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Commutators in Banach *-algebras

by

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Abstract. The set of commutators in a Banach *-algebra A, with continuous involution, is examined. Applications are made to the case where $A = B(\ell_2)$, the algebra of all bounded linear operators on ℓ_2.

1. Introduction. We present a study of commutators, elements of the form $[x, y] = xy - yx$ in Banach and Banach *-algebras. Commutators have been examined carefully in the case of $B(X)$, the algebra of all bounded linear operators on a Hilbert space X. We cite the book by Putnam [11], where further references can be found.

Let A be a Banach *-algebra with a continuous involution. Our results also apply to any Banach algebra if the conclusions involving the involution are ignored. Throughout we let E denote a closed linear subspace in A. We let $\mathfrak{C}$ denote the set of all commutators in A, and let $\mathfrak{Z}(E)$ denote the *center of A modulo E*, the set of all $a \in A$ such that $[a, x] \in E$ for all $x \in A$. In Herstein's book [9, p. 5], this notion was studied for ring theory under the notation $T(E)$. He showed [9, Lemma 1.4] that $T(E)$ is both a subring and a Lie ideal if E is a Lie ideal.

Suppose that A has an identity and that $E \not\supset \mathfrak{C}$. We show that the complement of $\mathfrak{Z}(E)$ contains a set Σ where $x^* \in \Sigma$ and $x^n \in \Sigma$ for all positive integers n whenever $x \in \Sigma$. This implies that the set $\mathfrak{D}(E)$ of all $[a, b] \in \mathfrak{C}$ such that $[a^k, b^r] \notin E$ for all positive integers k and r, is dense in $\mathfrak{C}$. We apply this when $\mathfrak{A} = B(\ell_2)$, the algebra of all bounded linear operators on ℓ_2. As shown in [3], $\mathfrak{C}$ is dense in $\mathfrak{A}$. Let $E \neq \mathfrak{A}$. Then $\mathfrak{D}(E)$ is dense in $\mathfrak{A}$ as well as in $\mathfrak{C}$. If $E = \mathfrak{K}(\ell_2)$, the algebra of all compact linear operators on ℓ_2, and $[a, b] \in \mathfrak{D}(E)$, then also every a^k and b^r lies in $\mathfrak{C}$.

In Section 3 we study relations between the center $\mathfrak{Z}$ of A, $\mathfrak{C}$ and the centralizer $\Gamma(\mathfrak{C})$ of $\mathfrak{C}$, the set of $a \in A$ with $[a, x] = 0$ for all $x \in \mathfrak{C}$. If A is

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semi-prime, then $\Gamma(\mathfrak{C}) = \mathfrak{Z}$. Also, if $\mathfrak{Z}$ is semi-simple, then $\mathfrak{Z} \cap \mathfrak{C} = (0)$ and $\Gamma(\mathfrak{C})$ is commutative.

In Section 4 we treat the case where E is a closed two-sided ideal in A. Particular attention is given to the case where E is a modular primitive ideal. These results are applied to the case $\mathfrak{A} = B(\ell_2)$. We make use of the remarkable result in [3] that here $\mathfrak{C}$ is dense in $\mathfrak{A}$. We show that $\mathfrak{C}$ contains a set Γ, dense in $B(\ell_2)$, such that $a^n \in \Gamma$ for all positive integers n whenever $a \in \Gamma$ and each $a \in \Gamma$ fails to be compact. Not only is $\mathfrak{C}$ dense in $B(\ell_2)$, but the subalgebra of $B(\ell_2)$ generated by $\mathfrak{C}$ is all of $B(\ell_2)$.

2. On the center modulo E. We retain the notation of the introduction. We make use of a notion of Herstein, that of the hypercenter of a ring [10]. For a ring R its *hypercenter* $\mathfrak{H}$ is the set of $w \in R$ such that, for each $x \in R$, there is a positive integer $n = n(x, w)$ with $[w, x^n] = 0$. For a Banach algebra A, we study a variant of this notion, the *hypercenter* $\mathfrak{H}(E)$ *modulo* E. By $\mathfrak{H}(E)$ we mean the set of $a \in A$ such that, for each $x \in A$, there is a positive integer $n = n(x, a)$ with $[a, x^n] \in E$.

