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On differentiability of strongly $\alpha(\cdot)$-paraconvex functions in non-separable Asplund spaces

by

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Abstract. In Rolewicz (2002) it was proved that every strongly $\alpha(\cdot)$-paraconvex function defined on an open convex set in a separable Asplund space is Fréchet differentiable on a residual set. In this paper it is shown that the assumption of separability is not essential.

1. Introduction. Properties of $\alpha(\cdot)$-paraconvex functions. Let $(X, \|\cdot\|)$ be a real Banach space. Let f be a real-valued convex continuous function defined on an open convex subset $\Omega \subset X$. Mazur (1933) proved that there is a subset $A_{\mathrm{G}} \subset \Omega$ of the first Baire category such that f is Gateaux differentiable on $\Omega \setminus A_{\mathrm{G}}$. Asplund (1968) showed that if additionally X is an Asplund space (in particular if X has a separable dual), then there is a subset $A_{\mathrm{F}} \subset \Omega$ of the first Baire category such that f is Fréchet differentiable on $\Omega \setminus A_{\mathrm{F}}$.

The result of Asplund was extended in Rolewicz (2002) to a larger class of functions called strongly $\alpha(\cdot)$-paraconvex, under the additional hypothesis that X is a separable Asplund space (i.e. it is a separable space with separable dual).

In the present paper we shall prove it without this additional hypothesis. First we recall the definitions of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions.

Let $(X, \|\cdot\|)$ be a real Banach space and f be a real-valued function defined on an open convex subset $\Omega \subset X$. Let $\alpha : [0, \infty) \to [0, \infty]$ be a nondecreasing function such that

$$\lim_{t \downarrow 0} \alpha(t)/t = 0. \tag{1.1}$$

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We say that f is *$\alpha(\cdot)$-paraconvex* if there is a constant $C>0$ such that for all $x,y\in\Omega$ and $0\le t\le 1$,

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+C\alpha(\|x-y\|). \tag{1.2}$$

For $\alpha(t)=t^2$ this definition was introduced in Rolewicz (1979a) and the t^2-paraconvex functions were called simply paraconvex. In Rolewicz (1979b) the notion was extended to the case of $\alpha(t)=t^\gamma, 1\le\gamma\le 2$, and the t^γ-paraconvex functions were called γ-paraconvex.

Observe that the convex functions can be treated as 0-paraconvex functions.

We say that f is *strongly $\alpha(\cdot)$-paraconvex* if there is a constant $C_1>0$ such that for all $x,y\in\Omega$ and $0\le t\le 1$,

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)+C_1\min\{t,(1-t)\}\alpha(\|x-y\|). \tag{1.3}$$

Obviously each strongly $\alpha(\cdot)$-paraconvex function is $\alpha(\cdot)$-paraconvex, but the converse is not true (Rolewicz (2000)).

The simplest examples of strongly $\alpha(\cdot)$-paraconvex functions are sums of convex and continuously differentiable functions, but the class of strongly $\alpha(\cdot)$-paraconvex functions is larger.

The notion of strongly $\alpha(\cdot)$-paraconvex functions can be treated as a uniformization of the notion of approximate convex functions introduced by Luc, Ngai and Théra (1999) (see Rolewicz (2001b)).

It is known that a convex function has a directional derivative at each point. The same holds for strongly $\alpha(\cdot)$-paraconvex functions.

PROPOSITION 1.1. *Let Ω be a convex subset of a Banach space X. Let $f:\Omega\to\mathbb{R}$ be a strongly $\alpha(\cdot)$-paraconvex function. Then at each $x\in\Omega$ the function f has a directional derivative in any direction h such that $x+h\in\Omega$.*

Proof. For simplicity we set $\widetilde{f}(t)=f(x_0+th)-f(x_0)$. We shall show that $\lim_{t\downarrow 0}\widetilde{f}(t)/t$ exists.

