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# Isometries between groups of invertible elements in Banach algebras

by

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**Abstract.** We show that if $T$ is an isometry (as metric spaces) from an open subgroup of the group of invertible elements in a unital semisimple commutative Banach algebra $A$ onto a open subgroup of the group of invertible elements in a unital Banach algebra $B$, then $T(1)^{-1}T$ is an isometrical group isomorphism. In particular, $T(1)^{-1}T$ extends to an isometrical real algebra isomorphism from $A$ onto $B$.

**1. Introduction.** A long tradition of inquiry seeks sufficient sets of conditions on (not only linear) isometries between Banach algebras in order that the algebras are algebraically isomorphic. The history of the problem probably dates back to a theorem of Banach [1, Theorem XI. 3], which is the original form of the Banach–Stone theorem. The latter states that the Banach spaces $C(X)$ and $C(Y)$ of complex-valued continuous functions on compact Hausdorff spaces $X$ and $Y$ respectively are isomorphic as Banach spaces if and only if $X$ and $Y$ are homeomorphic to each other, therefore if and only if $C(X)$ and $C(Y)$ are isomorphic as Banach algebras. One can say that the multiplication in the Banach algebra $C(X)$ can be recovered from the Banach space structure in the category of $C(K)$-spaces. Jarosz [3] generalized the theorem in the sense that the multiplication in a uniform algebra can be recovered from the Banach space structure in the category of unital Banach algebras (cf. [10, 4, 5]).

In this paper we consider a problem in the same vein. Suppose that $B$ is a unital Banach algebra. We say that the metric group structure of the group $B^{-1}$ of all invertible elements in $B$ can be recovered from the metric structure in the category of unital Banach algebras if $B^{-1}$ is isometrically isomorphic as a metric group to $B_1^{-1}$ whenever $B_1$ is a unital Banach algebra and $B^{-1}$ is isometric to $B_1^{-1}$ as a metric space. In this paper we show that the metric group structure of the group of invertible elements in a unital

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semisimple commutative Banach algebra can be recovered from the metric structure in the category of unital Banach algebras (Theorem 3.3). In this case the multiplication in the unital semisimple commutative Banach algebra can also be recovered. On the other hand, there exists a unital semisimple commutative Banach algebra whose multiplication cannot be recovered from the Banach space structure in the category of unital Banach algebras (see Example 3.2).

Throughout the paper we denote the unit element in a Banach algebra by 1, and for a complex number $\lambda$, $\lambda 1$ is abbreviated by $\lambda$. The maximal ideal space of a unital semisimple commutative Banach algebra $A$ is denoted by $\Phi_A$. We may suppose that $f \in A$ is a continuous function on $\Phi_A$ by identifying $f$ with its Gelfand transform. The spectral radius of $f \in A$ is equal to the supremum norm of $f$ on $\Phi_A$ and is denoted by $\|f\|_\infty$.

**2. Lemmata.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be real normed spaces. The theorem of Mazur and Ulam [9, 11] states that if they are isometric as metric spaces, then they are isometrically isomorphic as real normed spaces. Applying an idea of Väisälä [11], we prove the following local version.

LEMMA 2.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be real normed spaces, and $U_1$ and $U_2$ nonempty open subsets of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Suppose that $\mathcal{T}$ is a surjective isometry from $U_1$ onto $U_2$. If $f, g \in U_1$ satisfy $(1-r)f + rg \in U_1$ for every $r$ with $0 \le r \le 1$, then*

$$\mathcal{T}\left(\frac{f+g}{2}\right) = \frac{\mathcal{T}(f)+\mathcal{T}(g)}{2}.$$

*Proof.* Let $h, h' \in U_1$. Suppose that $\varepsilon > 0$ is such that $\|h - h'\|/2 < \varepsilon$, and

$$\{u \in B_1 : \|u - h\| < \varepsilon,\ \|u - h'\| < \varepsilon\} \subset U_1,$$
$$\{a \in B_2 : \|a - \mathcal{T}(h)\| < \varepsilon,\ \|a - \mathcal{T}(h')\| < \varepsilon\} \subset U_2.$$