We will make frequent use of the following fact. Let $p(t) = \sum_{j=0}^{n} a_j t^j$ be a polynomial in the real variable t with coefficients in A. If $p(t) \in E$ for an infinite subset of the reals, then every a_j is in E.

LEMMA 2.1. *$\mathfrak{H}(E)$ is the set of all $a \in A$ for which there is a positive integer $r = r(a)$ such that $[a, x^r] \in E$ for all $a \in A$.*

Proof. Let $a \in \mathfrak{H}(E)$. For each positive integer n, let $F_n = \{x \in A : [a, x^n] \in E\}$. Then A is the union of the closed sets F_n so that at least one of them, say F_r, contains a non-empty open set Ω. Let $b \in \Omega$ and y be any element of A. There is some $\varepsilon > 0$ such that $[a, (b + ty)^r] \in E$ for all real t, $0 \le t \le \varepsilon$. Hence $[a, y^r]$ is in E. ∎

LEMMA 2.2. *For a positive integer n, either $[a, x^n] \in E$ for all $x \in A$ or the set G_n of $x \in A$ with both $[a, x^n] \notin E$ and $[a, x^{*n}] \notin E$ is a dense open set in A.*

Proof. Clearly G_n is open. Suppose that G_n is not dense in A. Then there is a non-empty open set Ω in A such that, for each $x \in \Omega$, either $[a, x^n] \in E$ or $[a, x^{*n}] \in E$. Let $b \in \Omega$ and $y \in A$. There is some $\varepsilon > 0$ so that $b + ty \in \Omega$ for all real t, $0 \le t \le \varepsilon$. For each such t, either $[a, (b + ty)^n] \in E$ or $[a, (b + ty)^{*n}] \in E$. At least one of these possibilities holds for infinitely many values of t. Thus either $[a, y^n] \in E$ or $[a, y^{*n}] \in E$. Then A is the union of two closed sets, $F_1 = \{y \in A : y^n \in E\}$ and $F_2 = \{y \in A : y^{*n} \in E\}$. At least one of the sets F_1, F_2 must contain a non-empty open subset Γ of A. Say, $F_1 \supset \Gamma$. Let $w \in \Gamma$ and $y \in A$. There is an interval of reals of positive length with $[a, (w + ty)^n] \in E$ for each such real t. Hence $[a, y^n] \in E$ for all $y \in A$. Likewise, if $F_2 \supset \Gamma$, then $[a, y^{*n}] \in E$ for all $y \in A$. ∎

THEOREM 2.3. *Either $a \in \mathfrak{H}(E)$ or the set $S(a, E)$, of $x \in A$ such that both $[a, x^n] \notin E$ and $[a, x^{*n}] \notin E$ for all positive integers n, is dense in A.*

Proof. If $S(a, E)$ is dense, then $a \notin \mathfrak{H}(E)$ by Lemma 2.1. Suppose that $a \notin \mathfrak{H}(E)$. Then, by Lemma 2.1, for each positive integer n there is some $x \in A$ where $[a, x^n] \notin E$. By Lemma 2.2, each of the sets G_n of that lemma is dense and open. By the Baire category theorem, their intersection $\bigcap G_n = S(a, E)$ is dense in A. ∎

Notation. We will continue to use $S(a, E)$ to denote the set of $x \in A$ such that both $[a, x^n] \notin E$ and $[a, x^{*n}] \notin E$ for all positive integers n.

We treat the case $E = (0)$. Let $\mathfrak{Z}$ denote the center of A.

THEOREM 2.4. *Let A be a semi-prime Banach algebra and $E = (0)$. Either $a \in \mathfrak{Z}$ or $S(a, E)$ is dense in A.*

Proof. Suppose $S(a, E)$ is not dense in A. Then by Theorem 2.3, a is in the hypercenter $\mathfrak{H}$ of A. Herstein [10, Theorem 2] has shown that if a ring R has no nil ideals, then $\mathfrak{H} = \mathfrak{Z}$. For a Banach algebra A, Dixon [6] showed that the condition for A to have no nil ideals is equivalent to A being semi-prime. ∎

For the notion of a left or right approximate identity, see [7, p. 2].