The first step is to show that $\limsup_{t\downarrow 0}\widetilde{f}(t)/t$ is finite. Indeed, by strong $\alpha(\cdot)$-paraconvexity of f,

$$\frac{\widetilde{f}(t)}{t}\le\widetilde{f}(1)+C\,\frac{\alpha(t\|h\|)}{t}. \tag{1.4}$$

Thus

$$\limsup_{t\downarrow 0}\widetilde{f}(t)/t\le\widetilde{f}(1). \tag{1.5}$$

This means that there are a real a and a sequence $\{t_n\}$ tending to 0 such that

$$\lim_{n\to\infty}\widetilde{f}(t_n)/t_n=a.$$

The next step is to show that the limits $\lim_{n\to\infty} \widetilde{f}(t_n)/t_n$ are the same for all sequences tending to 0.

Indeed, let $\tau_m \to 0$ and

$$\lim_{t\downarrow 0} \widetilde{f}(\tau_m)/\tau_m = b.$$

Suppose that $\tau_m < t_n$. Then by strong $\alpha(\cdot)$-paraconvexity of f,

$$\begin{aligned} \frac{\widetilde{f}(\tau_m)}{\tau_m} &\le \frac{1}{\tau_m}\left(\frac{\tau_m}{t_n}\,\widetilde{f}(t_n) + C\,\frac{\tau_m}{t_n}\left(1-\frac{\tau_m}{t_n}\right)\alpha(t_n\|h\|)\right) \\ &= \frac{\widetilde{f}(t_n)}{t_n} + C\left(1-\frac{\tau_m}{t_n}\right)\frac{\alpha(t_n\|h\|)}{t_n}. \end{aligned} \tag{1.6}$$

Thus $b \le a$. Reversing the roles of $\{t_n\}$ and $\{\tau_m\}$ we get $a \le b$. Therefore $a = b$. ■

2. Uniform approximate subdifferentiability. The proof of the Asplund theorem in the classical case of convex functions consists of two parts:

(a) a convex function defined on an open set has a subgradient at each point,

(b) if a function f has a subgradient at each point, then there is a set $A_{\mathrm{F}} \subset \Omega$ of the first category such f is Fréchet differentiable at every point $x_0 \in \Omega \setminus A_{\mathrm{F}}$.

In the classical situation the first part is so trivial that it is not observed at all. But now we are in a different situation. It is necessary to define "subgradients" and to show that a strongly $\alpha(\cdot)$-paraconvex function has a "subgradient" at each point.

The definition can be found in the papers of Fabian (1989), Ioffe (1983), (1984), (1986), (1989), (1990), (2000) and Mordukhovich (1980), (1988). Namely, a linear functional $x^* \in X^*$ will be called an *approximate subgradient* of f at a point x if

$$\liminf_{h\to 0} \frac{(f(x+h)-f(x)) - x^*(h)}{\|h\|} \ge 0. \tag{2.1}$$

The set of all approximate subgradients of f at x is called the *approximate subdifferential* of f at x and denoted by $\partial f|_x$, as in the classical case. Thus $\partial f|_{(\cdot)}$ is a multifunction mapping the domain of $\partial f|_{(\cdot)}$ into 2^{X^*}.

Observe that (2.1) holds if and only there is a non-negative non-decreasing function β_x defined on $[0,\infty)$ and such that $\lim_{u\downarrow 0} \beta_x(u) = 0$ and

$$\frac{(f(x+h)-f(0)) - x^*(h)}{\|h\|} \ge -\beta_x(\|h\|). \tag{2.2}$$

Indeed, the function

$$\beta_x(s) = \sup_{\{h\,:\,\|h\|\le s\}} \left| \frac{(f(x+h)-f(x)) - x^*(h)}{\|h\|} \right| \tag{2.3}$$

has the required property.

Putting $\alpha_x(u) = u\beta_x(u)$ we can rewrite (2.2) in the form

$$f(x+h) - f(x) \ge x^*(h) - \alpha_x(\|h\|). \tag{2.4}$$

Unfortunately β_x (and hence α_x) can be different at each point and we are not able to use this definition for the problem of differentiation on a residual set. This prompts an idea of uniformization of this notion.