We will show that $\mathcal{T}\big(\frac{h+h'}{2}\big) = \frac{\mathcal{T}(h)+\mathcal{T}(h')}{2}$. Set $r = \|h - h'\|/2$ and let

$$L_1 = \{u \in \mathcal{B}_1 : \|u - h\| = r = \|u - h'\|\},$$
$$L_2 = \{a \in \mathcal{B}_2 : \|a - \mathcal{T}(h)\| = r = \|a - \mathcal{T}(h')\|\}.$$

Set also $c_1 = (h + h')/2$ and $c_2 = (\mathcal{T}(h) + \mathcal{T}(h'))/2$. Then $\mathcal{T}(L_1) = L_2$, $c_1 \in L_1 \subset U_1$, and $c_2 \in L_2 \subset U_2$. Let

$$\psi_1(x) = h + h' - x \qquad (x \in \mathcal{B}_1),$$
$$\psi_2(y) = \mathcal{T}(h) + \mathcal{T}(h') - y \qquad (y \in \mathcal{B}_2).$$

Then we see that $\psi_1(c_1) = c_1$, $\psi_1(L_1) = L_1$, and $\psi_2(L_2) = L_2$. Let $Q = \psi_1 \circ \mathcal{T}^{-1} \circ \psi_2 \circ \mathcal{T}$. A simple calculation shows that

$$2\|w - c_1\| = \|\psi_1(w) - w\| \qquad (w \in L_1)$$

and

$$\|\psi_1(z) - w\| = \|\psi_1 \circ Q^{-1}(z) - Q(w)\| \quad (z, w \in L_1).$$

Applying these equalities we see that

$$\begin{aligned}\|Q^{2^{k+1}}(c_1) - c_1\| &= \|\psi_1 \circ Q^{2^{k+1}}(c_1) - c_1\| \\ &= \|\psi_1 \circ Q^{2^k}(c_1) - Q^{2^k}(c_1)\| = 2\|Q^{2^k}(c_1) - c_1\|\end{aligned}$$

for every non-zero integer $k$, where $Q^{2^n}$ denotes the $2^n$-fold composition of $Q$. By induction we see that for every non-negative integer $n$,

$$\|Q^{2^n}(c_1) - c_1\| = 2^{n+1}\|c_2 - \mathcal{T}(c_1)\|.$$

Since $Q(L_1) = L_1$ and $L_1$ is bounded we see that $c_2 = \mathcal{T}(c_1)$, i.e., $\mathcal{T}\big(\frac{h+h'}{2}\big) = \frac{\mathcal{T}(h)+\mathcal{T}(h')}{2}$.

For $f$ and $g$ as in the statement, let

$$K = \{(1-r)f + rg : 0 \le r \le 1\}.$$

Since $K$ and $\mathcal{T}(K)$ are compact, there is $\varepsilon > 0$ with

$$d(K, \mathcal{B}_1 \setminus U_1) > \varepsilon, \qquad d(\mathcal{T}(K), \mathcal{B}_2 \setminus U_2) > \varepsilon,$$

where $d(\cdot,\cdot)$ denotes the distance of two sets. Then for every $h \in K$ we have

$$\{u \in \mathcal{B}_1 : \|u - h\| < \varepsilon\} \subset U_1, \qquad \{b \in \mathcal{B}_2 : \|b - \mathcal{T}(h)\| < \varepsilon\} \subset U_2.$$

Choose a natural number $n$ with $\|f - g\|/2^n < \varepsilon$. Let

$$h_k = \frac{k}{2^n}(g - f) + f$$

for each $0 \le k \le 2^n$. By the first part of the proof we have

$$\mathcal{T}(h_k) + \mathcal{T}(h_{k+2}) - 2\mathcal{T}(h_{k+1}) = 0 \tag{$k$}$$

for $0 \le k \le 2^n - 2$. For $0 \le k \le 2^n - 4$, adding the equations $(k)$, $(k+1)$ multiplied by 2, and $(k+2)$ we have

$$\mathcal{T}(h_k) + \mathcal{T}(h_{k+4}) - 2\mathcal{T}(h_{k+2}) = 0,$$

whence the equality

$$\mathcal{T}\left(\frac{f+g}{2}\right) = \frac{\mathcal{T}(f) + \mathcal{T}(g)}{2}$$

by induction on $n$. ■

Let $B$ be a unital Banach algebra. The *exponential spectrum* for $a \in B$ is

$$\sigma_{\exp B}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \exp B\},$$

where $\exp B$ denotes the principal component of $B^{-1}$; it is the set of $\exp a$ for all $a \in B$ when $B$ is commutative, and the set of all finite products of elements of the form $\exp a$ for $a \in B$ in general. A complex-valued function

$\varphi$ on $B$ is said to be a *selection from the exponential spectrum* if $\varphi(a) \in \sigma_{\exp B}(a)$ whenever $a \in B$.

