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Boundedness of the Hausdorff operators in H^p spaces, $0 < p < 1$

by

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Abstract. Sufficient conditions for the boundedness of the Hausdorff operators in the Hardy spaces H^p, $0 < p < 1$, on the real line are proved. Two related negative results are also given.

1. Introduction. We shall use the following notation throughout this paper. The function space L^p is the L^p space on $\mathbb{R}$ with respect to the Lebesgue measure. The space H^p is the Hardy space on $\mathbb{R}$ defined by Fefferman and Stein [FS, §11]. The Fourier transform is defined by

$$\widehat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

For $x \in \mathbb{R}$, we write $[x]$ to denote the largest integer not exceeding x. We use the letter c with or without subscripts to denote various positive constants, which may be different at different places.

As in [LM], given a function φ on $(0, \infty)$, the *Hausdorff operator* $\mathcal{H}_\varphi$ is defined by

$$(\mathcal{H}_\varphi f)(x) = \int_0^\infty \frac{1}{t} f\left(\frac{x}{t}\right) \varphi(t)\, dt, \tag{1.1}$$

where f denotes a function on $\mathbb{R}$. The assumptions on φ and f will be specified later on. Here we only observe that, if $f \in L^1$ and $\varphi \in L^1(0, \infty)$, then Fubini's theorem gives the formula

$$(\mathcal{H}_\varphi f)^\wedge(\xi) = \int_0^\infty \widehat{f}(t\xi) \varphi(t)\, dt. \tag{1.2}$$

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There is a rather simple result for the Hausdorff operator in L^p, $1 \le p \le \infty$. For these p, Minkowski's inequality in the integral form gives

$$\Big\| \int_0^\infty |t^{-1} f(t^{-1}x)\varphi(t)|\,dt \Big\|_{L^p_x} \le \int_0^\infty t^{-1} \|f(t^{-1}x)\|_{L^p_x} |\varphi(t)|\,dt = \int_0^\infty t^{-1+1/p} \|f\|_{L^p} |\varphi(t)|\,dt = A_{\varphi,p}\|f\|_{L^p},$$

where

$$A_{\varphi,p} = \int_0^\infty t^{-1+1/p} |\varphi(t)|\,dt. \tag{1.3}$$

From this inequality, we see that, if $1 \le p \le \infty$ and $A_{\varphi,p} < \infty$, then (1.1) gives a well-defined bounded linear operator $\mathcal{H}_\varphi$ in L^p.

If $1 \le p \le 2$ and $A_{\varphi,p} < \infty$, then using the L^p boundedness of $\mathcal{H}_\varphi$ and the Hausdorff–Young theorem, $\|\widehat{f}\|_{L^{p/(p-1)}} \le c_p \|f\|_{L^p}$, we easily see that (1.2) holds for all $f \in L^p$.

We shall consider the Hausdorff operator in the Hardy space H^p, $0 < p \le 1$. For H^p with $p = 1$, there is also a simple result. If $A_{\varphi,1} < \infty$ and $f \in H^1$, then $f \in L^1$ (since $H^1 \subset L^1$) and hence $\mathcal{H}_\varphi f$ is well-defined by (1.1) and the formula (1.2) holds. The first named author and Móricz proved the following.

THEOREM A ([LM]). *If $A_{\varphi,1} < \infty$, then the Hausdorff operator $\mathcal{H}_\varphi$ is bounded in H^1.*

In the present paper, we shall mainly consider the Hausdorff operators in H^p with $0 < p < 1$.

We shall give the precise definition of $\mathcal{H}_\varphi f$ for $f \in H^p$ with $0 < p < 1$ following [M2]. We make the following observation. If $f \in H^p$ with $0 < p < 1$, then $\widehat{f}$ is a continuous function satisfying $|\widehat{f}(\xi)| \le c_p \|f\|_{H^p} |\xi|^{1/p-1}$ (see e.g. [S, Chapt. III, §5.4, p. 128]), and hence

$$\int_0^\infty |\widehat{f}(t\xi)\varphi(t)|\,dt \le \int_0^\infty c_p \|f\|_{H^p} |t\xi|^{1/p-1} |\varphi(t)|\,dt = c_p\, A_{\varphi,p} \|f\|_{H^p} |\xi|^{1/p-1}, \tag{1.4}$$

where $A_{\varphi,p}$ for $0 < p < 1$ is given by (1.3) as well. Thus, if $0 < p < 1$, $A_{\varphi,p} < \infty$, and $f \in H^p$, then the right hand side of (1.2) gives a continuous function of $\xi \in \mathbb{R}$ that is uniformly $O(|\xi|^{1/p-1})$ and, hence, the tempered distribution $\mathcal{H}_\varphi f$ is well-defined through (1.2). Thus, including also the case $p = 1$ as mentioned above, we give the following definition.