Let, as before, $\alpha : [0,\infty) \to [0,\infty]$ be a non-decreasing function such that $\lim_{t\downarrow 0} \alpha(t)/t = 0$.

Let f be a real-valued function defined on an open subset Ω of a Banach space X. Let $x \in X$. A linear functional $x^* \in X^*$ such that

$$f(x+h) - f(x) \ge x^*(h) - \alpha(\|h\|) \tag{2.5}$$

is called a *uniform approximate subgradient* of f at x with modulus $\alpha(\cdot)$ (or briefly an $\alpha(\cdot)$-*subgradient* of f at x). The set of all $\alpha(\cdot)$-subgradients of f at x will be called the $\alpha(\cdot)$-*subdifferential* of f at x, and denoted by $\partial_\alpha f|_x$.

We say that f is $\alpha(\cdot)$-*subdifferentiable* if $\partial_\alpha f|_x \ne \emptyset$ for all $x \in \Omega$.

In a similar way, we say that $x^* \in X^*$ is an $\alpha(\cdot)$-*gradient* of f at x if

$$|f(x+h) - f(x) - x^*(h)| \le \alpha(\|h\|). \tag{2.6}$$

By linearity of x^* and property (1.1) of $\alpha(\cdot)$ the $\alpha(\cdot)$-gradient is unique. The notion of $\alpha(\cdot)$-gradient can be considered as a uniformization of Fréchet gradients.

We say that f is $\alpha(\cdot)$-*differentiable* if it has $\alpha(\cdot)$-gradients for all $x \in \Omega$.

3. $\alpha(\cdot)$-subdifferentiability of strongly $\alpha(\cdot)$-paraconvex functions. We shall show that every strongly $\alpha(\cdot)$-paraconvex function is $\alpha(\cdot)$-subdifferentiable.

In the case of convex functions on open convex sets, this is a trivial consequence of the Hahn–Banach theorem.

In the general case the proof is based on the following two propositions:

PROPOSITION 3.1 (Rolewicz (2000)). *Every strongly $\alpha(\cdot)$-paraconvex function is locally Lipschitzian.*

PROPOSITION 3.2 (Rolewicz (2001a)). *The $\alpha(\cdot)$-subdifferential of each strongly $\alpha(\cdot)$-paraconvex function is equal to its Clarke subdifferential.*

As a trivial consequence we obtain

PROPOSITION 3.3 (Rolewicz (2002)). *Every strongly $\alpha(\cdot)$-paraconvex function is $\alpha(\cdot)$-subdifferentiable.*

In the case of X with separable dual (i.e. a separable Asplund space) we have

THEOREM 3.4 (Rolewicz (2002)). *Let A be an open convex set in a separable Asplund space X. Let $f : A \to \mathbb{R}$ be $\alpha(\cdot)$-subdifferentiable. Then there is a residual set $\Omega \subset A$ such that f is Fréchet differentiable at every point $x_0 \in \Omega$.*

4. Main results. As in the case of convex functions (see Phelps (1989)), we have

PROPOSITION 4.1. *Let $(X, \|\cdot\|)$ be a real Banach space. Let f be a real-valued function defined on an open convex subset $\Omega \subset X$. Suppose that x^* is an $\alpha(\cdot)$-subgradient of f at $x \in \Omega$. Then x^* is the Fréchet gradient of f at x if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\frac{f(x+ty)+f(x-ty)-2f(x)}{t} < \varepsilon \tag{4.1}$$

for all $y \in X$ such that $\|y\| = 1$ and $0 < t < \delta$, in other words,

$$\lim_{t\to 0} \sup_{\{y\in X\,:\,\|y\|=1\}} \frac{f(x+ty)+f(x-ty)-2f(x)}{t} = 0.$$

Proof. Necessity. If x^* is the Fréchet gradient at x, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f(x+ty) - f(x) - x^*(ty) < \frac{\varepsilon}{2}\, t \tag{4.2}$$

for all $y \in X$ such that $\|y\| = 1$ and $0 < t < \delta$. Replacing y by $-y$ we obtain

$$f(x-ty) - f(x) + x^*(ty) < \frac{\varepsilon}{2}\, t. \tag{4.3}$$

Adding (4.2) and (4.3) yields (4.1).