To prove Theorem 3.3 below, we apply a lemma concerning complex-linearity of real-linear selections from the exponential spectrum, which is a version of a result due to Kowalski and Słodkowski [8, Lemma 2.1].

LEMMA 2.2. *Let $B$ be a unital Banach algebra. Suppose that $\varphi : B \to \mathbb{C}$ is a real-linear selection from the exponential spectrum. Then $\varphi$ is a complex homomorphism.*

*Proof.* A proof is similar to that for [8, Lemma 2.1]; however, in [8] the spectral maping theorem is applied, which is not suitable for the exponential spectrum, so we apply an alternative. For $x \in B$, let

$$\varphi_1(x) = \operatorname{Re}\varphi(x) - i\operatorname{Re}\varphi(ix), \qquad \varphi_2(x) = \operatorname{Im}\varphi(ix) + i\operatorname{Im}\varphi(x).$$

As in the proof of [8, Lemma 2.1], $\varphi_1$ and $\varphi_2$ are complex-linear selections from the exponential spectrum, hence they are complex homomorphisms by the original proof of the Gleason–Kahane–Żelazko theorem [7, 12] (cf. [6]). We will show that $\varphi_1 = \varphi_2$, which will force that $\varphi$ is a complex homomorphism. Suppose not. Then there is $a \in A$ with $\varphi_1(a) = 1$ and $\varphi_2(a) = 0$. Let

$$h(a) = \exp(\pi i a/2) - 1.$$

Since $\varphi_1$ and $\varphi_2$ are continuous, we see that

$$\begin{aligned}\varphi(h(a)) &= \operatorname{Re}\varphi_1(h(a)) + i\operatorname{Im}\varphi_2(h(a)) \\ &= \operatorname{Re}h(\varphi_1(a)) + i\operatorname{Im}h(\varphi_2(a)) = -1.\end{aligned}$$

Since $\varphi$ is a selection from the exponential spectrum, $-1 \in \sigma_{\exp B}(h(a))$. On the other hand, $h(a) + 1 \in \exp B$, so that $-1 \notin \sigma_{\exp B}(h(a))$, which is a contradiction proving that $\varphi_1 = \varphi_2$. ∎

## 3. Main results

THEOREM 3.1. *Let $A$ be a unital semisimple commutative Banach algebra, $B$ a unital Banach algebra, and*

$$\begin{aligned}\Omega_A &= \{f \in A : \|f - r\| < r \text{ for some positive real number } r\},\\ \Omega_B &= \{a \in B : \|a - r\| < r \text{ for some positive real number } r\}.\end{aligned}$$

*Suppose that $U$ is an open set such that $\Omega_A \subset U \subset A^{-1}$ and $(\mathbb{C}\setminus\{0\})U \subset U$, and that $V$ is an open set such that $\Omega_B \subset V \subset B^{-1}$, $(\mathbb{C}\setminus\{0\})V \subset V$, and $V\Omega_B \subset V$. Let $g \in B^{-1}$. If $T$ is a surjective isometry from $U$ onto $gV$, then $T$ extends to a real-linear isometry from $A$ onto $B$.*

*Proof.* Applying Lemma 2.1 we see that

$$T\left(\frac{f+g}{2}\right)=\frac{T(f)+T(g)}{2}$$

for every pair $f$ and $g$ in $\Omega_A$ since $\Omega_A$ is convex.

We will show that $\lim_{U\ni f\to 0} T(f)=0$. Since $T^{-1}$ is an isometry, the limit $u=\lim_{gV\ni a\to 0} T^{-1}(a)$ exists by a routine argument on Cauchy sequences. Then $\sigma(u)=\{0\}$. [Suppose not; let $0\neq\lambda\in\sigma(u)$. Then $-\lambda\in U$ since $|\lambda|\in\Omega_A$ and $(\mathbb{C}\setminus\{0\})U\subset U$. Let $T(-\lambda)=c_\lambda\in gV$. The inequality $(1-s)(1-r)+sr>0$ for all $0<r<1$ and $0\le s\le 1$, hence

$$(1-s)\{(1-r)c_\lambda\}+src_\lambda\in gV.$$

Applying Lemma 2.1 with $f=(1-r)c_\lambda$, $g=rc_\lambda$ we have

$$T^{-1}\left(\frac{c_\lambda}{2}\right)=T^{-1}\left(\frac{(1-r)c_\lambda+rc_\lambda}{2}\right)=\frac{T^{-1}((1-r)c_\lambda)+T^{-1}(rc_\lambda)}{2}.$$