Definition 1.1. If $0 < p \le 1$ and φ is a measurable function on $(0,\infty)$ with $A_{\varphi,p} < \infty$, then we define the continuous linear mapping $\mathcal{H}_\varphi : H^p \to \mathcal{S}'$ by (1.2).

Kanjin [K] proved the following Theorem B. In this theorem, we extend the function φ, originally defined on $(0,\infty)$, to the whole real line by setting $\varphi(t) = 0$ for $t \le 0$.

Theorem B ([K]). *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose $A_{\varphi,1} < \infty$, $A_{\varphi,2} < \infty$, and suppose $\widehat{\varphi}$ is a function of class C^{2M} on $\mathbb{R}$ with $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(2M)}(\xi)| < \infty$. Then the Hausdorff operator $\mathcal{H}_\varphi$ is bounded in H^p.*

This theorem contains assumptions on $\widehat{\varphi}$. It is the purpose of the present paper to give the conditions on φ, not on $\widehat{\varphi}$, that ensure the boundedness of $\mathcal{H}_\varphi$ in H^p.

Before we give the main results of this paper, we record the following theorem.

Theorem C. *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. If φ is a function on $(0,\infty)$ of class C^M and if $\operatorname{supp}\varphi$ is a compact subset of $(0,\infty)$, then $\mathcal{H}_\varphi$ is a bounded linear operator in H^p.*

In fact, this theorem is a direct corollary of Theorem B; it is easy to see that $\widehat{\varphi}$ satisfies the assumptions of Theorem B if φ satisfies the assumptions of Theorem C. In the next section, we shall give a proof of Theorem C that does not appeal to Theorem B.

Now, our purpose is to improve Theorem C. We shall give results applying to φ that do not have a compact support or are not of class C^M on the whole half line $(0,\infty)$. The following are the main theorems of the present paper.

Theorem 1.2. *Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and $\epsilon > 0$. Suppose φ is a function of class C^M on $(0,\infty)$ such that*

$$|\varphi^{(k)}(t)| \le \min\{t^\epsilon, t^{-\epsilon}\} t^{-1/p-k} \quad \text{for } k = 0, 1, \ldots, M.$$

Then $\mathcal{H}_\varphi$ is a bounded linear operator in H^p.

Theorem 1.3. *Let $0 < p < 1$, $M = [1/p - 1/2] + 1$, and $\epsilon, a > 0$. Suppose φ is a function on $(0,\infty)$ such that $\operatorname{supp}\varphi$ is a compact subset of $(0,\infty)$, φ is of class C^M on $(0,a) \cup (a,\infty)$, and*

$$|\varphi^{(k)}(t)| \le |t-a|^{\epsilon-1-k} \quad \text{for } k = 0, 1, \ldots, M.$$

Then $\mathcal{H}_\varphi$ is a bounded linear operator in H^p.

Proofs of these theorems will be given in the next section.

Our theorems and Kanjin's theorem are mutually independent; Theorems 1.2 and 1.3 do not imply Theorem B, and the latter does not imply the former. However, it is usually easier to check assumptions posed on the function itself than those posed on its Fourier transform. Besides, $2M$ smoothness assumed for the Fourier transform in Theorem B seems too restrictive and probably just fits the method of proof.

In the last section, we shall give two negative results, Theorems 3.2 and 3.3. Theorem 3.2 will show that $\epsilon > 0$ in Theorem 1.2 cannot be removed. Theorem 3.3 will show that, if $0 < p < 1$, then no simple nontrivial condition concerning only the absolute value of φ is sufficient for the H^p-boundedness of the Hausdorff operator $\mathcal{H}_\varphi$. This was discovered by the second named author some time ago, but was never published.

Remark 1.4. In the case where $\varphi(t) = \alpha(1-t)^{\alpha-1}$ for $0 < t < 1$ and $\varphi(t) = 0$ otherwise, the operator $\mathcal{H}_\varphi = \mathcal{C}_\alpha$ is called the *Cesàro operator* of order α. Giang and Móricz [GM] proved that $\mathcal{C}_\alpha$ is bounded in the Hardy space H^1. Kanjin [K] proved that $\mathcal{C}_\alpha$ is bounded in H^p provided α is a positive integer and $2/(2\alpha+1) < p < 1$. He used Theorem B. In [M2], the second named author proved that the Cesàro operator $\mathcal{C}_\alpha$ is bounded in H^p for every $\alpha > 0$ and every $0 < p < 1$. This result is covered by Theorems 1.2 and 1.3. Our proof of Theorems 1.2 and 1.3 will be based on the ideas elaborated in [M2]. A brief history of the study of the Cesàro operator related to some classical problems of Fourier series can be found in [K, §1].

Remark 1.5. We have restricted ourselves to functions φ defined on $(0,\infty)$ only for the sake of simplicity. If φ is defined on the whole $\mathbb{R}$, one can treat its parts on the positive and negative half-axes separately extending them by zero to the other half-axis, and then combine the results.

2. Proofs of Theorems 1.2 and 1.3. Before proceeding to the proofs, we record the following two lemmas. Their proofs are routine and left to the reader.