Sufficiency. By the property (1.1) of $\alpha(\cdot)$, for every $\varepsilon > 0$ there is $\delta_1 > 0$ such that

$$f(x+ty) - f(x) - x^*(ty) > -\varepsilon t \tag{4.4}$$

for all $y \in X$ such that $\|y\| = 1$ and $0 < t < \delta_1$.

Replacing y by $-y$ and multiplying by -1 we get

$$f(x) - f(x-ty) - x^*(ty) < \varepsilon t \tag{4.4$^-$}$$

On the other hand, (4.1) implies that there is $\delta_2 > 0$ such that

$$f(x+ty) - f(x) - x^*(ty) < \varepsilon t + f(x) - f(x-ty) - x^*(ty) \tag{4.5}$$

for $0 < t < \delta_2$. Thus for $0 < t < \delta = \min\{\delta_1, \delta_2\}$ by (4.4) and (4.5) we get

$$-\varepsilon t < f(x+ty)-f(x)-x^*(ty) < \varepsilon t+f(x)-f(x-ty)-x^*(ty) < 2\varepsilon t. \tag{4.6}$$

The arbitrariness of ε implies that x^* is the Fréchet gradient of f at x. ∎

If f is strongly $\alpha(\cdot)$-paraconvex we can replace the requirement that (4.1) holds for t small enough by the condition that such a t exists, and we obtain

PROPOSITION 4.2. *Let* $(X, \|\cdot\|)$ *be a real Banach space and* f *a strongly* $\alpha(\cdot)$*-paraconvex function defined on an open convex subset* $\Omega \subset X$*. The function* f *is Fréchet differentiable at a point* $x \in \Omega$ *if and only if for every* $\varepsilon > 0$ *there is* $t_\varepsilon > 0$ *such that*

$$\frac{f(x+t_\varepsilon y)+f(x-t_\varepsilon y)-2f(x)}{t_\varepsilon} < \varepsilon \tag{4.7}$$

for all $y \in X$ *such that* $\|y\| = 1$.

The proof is based on the following lemma:

LEMMA 4.3. *Let* $(X, \|\cdot\|)$ *be a real Banach space and* f *a strongly* $\alpha(\cdot)$*-paraconvex function defined on an open convex subset* $\Omega \subset X$*. Then for* $x \in \Omega$ *and all* $y \in X$ *of norm one*, $0 < s < 1$, *and* $t > 0$ *such that* $x \pm ty \in \Omega$ *we have*

$$\frac{f(x+sty)+f(x-sty)-2f(x)}{st} \le \frac{f(x+ty)+f(x-ty)-2f(x)}{t} + 2\,\frac{\alpha(t)}{t}. \tag{4.8}$$

Proof. Since f is strongly $\alpha(\cdot)$-paraconvex,

$$f(x+sty) \le (1-s)f(x) + sf(x+ty) + s\alpha(t), \tag{4.9}$$

$$f(x-sty) \le (1-s)f(x) + sf(x-ty) + s\alpha(t). \tag{4.10}$$

Adding (4.9) and (4.10) we get

$$f(x+sty)+f(x-sty) \le 2(1-s)f(x) + sf(x+ty) + sf(x-ty) + 2s\alpha(t). \tag{4.11}$$

Thus

$$f(x+sty)+f(x-sty)-2f(x) \le s[f(x+ty)+f(x-ty)-2f(x)] + 2s\alpha(t). \tag{4.12}$$

Dividing (4.12) by st we get (4.8). ■

Proof of Proposition 4.2. The necessity is obvious: it follows from Proposition 4.1 and the fact that each strongly $\alpha(\cdot)$-paraconvex function is $\alpha(\cdot)$-subdifferentiable.

Let $t_\varepsilon > 0$ be such that $t_\varepsilon + 2\alpha(t_\varepsilon)/t_\varepsilon < \varepsilon$. Then Lemma 4.3 and (4.8) yield (4.7). ■

PROPOSITION 4.4. *Let* $(X, \|\cdot\|)$ *be a real Banach space. Let* f *be a strongly* $\alpha(\cdot)$*-paraconvex function defined on an open convex subset* $\Omega \subset X$.