THEOREM 2.5. *Suppose that A has a left or a right approximate identity $\{e_\lambda\}$. Then $\mathfrak{H}(E) = \mathfrak{Z}(E)$ so that either $a \in \mathfrak{Z}(E)$ or $S(a, E)$ is dense in A.*

Proof. Clearly $\mathfrak{Z}(E) \subset \mathfrak{H}(E)$. Let $a \in \mathfrak{H}(E)$. By Lemma 2.1, there is a fixed positive integer n such that $[a, x^n] \in E$ for all $x \in A$. We show this holds for $n = 1$ so that $a \in \mathfrak{Z}(E)$.

Suppose that $n > 1$ and $[a, x^n] \in E$ for all $x \in A$. Then $[a, (x+te_\lambda)^n] \in E$ for each given $x \in A$, each e_λ and all real values of t. The coefficient of t in the polynomial $[a, (x + te_\lambda)^n]$ lies in E, so that

$$\Big[a, \sum_{j=0}^{n-1} x^j e_\lambda x^{n-1-j}\Big] \in E.$$

Taking the limit on e_λ, we see that $[a, x^{n-1}] \in E$ for all $x \in A$. Continuing in this way, we see that $a \in \mathfrak{Z}(E)$. ∎

Theorem 2.5 need not hold if A has no approximate identity. For example, take any A such that, for some positive integer n, $x^n = 0$ for all $x \in A$. However, when $E = (0)$, we have seen in Theorem 2.4 that the conclusion holds for every semi-prime A. We do not know if this is the case for all E, but we show that such is the case if A has a dense socle.

THEOREM 2.6. *Let A be a semi-prime Banach algebra with a dense socle Σ. Then $\mathfrak{H}(E) = \mathfrak{Z}(E)$ for all E.*

Proof. Let $a \in \mathfrak{H}(E)$. By Lemma 2.1, there is a positive integer n so that $[a, x^n] \in E$ for all $x \in A$. We show that the validity of this statement for some $n \geq 2$ implies its validity for $n = 1$.

Let Ap, where $p^2 = p$, be a minimal left ideal in A. Note that $[a, p] \in E$ as $p = p^n$. We have

$$t^{-1}[a, (p + ty)^n - p] \in E$$

for all real values of $t \neq 0$ and any $y \in A$. Also

$$(p + ty)^n = p + t[yp + (n - 2)pyp + py] + \cdots ,$$

where we have omitted all terms in the expansion of $(p + ty)^n$ involving higher powers of t. Therefore, if we let $t \to 0$, we see that

$$[a, yp + (n - 2)pyp + py] \in E.$$

However, $pyp = \lambda p$ for a scalar λ so that $[a, yp + py] \in E$ for all $y \in A$. Replace y by yp to see that $[a, yp + pyp] \in E$ or $[a, yp] \in E$ for all $y \in A$. Therefore $[a, w] \in E$ for all $w \in \Sigma$. As Σ is dense in A, we have $a \in \mathfrak{Z}(E)$. ∎

Note that E contains the set $\mathfrak{C}$ of all commutators if and only if $[x, y] \in E$ for all $x, y \in A$ or, equivalently, $\mathfrak{Z}(E) = A$. Thus if $E \not\supset \mathfrak{C}$, then $\mathfrak{Z}(E)$ is a proper closed linear subspace of A, so that its complement is dense in A. We denote that complement by $\mathfrak{N}(E)$.

PROPOSITION 2.7. *$\mathfrak{N}(E)$ is an open subset of A, any two elements of which are connected by one or two line segments in $\mathfrak{N}(E)$.*

Proof. We assume that $\mathfrak{N}(E)$ is not empty. For $a \in A$, $a \in \mathfrak{N}(E)$ if and only if $[a, b] \notin E$ for some $b \in A$. Let $a, b \in A$ with $[a, b] \notin E$ so that a and b lie in $\mathfrak{N}(E)$. For any scalars λ and μ where $\lambda \neq 0$, we have $[\lambda a + \mu b, b] \notin E$. Thus the line segment from a to b lies in $\mathfrak{N}(E)$.