Lemma 2.1. *If p and φ satisfy the assumptions of Definition* 1.1, *then for every $f \in H^p$ and for every $s > 0$, we have $\mathcal{H}_\varphi(f(s\,\cdot)) = (\mathcal{H}_\varphi f)(s\,\cdot)$.*

Lemma 2.2. *If $f \in H^p$, $0 < p \le 1$, and $s > 0$, then $\|s^{1/p} f(s\,\cdot)\|_{H^p} = \|f\|_{H^p}$.*

To prove Theorems 1.2 and 1.3, we use the one-dimensional version of the modified atomic decomposition for H^p given in [M1].

Definition 2.3 ([M1]). Let $0 < p \le 1$ and let M be a positive integer. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s)$ as the set of all those $f \in L^2$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \le s^{-1}$ and $\|\widehat{f}^{(k)}\|_{L^2} \le s^{k-1/p+1/2}$ for $k = 0, 1, \ldots, M$. We define $\mathcal{A}_{p,M}$ as the union of $\mathcal{A}_{p,M}(s)$ over all $0 < s < \infty$.

LEMMA 2.4 ([M1], [M2]). *Let $0 < p \le 1$ and M be a positive integer satisfying $M > 1/p - 1/2$. Then there exists a constant $c_{p,M}$, depending only on p and M, such that the following hold.*

(1) $\|f(\cdot - x_0)\|_{H^p} \le c_{p,M}$ *for all* $f \in \mathcal{A}_{p,M}$ *and all* $x_0 \in \mathbb{R}$.
(2) *Every* $f \in H^p$ *can be decomposed as*

$$f = \sum_{j=1}^{\infty} \lambda_j f_j(\cdot - x_j), \tag{2.1}$$

where $f_j \in \mathcal{A}_{p,M}$, $x_j \in \mathbb{R}$, $0 \le \lambda_j < \infty$, *and*

$$\Big(\sum_{j=1}^{\infty} \lambda_j^p\Big)^{1/p} \le c_{p,M}\|f\|_{H^p},$$

and the series in (2.1) *converges in* H^p. *If* $f \in H^p \cap L^2$, *then this decomposition can be made so that the series in* (2.1) *converges in* L^2 *as well.*

This lemma is given in [M1, Lemmas 2 and 3] except for the assertion on the L^2 convergence. A complete proof of part (2) can be found in [M2, §3].

The following lemma is easy to see and the proof is left to the reader.

LEMMA 2.5. *Let $0 < p \le 1$, M a positive integer, $s > 0$, and $f \in L^2$. Then $f \in \mathcal{A}_{p,M}(s)$ if and only if $s^{1/p} f(s\,\cdot) \in \mathcal{A}_{p,M}(1)$.*

Now, an essential point in the proofs of Theorems 1.2 and 1.3 is contained in the following lemma.

LEMMA 2.6. *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Let $0 < T < \infty$ and $0 < \delta \le 9/10$. Suppose φ is a function of class C^M on $(0, \infty)$ such that* $\operatorname{supp}\varphi \subset [(1-\delta)T, T]$ *and*

$$|\varphi^{(k)}(t)| \le T^{-1/p-k}\delta^{-1-k} \quad \text{for } k = 0, 1, \ldots, M.$$

Then there exists a constant c_p depending only on p (independent of T and δ) such that

$$\|\mathcal{H}_\varphi f\|_{H^p} \le c_p\|f\|_{H^p}. \tag{2.2}$$

Proof. Throughout this proof, we simply write c to denote positive constants that depend only on p.

Since $\mathcal{H}_\varphi$ is a continuous operator of H^p to $\mathcal{S}'$, it is sufficient to prove the estimate (2.1) for f in a dense subset of H^p. We shall prove (2.1) for $f \in H^p \cap L^2$. By Lemma 2.4(2) and by the L^2 boundedness of $\mathcal{H}_\varphi$, the estimate (2.1) for $f \in H^p \cap L^2$ will follow once we prove the estimate

$$\|\mathcal{H}_\varphi(f(\cdot - x_0))\|_{H^p} \le c \tag{2.3}$$

for all $f \in \mathcal{A}_{p,M}$ and all $x_0 \in \mathbb{R}$.

If $s > 0$, $f \in \mathcal{A}_{p,M}(s)$, and $x_0 \in \mathbb{R}$, and if we set $g(\cdot) = s^{1/p} f(s\,\cdot)$ and $y_0 = s^{-1}x_0$, then $g \in \mathcal{A}_{p,M}(1)$ (by Lemma 2.5),

$$\mathcal{H}_\varphi(f(\cdot - x_0))(x) = \mathcal{H}_\varphi(s^{-1/p}g(s^{-1}\cdot - y_0))(x) = s^{-1/p}\mathcal{H}_\varphi(g(\cdot - y_0))(s^{-1}x),$$

and

$$\|\mathcal{H}_\varphi(f(\cdot - x_0))\|_{H^p} = \|\mathcal{H}_\varphi(g(\cdot - y_0))\|_{H^p},$$

where we have used Lemmas 2.1 and 2.2. Hence the proof is reduced to showing the estimate

$$\|\mathcal{H}_\varphi(g(\cdot - y_0))\|_{H^p} \le c \tag{2.4}$$

for all $g \in \mathcal{A}_{p,M}(1)$ and all $y_0 \in \mathbb{R}$.