Then the set G (possibly empty) of points $x \in \Omega$ where f is Fréchet differentiable is a G_δ set.

Proof. We set

$$G_m^\delta(f) = \Big\{ x \in \Omega : \bar{B}(x,\delta) \subset \Omega, \\ \sup_{\{y \in X : \|y\|=1\}} \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} < \frac{1}{m} \Big\},$$

where as usual $\bar{B}(x,\delta) = \{y \in X : \|y-x\| \le \delta\}$ denotes the closed ball of radius δ with center at x. Let $G_m(f) = \bigcup_{\delta>0} G_m^\delta(f)$, i.e.

$$G_m(f) = \Big\{ x \in \Omega : \\ \inf_{\substack{\delta>0 \\ \bar{B}(x,\delta)\subset\Omega}} \sup_{\{y \in X : \|y\|=1\}} \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} < \frac{1}{m} \Big\}.$$

We shall show that the sets $G_m^\delta(f)$ are open.

Take any $x \in G_m^\delta(f)$. Since f is strongly $\alpha(\cdot)$-paraconvex, it is locally Lipschitz. Hence there are $\delta_1 > 0$ and $M > 0$ such that $|f(u)-f(v)| \le M\|u-v\|$, provided $u, v \in \bar{B}(x,\delta_1)$. Without loss of generality we may assume that $\bar{B}(x,\delta_1) \subset \Omega$.

Since $x \in G_m^\delta(f)$, there is r such that

$$\sup_{\{y \in X : \|y\|=1\}} \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} \le r < \frac{1}{m}.$$

Let $\delta_2 > 0$ be smaller than $\min\big\{\delta_1, \frac{\delta}{3M}\big(\frac{1}{m}-r\big)\big\}$. Then for any $z \in \bar{B}(x,\delta_2)$ and any y of norm one we have

$$\frac{f(z+\delta y)+f(z-\delta y)-2f(z)}{\delta} \le \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} \\ + \frac{|f(x+\delta y)-f(z+\delta y)|}{\delta} + \frac{|f(x-\delta y)-f(z-\delta y)|}{\delta} + \frac{|f(x)-f(z)|}{\delta} \\ \le r + 3M\delta_2 < 1/m.$$

Hence the set $G_m^\delta(f)$ is open. Therefore so is $G_m(f) = \bigcup_{\delta>0} G_m^\delta(f)$. By Propositions 4.1 and 4.2, $G = \bigcap_m G_m(f)$. Hence G is a G_δ set. ∎

COROLLARY 4.5. *Let $(X, \|\cdot\|)$ be a real Banach space and f a strongly $\alpha(\cdot)$-paraconvex function defined on an open convex subset $\Omega \subset X$. Suppose that the set G of points $x \in \Omega$ where f is Fréchet differentiable is dense in Ω. Then it is a residual set (i.e. its complement is of the first Baire category).*

Proof. Let $F = \Omega \setminus G$ and $F_m = \Omega \setminus G_m$. Since G_m is open, F_m is closed. Since G is dense, so is G_m. Thus F_m is nowhere dense and the set $F = \bigcup_m F_m$ is of the first Baire category. ■

THEOREM 4.6. *Let Ω be an open convex set in an Asplund space X. Let f be a strongly $\alpha(\cdot)$-paraconvex function defined on Ω. Then the set G of points where f is Fréchet differentiable is a dense G_δ set (hence residual).*

Proof. Suppose that G is **not** dense in Ω. We shall show that there is a separable subspace $E \subset X$ such that the set of points of Fréchet differentiability of the restriction $f|_{\Omega \cap E}$ is **not** dense in $\Omega \cap E$.