Let $v, w \in \mathfrak{N}(E)$. We show that there is $y \in A$ with $[v, y] \notin E$ and $[w, y] \notin E$. For suppose otherwise. Let $F_1 = \{x \in A : [v, x] \in E\}$ and $F_2 = \{x \in A : [w, x] \in E\}$. Then A is the union of the closed sets F_1 and F_2 so that at least one of them, say F_1, contains a non-empty open subset. Arguing as in the proof of Lemma 2.2, we see that $[v, x] \in E$ for all $x \in A$, so that $v \notin \mathfrak{N}(E)$.

Now let $y \in A$ with $[v, y] \notin E$ and $[w, y] \notin E$. The line segments joining v to y and y to w lie in $\mathfrak{N}(E)$. ∎

We say that a subset S of A is *power-closed* if $x^n \in S$ for all positive integers n whenever $x \in S$.

THEOREM 2.8. *Suppose that A has an identity e and that $E \not\supset \mathfrak{C}$. Then $\mathfrak{N}(E)$ contains a dense power-closed *-subset of A.*

Proof. Let S_1 be the set of all $x \in A$ for which both $x^n \in \mathfrak{R}(E)$ and $x^{*n} \in \mathfrak{R}(E)$ for all positive integers n. We show S_1 to be dense in A. Suppose otherwise. Then there exists a non-empty open set G where, for each $x \in G$, either there is a positive integer n with $x^n \in \mathfrak{Z}(E)$ or a positive integer m with $x^{*m} \in \mathfrak{Z}(E)$. For positive integers p and q, let

$$W_{p,q} = \{x \in A : x^p \notin \mathfrak{Z}(E) \text{ and } x^{*q} \notin \mathfrak{Z}(E)\}.$$

If every $W_{p,q}$ was dense in A, then so also would be their intersection by the Baire category theorem. But this would contradict the existence of G. Then we have the existence of a non-empty open set Ω in the complement of $W_{r,s}$, say. Let $a \in \Omega$ and $y \in A$. For some $\varepsilon > 0$ either $(a + ty)^r \in \mathfrak{Z}(E)$ or $(a + ty^*)^s \in \mathfrak{Z}(E)$ for each t, $0 \le t \le \varepsilon$. Arguing as in the proof of Lemma 2.2, we see that there is a positive integer n so that $y^n \in \mathfrak{Z}(E)$ for all $y \in A$.

We employ notation used in [13, p. 204]. Let B_r denote the sum of those terms in the expansion of $(a + b)^n$ for which the sum of the exponents of the b^i factors is r. Thus $B_0 = a^n$ and $B_1 = \sum_{k=0}^{n-1} a^k b a^{n-1-k}$. For any a and b in A and any real value of t, we see that $(a + tb)^n = \sum_{r=0}^{n} B_r t^r$ lies in $\mathfrak{Z}(E)$. Therefore each B_r is in $\mathfrak{Z}(E)$ and hence $[B_0, B_1] \in E$. We use this for $a = e + tx$ and $b = y$ to see, as in [13, p. 208], that

$$\Big[t^{-1}\{(e + tx)^n - e\}, \sum_{j=0}^{n-1} (e + tx)^j y (e + tx)^{n-1-j}\Big] \in E$$

for every real $t \ne 0$. We let $t \to 0$ to see that $[x, y] \in E$ for all $x, y \in A$. This contradicts $E \not\supset \mathfrak{C}$ so that S_1 is dense in A. ∎

We let $\mathfrak{D}(E)$ be the set of all $[a, b] \in \mathfrak{C}$ where $[a^k, b^r] \notin E$ for all positive integers k and r.

THEOREM 2.9. *Suppose that A has an identity and that $E \not\supset \mathfrak{C}$. Then $\mathfrak{D}(E)$ is dense in $\mathfrak{C}$.*

Proof. By Theorem 2.5 the set $S(w, E)$ is dense in A for each $w \notin \mathfrak{Z}(E)$. Recall that $S(w, E)$ is the intersection of countably many open dense subsets of A. We employ the set S_1 of Theorem 2.8.