We thus assume $g \in \mathcal{A}_{p,M}(1)$ and $y_0 \in \mathbb{R}$ and will prove (2.4). Writing

$$h = \mathcal{H}_\varphi(g(\cdot - y_0)),$$

we have

$$\widehat{h}(\xi) = \int_{(1-\delta)T}^{T} e^{-iy_0 t\xi}\, \widehat{g}(t\xi)\varphi(t)\, dt.$$

To prove (2.4), it suffices to derive $c^{-1}h(\cdot + Ty_0) \in \mathcal{A}_{p,M}(T)$ or, equivalently,

$$c^{-1}T^{1/p}h(T\cdot + Ty_0) \in \mathcal{A}_{p,M}(1). \tag{2.5}$$

Indeed, the latter will imply (2.4) by virtue of Lemma 2.4(1). Writing

$$F(\xi) = [T^{1/p}h(T\cdot + Ty_0)]^\wedge(\xi),$$

we have

$$\begin{aligned} F(\xi) &= T^{1/p-1}[h(\cdot + Ty_0)]^\wedge(T^{-1}\xi) = T^{1/p-1}e^{iy_0\xi}\,\widehat{h}(T^{-1}\xi) \\ &= T^{1/p-1}e^{iy_0\xi} \int_{(1-\delta)T}^{T} e^{-iy_0 tT^{-1}\xi}\, \widehat{g}(tT^{-1}\xi)\varphi(t)\, dt \\ &= \int_{1-\delta}^{1} e^{iy_0\xi(1-t)}\, \widehat{g}(t\xi)T^{1/p}\varphi(Tt)\, dt. \end{aligned}$$

From the last expression and assumption $g \in \mathcal{A}_{p,M}(1)$, it is obvious that $F(\xi) = 0$ for $|\xi| \le 1$.

We will prove $\|F^{(k)}\|_{L^2} \le c$ for $k = 0, 1, \dots, M$. By Leibniz' rule, the derivative $F^{(k)}(\xi)$ can be written as a linear combination of the expressions

$$G(\xi) = \int_{1-\delta}^{1} e^{iy_0\xi(1-t)}(iy_0)^m(1-t)^m t^n\, \widehat{g}^{(n)}(t\xi)T^{1/p}\varphi(Tt)\, dt,$$

where m and n are nonnegative integers satisfying $m + n = k$. We will prove $\|G\|_{L^2} \le c$ for all m, n. We shall consider the two cases $|y_0| \le 1$ and $|y_0| > 1$ separately.

We first proceed to the case $|y_0| \le 1$. By Minkowski's inequality and by the assumption $|\varphi(t)| \le T^{-1/p}\delta^{-1}$, we have

$$\|G\|_{L^2} \le \int_{1-\delta}^{1} |y_0|^m (1-t)^m t^n \|\widehat{g}^{(n)}(t\cdot)\|_{L^2}\, |T^{1/p}\varphi(Tt)|\, dt$$
$$\le \int_{1-\delta}^{1} |y_0|^m (1-t)^m t^n t^{-1/2}\delta^{-1}\, dt \le c.$$

Next, we consider the case $|y_0| > 1$. Since

$$e^{iy_0\xi(1-t)}(iy_0)^m = (-\xi^{-1}\partial_t)^m e^{iy_0\xi(1-t)},$$

we use integration by parts to represent $G(\xi)$ as

$$G(\xi) = \int_{1-\delta}^{1} e^{iy_0\xi(1-t)}(\xi^{-1}\partial_t)^m\big[(1-t)^m t^n \widehat{g}^{(n)}(t\xi)T^{1/p}\varphi(Tt)\big]\, dt.$$

Thus $G(\xi)$ can be written as a linear combination of the expressions

$$\int_{1-\delta}^{1} e^{iy_0\xi(1-t)}\xi^{-m}(1-t)^{m-m_1}t^{n-m_2}\xi^{m_3}\, \widehat{g}^{(n+m_3)}(t\xi)\, T^{1/p+m_4}\varphi^{(m_4)}(Tt)\, dt,$$

where m_1, m_2, m_3, and m_4 are nonnegative integers satisfying $m_1 + m_2 + m_3 + m_4 = m$ and $m_2 \le n$. We denote each such integral by $H(\xi)$. Notice that, since $\operatorname{supp}\widehat{g} \subset \{\xi : |\xi| \ge 1\}$, we have $|\xi^{-m+m_3}| \le 1$ on the support of H. Hence, by Minkowski's inequality and the assumption $|\varphi^{(k)}(t)| \le T^{-1/p-k}\delta^{-1-k}$, we have

$$\|H\|_{L^2} \le \int_{1-\delta}^{1} (1-t)^{m-m_1}t^{n-m_2}\|\widehat{g}^{(n+m_3)}(t\,\cdot)\|_{L^2}\, T^{1/p+m_4}|\varphi^{(m_4)}(Tt)|\, dt$$
$$\le \int_{1-\delta}^{1} \delta^{m-m_1}t^{n-m_2}t^{-1/2}\delta^{-1-m_4}\, dt \le c\delta^{m-m_1-m_4} \le c.$$