We denote as before by $G_m(f)$ the set of those x for which there is a $\delta > 0$ such that $\bar{B}(x,\delta) \subset \Omega$ and

$$\sup_{\{y\,:\,\|y\|=1\}} \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} < \frac{1}{m}.$$

By our assumption there are m and an open set $U \subset \Omega$ such that $U \cap G_m(f) = \emptyset$. Thus for $x \in U$ and all $\delta > 0$ such that $\bar{B}(x,\delta) \subset \Omega$,

$$\sup_{\{y\,:\,\|y\|=1\}} \frac{f(x+\delta y)+f(x-\delta y)-2f(x)}{\delta} \ge \frac{1}{m}. \tag{4.13}$$

Take any $x_1 \in U$. Take a decreasing sequence $\{\beta_j\}$ of positive numbers tending to 0 such that $\bar{B}(x_1,\beta_1) \subset \Omega$ and $\bar{B}(x_1,\beta_1) \cap G_m(f) = \emptyset$.

By (4.13) for j large enough we can find an element $y_{1,j}$ of norm one such that

$$\frac{f(x_1+\beta_j y_{1,j})+f(x_1-\beta_j y_{1,j})-2f(x_1)}{\beta_j} > \frac{1}{2m}. \tag{4.14}$$

Take $\delta > 0$ such that $\bar{B}(x,\delta_1) \subset \Omega$ and

$$\frac{\alpha(\delta)}{\delta} > \frac{1}{2m}. \tag{4.15}$$

By Lemma 4.3 and (4.15) there is a constant $\varepsilon_m > 0$ depending only on m such that for $\delta \ge \beta_j$ such that $\bar{B}(x_1,\delta) \subset \Omega$ and

$$\frac{f(x_1+\delta y_{1,j})+f(x_1-\delta y_{1,j})-2f(x_1)}{\delta} > \varepsilon_m. \tag{4.16}$$

Since $\beta_j \to 0$, for all $\delta > 0$ such that $\bar{B}(x_1,\delta) \subset \Omega$ we have

$$\sup_{j\ge 1} \frac{f(x_1+\delta y_{1,j})+f(x_1-\delta y_{1,j})-2f(x_1)}{\delta} > \varepsilon_m. \tag{4.17}$$

We denote by E_1 the closed linear span of the set $\{x_1, y_{1,1}, y_{1,2}, \ldots\}$. Of course $U \cap E_1 \ne \emptyset$ and E_1 is separable. Now we construct by induction a sequence $\{E_1, E_2, \ldots\}$ of separable spaces such that $E_k \subset E_{k+1}$ and $U \cap E_k \ne \emptyset$.

Suppose that we have constructed spaces $E_1, \ldots, E_k$. Let $\{x_{k,p}\}$ be dense in $U \cap E_k$. Then by (4.17) for each $x_{k,p}$ we can find a sequence $\{y_{p,j}\}$ of elements of norm one such that for all p for sufficiently small δ (depending on k and p)

$$\sup_{j\geq 1} \frac{f(x_{k,p}+\delta y_{p,j}) + f(x_{k,p}-\delta y_{p,j}) - 2f(x_{k,p})}{\delta} > \varepsilon_m. \tag{4.18}$$

We denote by E_{k+1} the closed linear span of the set $\{x_{k,p}, y_{p,j} : p = 1,2,\ldots,\ j = 1,2,\ldots\}$. Clearly $U \cap E_{k+1} \neq \emptyset$ and E_{k+1} is separable.

We put $E = \overline{\bigcup_k E_k}$. It is easy to see that the sequence $\{x_{k,p}\}$, $k,p = 1,2,\ldots,$ is dense in E. By construction, it is easy to see that if m_1 is such that $1/m_1 < \varepsilon_m$, then the points $x_{k,p}$, $k,p = 1,2,\ldots,$ do not belong to $G_{m_1}(f|_E)$. Since the set $G_{m_1}(f|_E)$ is open in E and disjoint from $E \cap U$ we conclude that $f|_E$ is not differentiable at any point of the open set $E \cap U$.

This is a contradiction, since $f|_E$ is a strongly $\alpha(\cdot)$-paraconvex function defined on an open set in the separable space E, and by Theorem 3.4 it is differentiable on a residual set. ■

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