Let $[a, b] \in \mathfrak{C}$. Fix attention on the positive integer n. There is $a_n \in S_1$ where $\|a - a_n\| < n^{-1}$. As S_1 is power-closed, we have $a_n^k \in S_1$ for each positive integer k, so that $S(a_n^k, E)$ is a dense *-subset of A. By the Baire category theorem, the set $Q_n = \bigcap_k S(a_n^k, E)$ is dense in A. By its definition, every $S(a_n^k, E)$ is power-closed. Therefore, so is Q_n. We select $b_n \in Q_n$ with $\|b - b_n\| < n^{-1}$. Then $[a_n^k, b_n^r] \notin E$ for all positive integers k and r. Also $[a_n, b_n] \to [a, b]$. ∎

Corollary 2.10. *Let $\mathfrak{A}$ be the algebra of all bounded linear operators on ℓ_2. Let E be a proper closed linear subspace of $\mathfrak{A}$. Then $\mathfrak{D}(E)$ is dense in $\mathfrak{A}$.*

Proof. In [3, Corollary 5.2] it is pointed out that $\mathfrak{C}$ is dense in $\mathfrak{A}$. Therefore $E \not\supset \mathfrak{C}$. We apply Theorem 2.9 to see that $\mathfrak{D}(E)$ is dense in $\mathfrak{A}$ as well as in $\mathfrak{C}$. ∎

3. Sets related to the center $\mathfrak{Z}$. We examine the sets $\mathfrak{Z}$, $\mathfrak{Z}(\mathfrak{Z}) = \{a \in A : [a, A] \subset \mathfrak{Z}\}$ and $\Gamma(\mathfrak{C})$, the centralizer of $\mathfrak{C}$, i.e., the set of $x \in A$ such that $[x, y] = 0$ for all $y \in \mathfrak{C}$.

First we show that properties of $\mathfrak{Z}$ alone can affect the nature of $\mathfrak{Z}(\mathfrak{Z})$, $\Gamma(\mathfrak{C})$ and $\mathfrak{C}$.

Theorem 3.1. *If $\mathfrak{Z}$ is a semi-prime algebra*, *then $\mathfrak{Z}(\mathfrak{Z}) = \mathfrak{Z}$.*

Proof. Let $a \in \mathfrak{Z}(\mathfrak{Z})$. Since $[a, [a, x]] = 0$ for all $x \in A$, arguments in [9, p. 4] show that $[a, x][a, y] = 0$ for all $x, y \in A$. Let $z \in \mathfrak{Z}$. Then $z[a, x] = [a, xz] = [a, x]z$ for all $x, y \in A$. Hence $[a, A]$ is an ideal in $\mathfrak{Z}$ with $uv = 0$ for all $u, v \in [a, A]$ so that $[a, A]^2 = (0)$. As $\mathfrak{Z}$ is semi-prime, we have $[a, A] = (0)$ or $a \in \mathfrak{Z}$. ∎

Let J denote the radical of A and $r(x)$ the spectral radius of $x \in A$.

Lemma 3.2. *Let $x, y \in A$. If $[x, y] \in \mathfrak{Z}$, then $[x, y] \in J$.*

Proof. Since x permutes with $[x, y]$, by the Kleinecke–Shirokov theorem [1, p. 91], we have $r([x, y]) = 0$. Let $v \in A$. As v also permutes with $[x, y]$ we have, by [12, Theorem 1.4.1],

$$r([x, y]v) \le r([x, y])r(v) = 0.$$

Therefore $[x, y]v$ is quasi-regular for each $v \in A$, so that $[x, y] \in J$. ∎

Theorem 3.3. *No non-zero idempotent lies in $\mathfrak{C} \cap \mathfrak{Z}$. If $\mathfrak{Z}$ is semi-simple*, *then $\mathfrak{C} \cap \mathfrak{Z} = (0)$.*

Proof. Let p be a non-zero idempotent. Since $p \notin J$, we see by Lemma 3.2 that $p \notin \mathfrak{C} \cap \mathfrak{Z}$. If $\mathfrak{Z}$ is semi-simple, then $\mathfrak{C} \cap \mathfrak{Z} = (0)$ by Lemma 3.2. ∎

This is an extension of the classical result [11, p. 2] that the identity in a Banach algebra cannot be a commutator. We note also that if A is semi-simple, then so is $\mathfrak{Z}$. For, let x_0 be in the radical of $\mathfrak{Z}$ and $y \in A$. By [12, Theorem 1.4.1], $r(x_0 y) \le r(x_0)r(y) = 0$. Thus $x_0 y$ is quasi-regular for each $y \in A$, so that $x_0 \in J$.