Letting $r\to 0$ we have

$$T^{-1}\left(\frac{c_\lambda}{2}\right)=\frac{-\lambda+u}{2},$$

which is a contradiction since $T^{-1}(c_\lambda/2)\in U\subset A^{-1}$ and $(-\lambda+u)/2\notin A^{-1}$ for $\lambda\in\sigma(u)$.] Since $A$ is semisimple and commutative, we see that $u=0$, that is, $\lim_{gV\ni a\to 0} T^{-1}(a)=0$. It turns out that

$$\lim_{U\ni f\to 0} T(f)=0 \tag{3.1}$$

since $T$ is isometry.

Next we will show that $T(-f)=-T(f)$ for every $f\in U$. Let $f\in U$. Then $-f\in U$, and for every integer $n$, $-f+(i/n)f\in U$. Moreover,

$$(1-r)f+r\left(-f+\frac{i}{n}f\right)\in U$$

for every $0\le r\le 1$ and every integer $n$. Then by Lemma 2.1,

$$T\left(\frac{i}{2n}f\right)=T\left(\frac{f+(-f+\frac{i}{n}f)}{2}\right)=\frac{T(f)+T(-f+\frac{i}{n}f)}{2}.$$

Letting $n\to\infty$ we have $T(-f)=-T(f)$ by (3.1).

Next we will show that

$$T\left(\frac{f}{2}\right)=\frac{T(f)}{2} \tag{3.2}$$

for every $f\in U$. Let $f\in U$. Then for every $1>\varepsilon>0$ and every $0\le r\le 1$,

$$(1-r)f+r\varepsilon f\in U.$$

Hence $T\left(\frac{f+\varepsilon f}{2}\right)=\frac{T(f)+T(\varepsilon f)}{2}$ by Lemma 2.1, and letting $\varepsilon\to 0$ yields (3.2).

Let $f \in U$. Suppose that $T(kf) = kT(f)$ for a positive integer $k$. Then

$$T\left(\frac{f+kf}{2}\right) = \frac{T(f)+T(kf)}{2} = \frac{(k+1)T(f)}{2},$$

and by (3.2),

$$T\left(\frac{f+kf}{2}\right) = \frac{T((k+1)f)}{2},$$

hence by induction $T(nf) = nT(f)$ for every positive integer $n$. Then for any positive integers $m$ and $n$,

$$mT\left(\frac{n}{m}f\right) = T\left(m\frac{n}{m}f\right) = T(nf) = nT(f),$$

so $T(\frac{n}{m}f) = \frac{n}{m}T(f)$. By continuity of $T$, $T(rf) = rT(f)$ for every $f \in U$ and $r > 0$. Hence

$$T(rf) = rT(f) \tag{3.3}$$

for every $f \in U$ and any non-zero real number $r$ since $T(-f) = -T(f)$.

Applying Lemma 2.1 and (3.2) we see that

$$T(f+g) = T(f) + T(g) \tag{3.4}$$

for every pair $f$ and $g$ in $U$ whenever $(1-r)f + rg \in U$ for every $0 \le r \le 1$. In particular, (3.4) holds if $f, g \in \Omega_A$.

Define the map $T_U : A \to B$ by $T_U(0) = 0$ and

$$T_U(f) = T(f + 2\|f\|) - T(2\|f\|)$$

for a non-zero $f \in A$. The map $T_U$ is well-defined since $f + 2\|f\|$ and $2\|f\|$ are in $\Omega_A$ for every non-zero $f \in A$ and $T$ is defined on $U \supset \Omega_A$. If, in particular, $f \in \Omega_A$, then $T(f + 2\|f\|) = T(f) + T(2\|f\|)$, so that $T_U(f) = T(f)$.