Thus we have proved (2.5), and the proof of Lemma 2.6 is complete. ∎

Notice that Theorem C immediately follows from Lemma 2.6. In fact, with the aid of appropriate partition of unity, the case of Theorem C reduces to that of Lemma 2.6.

We are now in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We take a function $\eta \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\eta \subset [2^{-1}, 2]$ and $\sum_{j=-\infty}^{\infty}\eta(t/2^j) = 1$ for all $t > 0$. We decompose $\varphi(t)$ as

$$\varphi(t) = \sum_{j=-\infty}^{\infty}\varphi(t)\eta(t/2^j) = \sum_{j=-\infty}^{\infty}\varphi_j(t).$$

Then, using the inequality (1.4), it is easy to see that $\mathcal{H}_\varphi f$ for $f \in H^p$ can be decomposed as

$$\mathcal{H}_\varphi f = \sum_{j=-\infty}^{\infty} \mathcal{H}_{\varphi_j} f$$

with the series converging in $\mathcal{S}'$.

We have $\operatorname{supp} \varphi_j \subset [2^{j-1}, 2^{j+1}]$, and our assumption on φ yields the estimate

$$\left| \left(\frac{d}{dt}\right)^k \varphi_j(t) \right| \leq c 2^{-|j|\epsilon + j(-1/p-k)}$$

for $k = 0, 1, \ldots, M$. Hence applying Lemma 2.6 to each φ_j with $T = 2^{j+1}$ and $\delta = 3/4$, we obtain

$$\|\mathcal{H}_{\varphi_j} f\|_{H^p} \leq c 2^{-|j|\epsilon}.$$

Now summation over j gives the desired result. ∎

Proof of Theorem 1.3. We define $\varphi^{(1)}$ by

$$\varphi^{(1)}(t) = \begin{cases} \varphi(t) & \text{if } t < a, \\ 0 & \text{if } t > a, \end{cases}$$

and define $\varphi^{(2)} = \varphi - \varphi^{(1)}$. Thus $\mathcal{H}_\varphi = \mathcal{H}_{\varphi^{(1)}} + \mathcal{H}_{\varphi^{(2)}}$.

We first consider $\mathcal{H}_{\varphi^{(1)}}$. Using the same function η as in the proof of Theorem 1.2, we decompose $\varphi^{(1)}$ as

$$\varphi^{(1)}(t) = \sum_{j=-\infty}^{N} \varphi(t)\eta\left(\frac{a-t}{2^j}\right) = \sum_{j=-\infty}^{N} \varphi_j^{(1)}(t),$$

where N is the smallest integer such that $2^N \geq a$. Then $\mathcal{H}_{\varphi^{(1)}} f$ for $f \in H^p$ can be decomposed as

$$\mathcal{H}_{\varphi^{(1)}} f = \sum_{j=-\infty}^{N} \mathcal{H}_{\varphi_j^{(1)}} f$$

with the series converging in $\mathcal{S}'$. We have $\operatorname{supp} \varphi_j^{(1)} \subset [a - 2^{j+1}, a - 2^{j-1}]$ and

$$\left| \left(\frac{d}{dt}\right)^k \varphi_j(t) \right| \leq c 2^{j(\epsilon-1-k)}$$

for $k = 0, 1, \ldots, M$. If j is sufficiently small or, to be precise, if $2^{j+1} \leq 9a/10$, then applying Lemma 2.6 to $\varphi_j^{(1)}$ with $T = a$ and $\delta = 2^{j+1}/a$, we obtain

$$\|\mathcal{H}_{\varphi_j^{(1)}} f\|_{H^p} \leq c 2^{j\epsilon} \|f\|_{H^p}.$$

Now, summing up the estimates obtained yields

$$\Big\| \sum_{2^{j+1} \le 9a/10} \mathcal{H}_{\varphi_j^{(1)}} f \Big\|_{H^p} \le c\|f\|_{H^p}.$$

There are only finitely many j's that satisfy $2^{j+1} > 9a/10$ and $j \le N$. For those j's we can make use of Theorem C. Thus we conclude that $\mathcal{H}_{\varphi^{(1)}} = \sum_{j=-\infty}^{N} \mathcal{H}_{\varphi_j^{(1)}}$ is bounded in H^p.