Corollary 3.4. *If $\mathfrak{Z}$ is semi-prime*, *then $\Gamma(\mathfrak{Z})$ is commutative.*

Proof. Let $a, b \in \Gamma(\mathfrak{C})$. By the Jacobi identity $[a, [b, x]] + [b, [x, a]] + [x, [a, b]] = 0$ for all $x \in A$. Hence $[x, [a, b]] = 0$ for all $x \in A$, so that

$[a,b] \in \mathfrak{Z}$. As $\mathfrak{Z}$ is semi-simple, $[a,b]=0$ by Theorem 3.3. Thus $\Gamma(\mathfrak{C})$ is commutative. ∎

THEOREM 3.5. *$\Gamma(\mathfrak{C}) \supset \mathfrak{Z}(\mathfrak{Z})$. If $J \subset \mathfrak{Z}$, then $\Gamma(\mathfrak{C}) = \mathfrak{Z}(\mathfrak{Z})$.*

Proof. Let $a \in \mathfrak{Z}(\mathfrak{Z})$. By the Jacobi identity $[a,[x,y]]+[x,[y,a]]+[y,[a,x]] = 0$ for all $x,y \in A$. But as $[y,a] \in \mathfrak{Z}$ and $[a,x] \in \mathfrak{Z}$, we see that $[a,[x,y]] = 0$ for all $x,y \in A$, or $a \in \Gamma(\mathfrak{C})$.

Next let $a \in \Gamma(\mathfrak{C})$. Then $[a,[a,x]] = 0$ for all $x \in A$. Arguments of Herstein [9, p. 4] show that $[a,x]A[a,x] = (0)$ for all $x \in A$. Then $[a,x] \in J \subset \mathfrak{Z}$. ∎

There are interesting examples of A where $J \neq (0)$ is the set of $x \in A$ where $xA = Ax = (0)$. Of course, in that case $J \subset \mathfrak{Z}$. The prototype of such instances is an example of C. Feldman [12, p. 297]. That example is commutative. More elaborate examples, where $J \neq (0)$, $J \subset \mathfrak{Z}$, which are not commutative, are given in [14]; the Feldman example is a special case.

THEOREM 3.6. *If A is a semi-prime algebra, then $\mathfrak{Z}(\mathfrak{Z}) = \Gamma(\mathfrak{C}) = \mathfrak{Z}$.*

Proof. This is valid for any algebra, not just a Banach algebra. By [9, Lemma 1.5, p. 11] we see that $\Gamma(\mathfrak{C}) = \mathfrak{Z}$. By [15, Theorem 3.1], we have $\mathfrak{Z}(\mathfrak{Z}) = \mathfrak{Z}$. ∎

4. On the center modulo an ideal. All ideals considered here are two-sided unless otherwise specified. Henceforth K will denote a *closed* ideal in A, and π will denote the natural homomorphism of A onto A/K. We recall the notation $\mathfrak{Z}(K)$ and its complement $\mathfrak{N}(K)$ of Section 2. We examine properties of $\mathfrak{N}(K)$, motivated by the example of $A = B(\ell_2)$, the algebra of all bounded linear operators on ℓ_2. Let K be the subset of its compact operators. It follows from [3] that $\mathfrak{N}(K)$ is the set of all elements of $\mathfrak{C}$ which are not compact.