We will show that $T_U$ is real-linear. Let $f \in A \setminus \{0\}$. Then $f + r \in \Omega_A$ for every $r \ge 2\|f\|$, whence by (3.4),

$$T(f + 2\|f\|) + T(r) = T(f + 2\|f\| + r) = T(f + r) + T(2\|f\|),$$

so that

$$T_U(f) = T(f + r) - T(r) \tag{3.5}$$

for every $r \ge 2\|f\|$. Let $f, g \in A$. Then $T_U(f+g) = T_U(f) + T_U(g)$ if $f = 0$ or $g = 0$. Suppose that $f \ne 0$ and $g \ne 0$. Then by (3.4) and (3.5) we have

$$\begin{aligned} T_U(f+g) &= T(f + g + 2\|f\| + 2\|g\|) - T(2\|f\| + 2\|g\|) \\ &= T(f + 2\|f\|) + T(g + 2\|g\|) - T(2\|f\|) - T(2\|g\|) \\ &= T_U(f) + T_U(g). \end{aligned}$$

If $f = 0$ or $r = 0$ then $T_U(rf) = rT_U(f)$. Suppose that $f \ne 0$ and $r \ne 0$. If

$r > 0$, then by (3.3),

$$\begin{aligned} T_U(rf) &= T(rf + 2\|rf\|) - T(2\|rf\|) \\ &= T(r(f + 2\|f\|)) - T(r2\|f\|) \\ &= rT(f + 2\|f\|) - rT(2\|f\|) = rT_U(f). \end{aligned}$$

If $r < 0$, then

$$T_U(rf) = (-r)(T(-f + 2\|f\|) - T(2\|f\|)).$$

Since $-f + 2\|f\|$, $f + 2\|f\| \in \Omega_A$ we have

$$T(-f + 2\|f\|) - T(2\|f\|) = -T(f + 2\|f\|) + T(2\|f\|).$$

It follows that

$$T_U(rf) = (-r)(-T(f + 2\|f\|) + T(2\|f\|)) = rT_U(f).$$

We now show that $T_U$ is surjective. Let $a \in B$. Then

$$(T(1))^{-1}a + r \in \Omega_B \subset V,$$

so

$$a + T(r) = a + rT(1) \in T(1)\Omega_B \subset gV\Omega_B \subset gV$$

whenever $\|(T(1))^{-1}a\| < r$ and $\|a\| < r$ for $T(1) \in gV$. We also have

$$\|T^{-1}(a + T(r)) - r\| = \|a + T(r) - T(r)\| < r,$$

thus $T^{-1}(a + T(r)) \in \Omega_A$. Let $f = T^{-1}(a + T(r)) - r \in A$. Then $f + r = T^{-1}(a + T(r)) \in \Omega_A$. Hence by (3.4) we see that

$$T(f + r) + T(2\|f\|) = T(f + 2\|f\| + r) = T(f + 2\|f\|) + T(r),$$

so we have

$$a = T(f + r) - T(r) = T(f + 2\|f\|) - T(2\|f\|) = T_U(f).$$

We next show that $T_U$ is an isometry. Since $T_U$ is linear, it is sufficient to show that $\|T_U(f)\| = \|f\|$ for every $f \in A$. If $f = 0$, the equatlity clearly holds. Suppose that$f \neq 0$. Then

$$\|T_U(f)\| = \|T(f + 2\|f\|) - T(2\|f\|)\| = \|f + 2\|f\| - 2\|f\|\,\| = \|f\|.$$

Finally, we show that $T_U$ is an extension of $T$, i.e., $T_U(f) = T(f)$ for every $f \in U$. Put $P = T_U^{-1} \circ T : U \to A$. Let $f \in U$. Then $P(f + 2\|f\|) = f + 2\|f\|$ since $f + 2\|f\| \in \Omega_A$ and $T = T_U$ on $\Omega_A$. Thus we have

$$\begin{aligned} 2\|f\| &= \|T(f + 2\|f\|) - T(f)\| \\ &= \|f + 2\|f\| - P(f)\| \geq \|P(f) - f - 2\|f\|\,\|_\infty, \end{aligned}$$

so that the range of $P(f) - f$ on the maximal ideal space $\Phi_A$ is contained in the closed disk in the complex plane with radius $2\|f\|$ and center $2\|f\|$. Applying $P(-f + 2\|f\|) = -f + 2\|f\|$ in the same way yields

$$2\|f\| \geq \|P(f) - f + 2\|f\|\,\|_\infty,$$

since $T(-f) = -T(f)$, so that the range of $P(f) - f$ is in the closed unit disk with radius $2\|f\|$ and center $-2\|f\|$. It follows that $\sigma(P(f) - f) = (P(f) - f)(\Phi_A) = \{0\}$. Since $A$ is semisimple and commutative, we see that $P(f) = f$ for every $f \in U$; hence $T_U(f) = T(f)$ for every $f \in U$. ∎

Two unital semisimple commutative Banach algebras which are isometrically isomorphic to each other as Banach spaces need not be isometrically isomorphic to each other as Banach algebras.