The estimate for $\mathcal{H}_{\varphi^{(2)}}$ can be obtained in a similar way. We decompose $\varphi^{(2)}$ as

$$\varphi^{(2)}(t) = \sum_{j=-\infty}^{N'} \varphi(t)\eta\Big(\frac{t-a}{2^j}\Big) = \sum_{j=-\infty}^{N'} \varphi_j^{(2)}(t),$$

where N' is the smallest integer such that the interval $[a+2^{N'}, \infty)$ does not intersect $\operatorname{supp}\varphi$. Then $\mathcal{H}_{\varphi^{(2)}} f$ for $f \in H^p$ can be decomposed as

$$\mathcal{H}_{\varphi^{(2)}} f = \sum_{j=-\infty}^{N'} \mathcal{H}_{\varphi_j^{(2)}} f$$

with the series converging in $\mathcal{S}'$. We have $\operatorname{supp}\varphi_j^{(2)} \subset [a+2^{j-1}, a+2^{j+1}]$ and

$$\Big|\Big(\frac{d}{dt}\Big)^k \varphi_j^{(2)}(t)\Big| \le c2^{j(\epsilon-1-k)}$$

for $k = 0, 1, \ldots, M$. If j is sufficiently small, say $j \le -N''$, then applying Lemma 2.6 to $\varphi_j^{(2)}$ with $T = a + 2^{j+1}$ and $\delta = 3 \cdot 2^{j-1}/(a+2^{j+1})$, we obtain

$$\|\mathcal{H}_{\varphi_j^{(2)}} f\|_{H^p} \le c2^{j\epsilon}\|f\|_{H^p}.$$

Now, summation over j gives

$$\Big\| \sum_{j=-\infty}^{-N''} \mathcal{H}_{\varphi_j^{(2)}} f \Big\|_{H^p} \le c\|f\|_{H^p}.$$

There are only finitely many j's that satisfy $-N'' < j \le N'$. As above, using Theorem C for those j's, we conclude that $\mathcal{H}_{\varphi^{(2)}} = \sum_{j=-\infty}^{N'} \mathcal{H}_{\varphi_j^{(2)}}$ is bounded in H^p. This completes the proof of Theorem 1.3. ∎

3. Two negative results. In this section, we shall give two negative results concerning the H^p-boundedness, $0 < p < 1$, of Hausdorff operators.

The first result is that the Hausdorff operator $\mathcal{H}_\varphi$ for $\varphi(t) = t^{-1/p}$ is not bounded in H^p. To prove this, we have to extend the definition of $\mathcal{H}_\varphi$ since $\varphi(t) = t^{-1/p}$ does not satisfy the assumption $A_{\varphi,p} < \infty$ of Definition 1.1.

Lemma 3.1. *If* $0 < p \le 1$, *then there exists a constant* c_p *such that*

$$\Big|\int_0^\infty |\widehat{f}(t\xi)|t^{-1/p}\,dt\Big| \le c_p\|f\|_{H^p}|\xi|^{1/p-1}. \tag{3.1}$$

Proof. We again write $M = [1/p-1/2]+1$. By Lemma 2.4(2), it suffices to prove the estimate

$$\int_0^\infty |\widehat{f}(t\xi)|t^{-1/p}\,dt \le c_p|\xi|^{1/p-1}$$

for all f of the form $f = s^{-1/p}g(s^{-1}(\cdot - x_0))$ with $g \in \mathcal{A}_{p,M}(1)$ and $x_0 \in \mathbb{R}$. For such f, we have

$$\widehat{f}(\xi) = e^{-ix_0\xi}s^{-1/p+1}\,\widehat{g}(s\xi),$$

and hence

$$\begin{aligned}\int_0^\infty |\,\widehat{f}(t\xi)|t^{-1/p}\,dt &= \int_{(s|\xi|)^{-1}}^\infty s^{-1/p+1}\,|\widehat{g}(st\xi)|\,t^{-1/p}\,dt\\ &\le s^{-1/p+1}\|\widehat{g}(s\xi\cdot)\|_{L^2}\Big(\int_{(s|\xi|)^{-1}}^\infty t^{-2/p}\,dt\Big)^{1/2}\\ &\le c_p s^{-1/p+1}(s|\xi|)^{-1/2}(s|\xi|)^{1/p-1/2} = c_p|\xi|^{1/p-1}.\end{aligned}$$

Lemma 3.1 is proved. ■

Using this lemma, we see that for $\varphi(t) = t^{-1/p}$ the Hausdorff operator $\mathcal{H}_\varphi$ is well-defined through (1.2) as a continuous linear operator from H^p to $\mathcal{S}'$.

We then prove the following.

Theorem 3.2. *Let* $0 < p \le 1$ *and* $\varphi(t) = t^{-1/p}$. *Then the Hausdorff operator* $\mathcal{H}_\varphi$ *is not bounded in* H^p.