THEOREM 4.1. *Suppose that A/K is semi-simple and $a \in A$. Either $[a,A] \subset K$, or the set of $x \in A$ such that $[a,x^n] \notin K$ and $[a,x^{*n}] \notin K$ for all positive integers n is dense in A.*

Proof. An equivalent statement is that $\mathfrak{N}(K)$ is the set of all $a \in A$ with the stated properties. First, note that $\pi(\mathfrak{H}(K))$ is the hypercenter $\mathfrak{H}^\#$ of A/K. As A/K is semi-simple, $\mathfrak{H}^\#$ is the center $\mathfrak{Z}^\#$ of A/K by [10, Lemma 2]. Now $\pi^{-1}(\mathfrak{Z}^\#) = \{y \in A : [y,A] \subset K\} = \mathfrak{Z}(K)$. Thus

$$\mathfrak{Z}(K) \subset \mathfrak{H}(K) \subset \pi^{-1}(\mathfrak{Z}^\#) = \mathfrak{Z}(K).$$

Therefore $\mathfrak{Z}(K) = \mathfrak{H}(K)$. We now apply Theorem 2.3 to see that if $a \notin \mathfrak{Z}(K)$, then a has the required properties. ∎

THEOREM 4.2. *Suppose that A/K is semi-simple. If $a \in \mathfrak{Z}(K) \cap \mathfrak{C}$, then $a \in K$.*

Proof. In other words, if $a \in \mathfrak{C}$ is not in K, then $a \in \mathfrak{R}(K)$. Note that $\pi(a)$ is a commutator lying in the center $\mathfrak{Z}^{\#}$ of A/K (see the preceding proof). Since A/K is semi-simple, $\pi(a) = 0$ by Theorem 3.3. ∎

If the ideal $K \neq A$ of Theorem 4.2 is modular and j is an identity for A modulo K, then, by Theorem 4.2, $j \notin \mathfrak{C}$ since $j \in \mathfrak{Z}(K)$ and $j \notin K$.

Henceforth, let P denote a modular primitive ideal where j is an identity for A modulo P.

LEMMA 4.3. *$\mathfrak{Z}(P)$ is the set of elements in A of the form $\lambda j + y$ where λ is a scalar and $y \in P$. Also, $A = \mathfrak{Z}(P)$ if and only if $P \supset \mathfrak{C}$.*

Proof. Let π be the canonical homomorphism of A onto A/P. A/P is a primitive algebra with $\pi(j)$ as its identity. By [12, Corollary 2.4.5], the center $\mathfrak{Z}^{\#}$ of A/P is the set of scalar multiples of its identity $\pi(j)$. As in the proof of Theorem 4.1, $\mathfrak{Z}(P) = \pi^{-1}(\mathfrak{Z}^{\#})$ so that $\mathfrak{Z}(P)$ is the set of elements $\lambda j + y$ where λ is a scalar and $y \in P$.

Suppose $A = \mathfrak{Z}(P)$. By Theorem 4.2, we have $P \supset \mathfrak{C}$. Conversely, suppose that $P \supset \mathfrak{C}$, so that $[\pi(x), \pi(y)] = 0$ for all $x, y \in A$. Thus A/P is commutative and is the set of all $\lambda\pi(j)$ elements. Then $\mathfrak{Z}(P) = A$ by the description above of $\mathfrak{Z}(P)$. ∎

Thus, if $P \not\supset \mathfrak{C}$ then $\mathfrak{R}(P)$ is the set of elements not of the form $\lambda j + y$, $\lambda \neq 0$, $y \in P$ and where $\lambda j + y \notin P$.

THEOREM 4.4. *Suppose that P is a modular maximal ideal in A and that $P \not\supset \mathfrak{C}$. Then any $a \in A$ is of the form $x + y$ where x is in the subalgebra Q generated by $\mathfrak{C}$, and $y \in P$.*

Proof. Let Γ_0 be the subalgebra of A/P generated by its commutators. Here A/P is a simple algebra which, as $P \not\supset \mathfrak{C}$, is not commutative. By a corollary of Herstein [9, p. 6], we have $\Gamma_0 = A/P$. Now Γ_0 is the set of all $[\pi(x), \pi(y)]$, $x, y \in A$, where π is the natural homomorphism of A onto A/P. Then any $a \in A$ is of the form $x + y$, $x \in Q$ and $y \in P$. ∎

Henceforth, we confine attention to the algebra of all bounded linear operators on ℓ_2, which we denote by $\mathfrak{A}$. Let $\mathfrak{K}$ denote the subset of all compact operators.