Example 3.2. Let $W$ be the Wiener algebra,

$$W = \Big\{ f \in C(\mathbb{T}) : \|f\| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \Big\},$$

where $\mathbb{T}$ is the unit circle in the complex plane and $\hat{f}(n)$ denotes the $n$th Fourier coeficient, and let

$$W_+ = \{f \in W : \hat{f}(n) = 0 \text{ for every } n < 0\}.$$

Then

$$(T_W(f))(e^{i\theta}) = \sum_{n=0}^{\infty} \hat{f}(n) e^{2ni\theta} + \sum_{n=1}^{\infty} \hat{f}(-n) e^{(2n-1)i\theta}$$

defines an isometrical Banach space isomorphism from $W$ onto $W_+$. On the other hand, $W$ is not isomorphic to $W_+$ as a complex algebra since the maximal ideal space of $W$ is $\mathbb{T}$ and that of $W_+$ is the closed unit disk, which is not homeomorphic to $\mathbb{T}$.

Novertheless, isometries between two groups of invertible elements in unital semisimple commutative Banach algebras induce isometrical group isomorphisms.

Theorem 3.3. *Let $A$ be a unital semisimple commutative Banach algebra and $B$ a unital Banach algebra. Let $\mathfrak{A}$ and $\mathfrak{B}$ be open subgroups of $A^{-1}$ and $B^{-1}$ respectively. Suppose that $T$ is a surjective isometry* (*as a map between metric spaces*) *from $\mathfrak{A}$ onto $\mathfrak{B}$. Then $B$ is semisimple and commutative, and $(T(1))^{-1}T$ extends to an isometrical real algebra isomorphism from $A$ onto $B$. In particular, $A^{-1}$ is isometrically isomorphic to $B^{-1}$ as a metric group.*

*Proof.* Since $\mathfrak{A}$ (resp. $\mathfrak{B}$) is an open subgroup of $A^{-1}$ (resp. $B^{-1}$), $\exp A \subset \mathfrak{A}$ (resp. $\exp B \subset \mathfrak{B}$), whence $\Omega_A \subset \mathfrak{A}$ (resp. $\Omega_B \subset \mathfrak{B}$). Applying Theorem 3.1 with $U = \mathfrak{A}$, $V = \mathfrak{B}$, and $g = 1$, we obtain a surjective real-linear isometry $T_U$ from $A$ onto $B$ which is an extension of $T$.

We now show that $|T_U^{-1}(1)| = 1$ on $\Phi_A$. Since $T_U^{-1}$ is a linear isometry, we have $\|T_U^{-1}(1)\| = 1$, hence $|T_U^{-1}(1)| \le 1$ on $\Phi_A$. Suppose that there exists

$x \in \Phi_A$ such that $|T_U^{-1}(1)(x)| < 1$. Since $T_U$ is an isometry,

$$1 > |T_U^{-1}(1)(x)| = \|T_U^{-1}(1) - (T_U^{-1}(1) - (T_U^{-1}(1))(x))\| \\ = \|1 - T_U(T_U^{-1}(1) - (T_U^{-1}(1))(x))\|,$$

so that $T_U(T_U^{-1}(1) - (T_U^{-1}(1))(x)) \in \exp B \subset \mathfrak{B}$. Since $T_U = T$ on $\mathfrak{B}$ we have

$$T_U^{-1}(1) - (T_U^{-1}(1))(x) \in \mathfrak{A} \subset A^{-1},$$

which is a contradiction since $(T_U^{-1}(1) - (T_U^{-1}(1))(x))(x) = 0$. Hence

$$|T_U^{-1}(1)| = 1 \text{ on } \Phi_A. \tag{3.6}$$

Siimilarly,

$$|T_U^{-1}(i)| = 1 \quad \text{ on } \Phi_A. \tag{3.7}$$

Define $S : B \to A$ by $S(a) = (T^{-1}(1))^{-1}T_U^{-1}(a)$ for $a \in B$. Then $S$ is a bounded real-linear bijection from $B$ onto $A$ such that $S(\mathfrak{B}) = \mathfrak{A}$. Let $a \in B$. Then

$$\|S(a)\| = \|T^{-1}(1)\| \, \|(T^{-1}(1))^{-1}T_U^{-1}(a)\| \geq \|T_U^{-1}(a)\| = \|a\|. \tag{3.8}$$