Proof. Take a function f that has the following properties: f is bounded on $\mathbb{R}$ with compact support,

$$\int_{\mathbb{R}} x^k f(x)\,dx = 0 \quad \text{for } k = 0,1,\ldots,[1/p-1],$$

and

$$A = \int_0^\infty x^{1/p-1}f(x)\,dx \ne 0.$$

Then f is a constant multiple of an H^p-atom and hence $f \in H^p(\mathbb{R})$ (see e.g. [S, p. 106]). For $0 < \epsilon < M < \infty$, define $\varphi_{\epsilon,M}(t)$ by $\varphi_{\epsilon,M}(t) = t^{-1/p}$ for $\epsilon < t < M$ and $\varphi_{\epsilon,M}(t) = 0$ otherwise. Then, with the aid of (3.1), we see that $\mathcal{H}_{\varphi_{\epsilon,M}}(f)$ converges to $\mathcal{H}_\varphi(f)$ in $\mathcal{S}'$ as $\epsilon \to 0$ and $M \to \infty$. On the other hand, for $x > 0$,

$$(\mathcal{H}_{\varphi_{\epsilon,M}} f)(x) = \int_{\epsilon}^{M} t^{-1-1/p} f\left(\frac{x}{t}\right) dt = x^{-1/p} \int_{x/M}^{x/\epsilon} u^{1/p-1} f(u)\, du$$

converges to

$$x^{-1/p} \int_{0}^{\infty} u^{1/p-1} f(u)\, du = Ax^{-1/p}$$

and the convergence is uniform on every compact subset of $(0,\infty)$. Hence the distribution $\mathcal{H}_\varphi f$ coincides with the function $Ax^{-1/p}$ on $(0,\infty)$. Since the last function is not of class L^p on $(0,\infty)$, we see that $\mathcal{H}_\varphi f$ does not belong to H^p. ∎

Our second negative result reads as follows.

THEOREM 3.3. *There exists a function φ on $(0,\infty)$ that has the following properties*: *φ is bounded*, $\operatorname{supp}\varphi$ *is a compact subset of* $(0,\infty)$, *and*, *for every $p \in (0,1)$, the Hausdorff operator $\mathcal{H}_\varphi$ is not bounded in H^p.*

In the rest of this paper, we shall use the following notation: if $P(x,y,\dots)$ is a proposition containing variables $x,y,\dots$, then we define $\mathbf{1}\{P(x,y,\dots)\}$ to be 1 if $P(x,y,\dots)$ is true, and 0 otherwise.

To prove Theorem 3.3, we use the following function. Let $0 < a < b < \infty$ and let N be a positive integer satisfying $N \ge 2$. We write $h = b - a$ and

$$t_j = a + jh/N \qquad (j = 1,\dots,N-1)$$

and define the function $\psi_{a,b,N}(t)$ for $t \in (0,\infty)$ by

$$\psi_{a,b,N}(t) = \sum_{j=1}^{N-1} (-t\mathbf{1}\{t_j - h/2N < t < t_j\} + t\mathbf{1}\{t_j < t < t_j + h/2N\}).$$

For this function, we have the following.

LEMMA 3.4. *Let $0 < a < b < \infty$ and let N be a positive integer satisfying $N \ge 2$. Let φ be a bounded function on $(0,\infty)$ such that* $\operatorname{supp}\varphi$ *is a compact subset of* $(0,\infty)$ *and $\varphi(t) = \psi_{a,b,N}(t)$ if $t \in (a,b)$. Then, for each p satisfying $1/2 < p < 1$, we have*

$$\sup\{\|\mathcal{H}_\varphi f\|_{H^p}/\|f\|_{H^p} \mid f \in H^p,\ f \neq 0\} \ge c_0(a,b,p)N^{(1-p)/p}, \tag{3.2}$$

where $c_0(a,b,p)$ is a positive constant depending only on a, b, and p.

Proof. We take an ϵ such that $0 < \epsilon < h/8b$ and define the function $f_{N,\epsilon}$ on $\mathbb{R}$ by

$$f_{N,\epsilon}(x) = \mathbf{1}\{N - \epsilon < x < N\} - \mathbf{1}\{N < x < N + \epsilon\}.$$

We take a p satisfying $1/2 < p < 1$. Since $f_{N,\epsilon}$ is a constant multiple of an H^p-atom, we have

$$\|f_{N,\epsilon}\|_{H^p} = c_1(p,\epsilon) \in (0,\infty). \tag{3.3}$$

Notice that $\|f_{N,\epsilon}\|_{H^p}$ does not depend on N. We shall prove (3.2) by testing $f = f_{N,\epsilon}$.