COROLLARY 4.5. *The subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}$ is all of $\mathfrak{A}$.*

Proof. Here $\mathfrak{K}$ is a modular maximal ideal in $\mathfrak{A}$. Also, $\mathfrak{C} \supset \mathfrak{K}$ as shown in [2]. We apply Theorem 4.4. ∎

THEOREM 4.6. *In $\mathfrak{A}$, $\mathfrak{C} = \mathfrak{K} \cup \mathfrak{R}(\mathfrak{K})$.*

Proof. Let I denote the identity of $\mathfrak{A}$. By Lemma 4.3, $\mathfrak{R}(\mathfrak{K})$ is the set of elements of $\mathfrak{A}$ not of the form $\lambda I + T$ where λ is a scalar and $T \in \mathfrak{K}$. Then

$\mathfrak{N}(\mathfrak{K}) \cup \mathfrak{K}$ is the set of those elements not of the form $\lambda I + T$ where $\lambda \neq 0$ and $T \in \mathfrak{K}$. This, however, is $\mathfrak{C}$, as shown in [3]. ∎

THEOREM 4.7. *$\mathfrak{N}(\mathfrak{K})$ and hence $\mathfrak{C}$ contains a power-closed *-subset dense in $\mathfrak{A}$.*

Proof. By Theorem 2.8, $\mathfrak{N}(\mathfrak{K})$ possesses such a dense subset. By Theorem 4.6, $\mathfrak{N}(\mathfrak{K}) \subset \mathfrak{C}$. ∎

THEOREM 4.8. *The subset of $\mathfrak{C}$, consisting of all $[T,U]$ with $[T^k, U^r] \notin \mathfrak{K}$ for all positive integers k and r, is dense in $\mathfrak{A}$. Every T^k and U^r lies in $\mathfrak{C}$.*

Proof. By Corollary 2.10, the subset in question is dense in $\mathfrak{C}$ and therefore dense in $\mathfrak{A}$. By Theorem 4.6, every T^k and U^r is in $\mathfrak{C}$ as $T^k, U^r \notin \mathfrak{Z}(\mathfrak{K})$. ∎

THEOREM 4.9. *$\mathfrak{C}$ is a connected subset of $\mathfrak{A}$, any two elements of which are connected by one or two line segments lying entirely in $\mathfrak{C}$.*

Proof. By Theorem 4.6, we have $\mathfrak{C} = \mathfrak{K} \cup \mathfrak{N}(\mathfrak{K})$. Any two elements of $\mathfrak{K}$ are connected by a line segment in $\mathfrak{K}$. By Proposition 2.7, any two elements of $\mathfrak{N}(\mathfrak{K})$ are connected in $\mathfrak{N}(\mathfrak{K})$ by one or two line segments. Now let $T \in \mathfrak{K}$ and $U \in \mathfrak{N}(\mathfrak{K})$. We claim that $\alpha T + \beta U \in \mathfrak{C}$ for any scalars α and β. For otherwise, by [3], there exists a scalar $\gamma \neq 0$ and $W \in \mathfrak{K}$ so that $\alpha T + \beta U = \gamma I + W$. Then $\beta U = \gamma I + (W - \alpha T)$ where $W - \alpha T \in \mathfrak{K}$. This is impossible as $U \in \mathfrak{C}$. ∎

THEOREM 4.10. *$\mathfrak{K}$ and (0) are the only closed Lie ideals of $\mathfrak{K}$.*

Proof. $\mathfrak{K}$ is a primitive Banach algebra with a dense socle $\mathfrak{G}$. The center $\mathfrak{Z}$ of $\mathfrak{K}$ is (0). By [5, Theorem 6.1], any closed Lie ideal $\mathfrak{L}$ of $\mathfrak{K}$ must contain $[T,U]$ for all $T \in \mathfrak{G}$ and $U \in \mathfrak{K}$. Thus $\mathfrak{L}$ contains all commutators of $\mathfrak{K}$. Then, as shown in [2], $\mathfrak{L} = \mathfrak{K}$. ∎

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