n \|\sum_{k=0}^{n} T^kz\| < \infty\}$ is closed.

V. Fonf, M. Lin and A. Rubinov [2] proved that if $X$ is separable and does not contain an isomorphic copy of an infinite-dimensional dual Banach space, then the uniform ergodicity of a power-bounded $T$ is equivalent to

$$(I-T)X = \{z \in X : \sup_n \|\sum_{k=0}^{n} T^kz\| < \infty\}.$$ 

In [3], S. Grabiner and J. Zemánek give the following generalization of Lin’s theorem. Under the hypothesis of boundedness of $A_n(T)$ or convergence to zero of $T^n/n$ in some operator topology, they prove that if $(I-T)^n X$ is closed for some $n \geq 2$ ($n \geq 1$ if $T^n/n$ converges to zero in the uniform operator topology) or if $(I-T)X + \text{Ker}(I-T)$ is closed for some $n \geq 1$, then $X$ is the direct sum of the closed subspaces $(I-T)X$ and $\text{Ker}(I-T)$. In this case the sequence $A_n(T)$ converges in some operator topology if and only if $T^n/n$ converges to zero in the same operator topology.

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Very recently, E. Ed-dari [1] obtained an improvement of a result of T. Yoshimoto [6] about the uniform ergodic theorem with Cesàro means of order \(\alpha\). Ed-dari proved that for every \(\alpha > 0\) the sequence

\[
M_n^\alpha(T) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k,
\]

where \(A_n^\alpha, n = 0, 1, \ldots\), are the \((C, \alpha)\) coefficients of order \(\alpha\), converges in the uniform operator topology to an operator \(E \in \mathcal{L}(X)\) if and only if

\[
\|(\lambda - 1)R(\lambda, T) - E\| \to 0 \quad \text{as} \quad \lambda \to 1^+ \quad \text{and} \quad \lim_{n \to \infty} \|T^n\| = 0.
\]

Let \(T \in \mathcal{L}(X)\) be such that \(\sup_n \|T^n/n^w\| < \infty\) for some \(0 \leq w < 1\). We shall denote \(\{z \in X : \lim_{n \to \infty} \sum_{k=1}^{n} T^k z/k \text{ exists}\}\) by \(X_1\). In this paper we study the relationship between \(X_1\) and \((I - T)X\), and prove that \(X_1 = \text{cl}(I - T)X\) is equivalent to the uniform ergodicity of \(T\).

### 2. Results.

Throughout this section, \(T\) is a linear operator in \(\mathcal{L}(X)\) with \(\sup_n \|T^n/n^w\| < \infty\) for some \(0 \leq w < 1\).

**Lemma.** \((I - T)X \subset X_1 \subset \text{cl}(I - T)X\). Therefore \(X_1\) is closed if and only if \(X_1 = \text{cl}(I - T)X\).

**Proof.** Let \(z \in (I - T)X\). Then \(z = (I - T)x\). From

\[
\left\| \sum_{k=n+1}^{n+p} \frac{T^k z}{k} \right\| = \left\| \frac{T^{n+1} x}{n+1} - \frac{T^{n+p+1} x}{n+p} - \sum_{k=n+2}^{n+p} \frac{T^k x}{k(k-1)} \right\|
\]

and the boundedness of \(\|T^n/n^w\|\), we see that \((\sum_{k=1}^{n} T^k z/k)\) is a Cauchy sequence, and thus \(z \in X_1\).

Now, let \(z \in X_1\) and \(u \in \text{Ker}(I - T^*)\), where \(T^*\) is the adjoint operator of \(T\). Let \(z_0\) denote the limit of \(\sum_{k=1}^{n} T^k z/k\). Then we have

\[
\langle u, z_0 \rangle = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{T^k z}{k} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} \langle u, z \rangle.
\]

We conclude that \(\langle u, z \rangle = 0\). Hence \(z \in \text{cl}(I - T)X\) by the Hahn–Banach theorem. 

We can now state our result.

**Theorem.** The following conditions are equivalent:

(i) \(T\) is uniformly ergodic.

(ii) \(X_1\) is closed.

**Proof.** (i)\(\Rightarrow\)(ii) by Lin’s theorem [5] and the previous lemma.

(ii)\(\Rightarrow\)(i). By the lemma, \(X_1 = \text{cl}(I - T)X\) and therefore \(X_1\) is invariant under \(T\). Let \(S\) be the restriction of \(T\) to \(X_1\). We define \(S_n = \sum_{k=1}^{n} S^k/k\),
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By the principle of uniform boundedness, there exists a constant $K > 0$ such that $\sup_n \|S_n\| \leq K$. For each positive integer $n$ we define on $X_1$ the operator $B_n$ by

$$B_n = \frac{1}{a_n} S_n,$$

where $a_n = \sum_{k=1}^{n} \frac{1}{k}$.

Then $B_n$ converges to 0 as $n \to \infty$ in the uniform operator topology.

Put $A_n = n^{-1} \sum_{k=1}^{n} S^k$. Making use of the partial summation formula of Abel, we obtain

$$A_n = \frac{n+1}{n} S_n - \frac{1}{n} \sum_{k=1}^{n} S_k.$$

Thus there exists a constant $C > 0$ such that $\sup_n \|A_n\| \leq C$.

Since $\|S^n\| \leq \|T^n\|$, we also have

$$\lim_{n \to \infty} \|A_n S^j - A_n\| = 0 \quad \text{for any } j \geq 0. \quad (1)$$

Now, for any given $\varepsilon > 0$, choose an integer $k$ so large that $\|B_k\| < \varepsilon$. By (1) we may find an $N$ such that $\|A_n S^j - A_n\| < \varepsilon$ holds for $n \geq N$ and $j = 1, \ldots, k$. As $B_k$ is a convex combination of $S^j$, $1 \leq j \leq k$, for $n \geq N$ we therefore obtain $\|A_n B_k - A_n\| < \varepsilon$. Hence

$$\|A_n\| \leq \|A_n - A_n B_k\| + \|A_n B_k\| < \varepsilon(1 + C).$$

Consequently, $S$ is uniformly ergodic. Fix $n$ such that $\|A_n\| < 1$. Then $I - A_n$ is invertible, and hence so is $I - S$. Therefore

$$\text{cl } (I - T) X = X_1 = (I - S) X_1 \subset (I - T) X.$$

Thus, $(I - T) X$ is closed and we may apply Lin’s theorem [5] again to conclude that $T$ is uniformly ergodic. This completes the proof of the theorem.

**Remark 1.** For $T$ power-bounded, the uniform ergodicity of $S$ in the proof of the theorem follows from Krengel [4, p. 88].

**Remark 2.** Inspecting the above proofs we see that for an operator $T$ which satisfies $\lim_{n \to \infty} \|T^n/n\| = 0$, we just used the condition $\sup_n \|T^n/n^w\| < \infty$ to prove $(I-T)X \subset X_1$. Therefore the theorem is also valid for an operator $T \in \mathcal{L}(X)$ satisfying $(I-T)X \subset X_1$ together with $\lim_{n \to \infty} \|T^n/n\| = 0$.

**References**


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