We take a number δ such that $0 < \delta < 1/4$ and $\epsilon a/4 > \delta h$. We shall consider $(\mathcal{H}_\varphi f_{N,\epsilon})(x)$ for x in the intervals

$$Nt_j < x < Nt_j + \delta h \quad (j = 1, \dots, N-1). \tag{3.4}$$

For these x, we have

$$\begin{aligned}
&t_j < \frac{x}{N} < t_j + \frac{\delta h}{N} < t_j + \frac{h}{4N}, \\
&\frac{x}{N-\epsilon} - \frac{x}{N} < \frac{2\epsilon x}{N^2} < \frac{2\epsilon b}{N} < \frac{h}{4N}, \\
&\frac{x}{N} - \frac{x}{N+\epsilon/2} > \frac{\epsilon x}{4N^2} > \frac{\epsilon a}{4N} > \frac{\delta h}{N}, \\
&\frac{x}{N} - \frac{x}{N+\epsilon} < \frac{\epsilon x}{N^2} < \frac{\epsilon b}{N} < \frac{h}{8N},
\end{aligned}$$

and hence

$$t_j - \frac{h}{8N} < \frac{x}{N+\epsilon} < \frac{x}{N+\epsilon/2} < t_j < \frac{x}{N} < \frac{x}{N-\epsilon} < t_j + \frac{h}{2N}.$$

Using then the formula (1.1), we have

$$\begin{aligned}
(\mathcal{H}_\varphi f_{N,\epsilon})(x) &= -\int_{x/(N+\epsilon)}^{x/N} \frac{\varphi(t)}{t}\,dt + \int_{x/N}^{x/(N-\epsilon)} \frac{\varphi(t)}{t}\,dt \\
&= -\int_{x/(N+\epsilon)}^{x/N} \frac{\psi_{a,b,N}(t)}{t}\,dt + \int_{x/N}^{x/(N-\epsilon)} \frac{\psi_{a,b,N}(t)}{t}\,dt \\
&= \int_{x/(N+\epsilon)}^{x/(N+\epsilon/2)} dt - \int_{x/(N+\epsilon/2)}^{x/N} \frac{\psi_{a,b,N}(t)}{t}\,dt + \int_{x/N}^{x/(N-\epsilon)} dt \\
&= I - II + III, \quad \text{say.}
\end{aligned}$$

Obviously $I > 0$. For II and III, we have

$$|II| \le \frac{x}{N} - \frac{x}{N+\epsilon/2} < \frac{\epsilon x}{2N^2}, \qquad III = \frac{x}{N-\epsilon} - \frac{x}{N} > \frac{\epsilon x}{N^2}.$$

Thus, for x satisfying (3.4), we have

$$(\mathcal{H}_\varphi f_{N,\epsilon})(x) > \frac{\epsilon x}{2N^2} > \frac{\epsilon a}{2N}.$$

Hence

$$\|\mathcal{H}_\varphi f_{N,\epsilon}\|_{H^p}^p \ge \sum_{j=1}^{N-1} \int_{Nt_j}^{Nt_j+\delta h} |\mathcal{H}_\varphi f_{N,\epsilon}(x)|^p\,dx \tag{3.5}$$
$$\ge \sum_{j=1}^{N-1} \left(\frac{\epsilon a}{2N}\right)^p \delta h \ge \left(\frac{\epsilon a}{2}\right)^p \frac{\delta h}{2}\, N^{1-p}.$$

Now (3.3) and (3.5) yield (3.2). ∎

Proof of Theorem 3.3. Take a sequence $\{(a_n, b_n)\}$ of disjoint intervals such that $1 \le a_n < b_n \le 2$. Take a sequence $\{p_n\}$ such that $1/2 < p_n < 1$ and $\lim_{n\to\infty} p_n = 1$. Take positive integers N_n such that $N_n \ge 2$ and

$$c_0(a_n, b_n, p_j) N_n^{(1-p_j)/p_j} > n \quad \text{for } j = 1, \ldots, n, \tag{3.6}$$

where $c_0(a, b, p)$ is the constant in (3.2). We shall prove that the function

$$\varphi(t) = \sum_{n=1}^{\infty} \psi_{a_n,b_n,N_n}(t)$$

has the desired properties.

Since the intervals (a_n, b_n) are disjoint and contained in $[1, 2]$, it is obvious that φ is bounded and $\operatorname{supp}\varphi \subset [1, 2]$. For positive integers j and n satisfying $n \ge j$, Lemma 3.4 and (3.6) give

$$\sup\{\|\mathcal{H}_\varphi f\|_{H^{p_j}}/\|f\|_{H^{p_j}} \mid f \in H^{p_j},\ f \ne 0\} \ge c_0(a_n, b_n, p_j) N_n^{(1-p_j)/p_j} > n.$$

This implies that $\mathcal{H}_\varphi$ is not bounded in H^{p_j} for each positive integer j. On the other hand, $\mathcal{H}_\varphi$ is bounded in H^1 by virtue of Theorem A. Hence $\mathcal{H}_\varphi$ cannot be bounded in H^p for any $p < 1$, since otherwise interpolation would imply that $\mathcal{H}_\varphi$ is bounded in H^{p_j} for p_j close enough to 1. ∎

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