Next we show that $(S(i))(\Phi_A) \subset i\mathbb{R}$. Let $x \in \Phi_A$. For every $r > 0$ we see that

$$|r \pm (S(i))(x)| = |r(T^{-1}(1))(x) \pm (T_U^{-1}(i))(x)| \\ = |(T_U^{-1}(r \pm i))(x)| \leq \|T_U^{-1}(r \pm i)\| = |r \pm i|$$

since $T_U^{-1}$ is real-linear and (3.6) holds, hence $(S(i))(\Phi_A) \subset i\mathbb{R}$, so that

$$(S(i))(\Phi_A) \subset \{i, -i\}$$

by (3.6) and (3.7).

Let

$$\Phi_{A+} = \{x \in \Phi_A : S(i)(x) = i\}, \quad \Phi_{A-} = \{x \in \Phi_A : S(i)(x) = -i\}.$$

Then $\Phi_{A+}$ and $\Phi_{A-}$ are (possibly empty) closed and open subsets of $\Phi_A$ respectively and

$$\Phi_A = \Phi_{A+} \cup \Phi_{A-}, \quad \Phi_{A+} \cap \Phi_{A-} = \emptyset.$$

Define a function $\iota : C(\Phi_A) \to C(\Phi_A)$ by

$$(\iota(f))(x) = \begin{cases} f(x), & x \in \Phi_{A+}, \\ \overline{f(x)}, & x \in \Phi_{A-}. \end{cases}$$

Then $\iota$ is a real-linear bijection. Note that $\iota(S(i)) = i$ and $\iota(A)$ is a complex algebra. Define the norm $\|\cdot\|_{\pm}$ on $\iota(A)$ by

$$\|\iota(f)\|_{\pm} = \max\{\|f\|_+, \|f\|_-\},$$

where

$$\|f\|_+ = \inf\{\|g\| : g \in A,\ g = f \text{ on } \Phi_{A+}\},$$
$$\|f\|_- = \inf\{\|h\| : h \in A,\ h = f \text{ on } \Phi_{A-}\}.$$

Applying the Shilov idempotent theorem and a routine argument we see that $\iota(A)$ is a unital semisimple commutative Banach algebra with respect to the norm $\|\cdot\|_\pm$. Define $\tilde{S} : B \to \iota(A)$ by $\tilde{S}(a) = \iota(S(a))$ for $a \in B$. Then $\tilde{S}$ is a bounded real-linear bijection from $B$ onto $A$ such that $\tilde{S}(1) = 1$ and $\tilde{S}(i) = i$.

Let $\phi \in \Phi_{\iota(A)}$. We now show that $\phi \circ \tilde{S}$ is a real-linear selection from the exponential spectrum $\sigma_{\exp B}$, where

$$\sigma_{\exp B}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \exp B\}$$

for $a \in B$. We only need to show that $\phi \circ \tilde{S}(a) \in \sigma_{\exp B}(a)$ for every $a \in B$. Let $a \in B$ and put $\lambda = \phi \circ \tilde{S}(a)$. Then $\tilde{S}(a) - \lambda \notin (\iota(A))^{-1}$ since $\phi \in \Phi_{\iota(A)}$. Suppose that $\lambda \notin \sigma_{\exp B}(a)$. Then

$$\tilde{S}(a - \lambda) \in \tilde{S}(\exp B) \subset \iota(\mathfrak{A}) \subset \iota(A^{-1}).$$

Note that $\iota(A^{-1}) = (\iota(A))^{-1}$. Since $\tilde{S}(1) = 1$, $\tilde{S}(i) = i$, and $\tilde{S}$ is real-linear,

$$\tilde{S}(a - \lambda) = \tilde{S}(a) - \lambda,$$

so $\tilde{S}(a) - \lambda \in (\iota(A))^{-1}$, which is a contradiction.

By Lemma 2.2 we see that $\phi \circ \tilde{S}$ is a complex homomorphism. It follows that $\tilde{S}$ is a (complex) algebra isomorphism from $B$ onto $\iota(A)$. In particular, $B$ is semisimple and commutative.

Thus $S = (T^{-1}(1))^{-1}T_U^{-1}$ is a real algebra isomorphism from $B$ onto $A$. Since $B$ is semisimple and commutative, we see similarly that $(T(1))^{-1}T_U$ is a real algebra isomorphism from $A$ onto $B$ such that

$$\|(T(1))^{-1}T_U(f)\| \geq \|f\| \tag{3.9}$$

for every $f \in A$.

We now show that

$$((T^{-1}(1))^{-1}T_U^{-1})^{-1} = (T(1))^{-1}T_U. \tag{3.10}$$

Let $f \in A$ and put

$$a = ((T^{-1}(1))^{-1}T_U^{-1})^{-1}(f).$$

Then $a = T_U(T^{-1}(1)f)$. On the other hand, since $(T^{-1}(1))^{-1}T_U^{-1}$ is multiplicative, we see that

$$\begin{aligned}
T(1)T_U(T^{-1}(1)f) &= T_U(T^{-1}(1)(T^{-1}(1))^{-1})T_U(T^{-1}(1)f) \\
&= ((T^{-1}(1))^{-1}T_U^{-1})^{-1}((T^{-1}(1))^{-1})((T^{-1}(1))^{-1}T_U^{-1})^{-1}(f) \\
&= ((T^{-1}(1))^{-1}T_U^{-1})^{-1}((T^{-1}(1))^{-1}f) \\
&= T_U(T^{-1}(1)(T^{-1}(1))^{-1}f) = T_U(f).
\end{aligned}$$

Hence

$$(T(1))^{-1}T_U(f) = T_U(T^{-1}(1)f) = ((T^{-1}(1))^{-1}T_U^{-1})^{-1}(f)$$

for every $f \in A$, that is, (3.10) holds. Then by (3.8) and (3.9) we see that

$$\|(T(1))^{-1}T_U(f)\| = \|f\|$$

for every $f \in A$. Thus $(T(1))^{-1}T_U$ is an isometrical real algebra isomorphism from $A$ onto $B$, hence $(T(1))^{-1}T_U(A^{-1}) = B^{-1}$, and we see that $A^{-1}$ is isometrically isomorphic to $B^{-1}$ as a metric group. ∎

Theorem 3.3 shows that the metric group structure of the group of invertible elements in the unital semisimple commutative Banach algebra can be recovered from the metric structure in the category of unital Banach algebras.

PROBLEM. *In which unital Banach algebras can the metric group structure of the group of invertible elements be recovered from the metric structure in the category of unital Banach algebras?*

The conclusion of Theorem 3.3 does not hold if $A$ is commutative but not semisimple as the following example shows.

EXAMPLE 3.4. Let

$$A_0 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Let

$$A = \left\{ \begin{pmatrix} \alpha & a & b \\ 0 & \alpha & c \\ 0 & 0 & \alpha \end{pmatrix} : \alpha, a, b, c \in \mathbb{C} \right\}$$

be the unitization of $A_0$, where the multiplication (in $A_0$) is the zero multiplication: $MN = 0$ for all $M, N \in A_0$. Let $B = A$ as sets, while the multiplication in $B$ is the usual matrix multiplication. Then $A$ and $B$ are unital Banach algebras under the usual operator norm. Note that $A$ is commutative, but not semisimple. Note also that $A^{-1} = \left\{ \left(\begin{smallmatrix} \alpha & a & b \\ 0 & \alpha & c \\ 0 & 0 & \alpha \end{smallmatrix}\right) \in A : \alpha \neq 0 \right\}$ and $B^{-1} = \left\{ \left(\begin{smallmatrix} \alpha & a & b \\ 0 & \alpha & c \\ 0 & 0 & \alpha \end{smallmatrix}\right) \in B : \alpha \neq 0 \right\}$. Put $F = \left(\begin{smallmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Define $T : A^{-1} \to B^{-1}$ by $T(M) = M + F$. Then $T$ is well-defined and surjective (affine) isometry from $A$ onto $B$. On the other hand. $A^{-1}$ is not (group) isomorphic to $B^{-1}$.

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