Commutators in Banach ∗-algebras

by

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Abstract. The set of commutators in a Banach ∗-algebra A, with continuous involution, is examined. Applications are made to the case where A = B(ℓ₂), the algebra of all bounded linear operators on ℓ₂.

1. Introduction. We present a study of commutators, elements of the form [x, y] = xy − yx in Banach and Banach ∗-algebras. Commutators have been examined carefully in the case of B(X), the algebra of all bounded linear operators on a Hilbert space X. We cite the book by Putnam [11], where further references can be found.

Let A be a Banach ∗-algebra with a continuous involution. Our results also apply to any Banach algebra if the conclusions involving the involution are ignored. Throughout we let E denote a closed linear subspace in A. We let C denote the set of all commutators in A, and let ʮ(E) denote the center of A modulo E, the set of all a ∈ A such that [a, x] ∈ E for all x ∈ A. In Herstein’s book [9, p. 5], this notion was studied for ring theory under the notation T(E). He showed [9, Lemma 1.4] that T(E) is both a subring and a Lie ideal if E is a Lie ideal.

Suppose that A has an identity and that E ⊄ C. We show that the complement of ʮ(E) contains a set Σ where xⁿ ∈ Σ and xⁿ ∈ Σ for all positive integers n whenever x ∈ Σ. This implies that the set D(E) of all [a, b] ∈ C such that [aᵏ, bʳ] ∉ E for all positive integers k and r, is dense in C. We apply this when ʮ = B(ℓ₂), the algebra of all bounded linear operators on ℓ₂. As shown in [3], C is dense in ʮ. Let E ≠ ʮ. Then D(E) is dense in ʮ as well as in C. If E = ℜ(ℓ₂), the algebra of all compact linear operators on ℓ₂, and [a, b] ∈ D(E), then also every aᵏ and bʳ lies in C.

In Section 3 we study relations between the center ʮ of A, C and the centralizer Γ(C) of C, the set of a ∈ A with [a, x] = 0 for all x ∈ C. If A is

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semi-prime, then $\Gamma(\mathfrak{C}) = \mathfrak{Z}$. Also, if $\mathfrak{Z}$ is semi-simple, then $\mathfrak{Z} \cap \mathfrak{C} = (0)$ and $\Gamma(\mathfrak{C})$ is commutative.

In Section 4 we treat the case where $E$ is a closed two-sided ideal in $A$. Particular attention is given to the case where $E$ is a modular primitive ideal. These results are applied to the case $\mathfrak{A} = B(\ell_2)$. We make use of the remarkable result in [3] that here $\mathfrak{C}$ is dense in $\mathfrak{A}$. We show that $\mathfrak{C}$ contains a set $\Gamma$, dense in $B(\ell_2)$, such that $a^n \in \Gamma$ for all positive integers $n$ whenever $a \in \Gamma$ and each $a \in \Gamma$ fails to be compact. Not only is $\mathfrak{C}$ dense in $B(\ell_2)$, but the subalgebra of $B(\ell_2)$ generated by $\mathfrak{C}$ is all of $B(\ell_2)$.

2. On the center modulo $E$. We retain the notation of the introduction. We make use of a notion of Herstein, that of the hypercenter of a ring [10]. For a ring $R$ its hypercenter $\mathfrak{H}$ is the set of $w \in R$ such that, for each $x \in R$, there is a positive integer $n = n(x,w)$ with $[w,x^n] = 0$. For a Banach algebra $A$, we study a variant of this notion, the hypercenter $\mathfrak{H}(E)$ modulo $E$. By $\mathfrak{H}(E)$ we mean the set of $a \in A$ such that, for each $x \in A$, there is a positive integer $n = n(x,a)$ with $[a,x^n] \in E$.

We will make frequent use of the following fact. Let $p(t) = \sum_{j=0}^n a_j t^j$ be a polynomial in the real variable $t$ with coefficients in $A$. If $p(t) \in E$ for an infinite subset of the reals, then every $a_j$ is in $E$.

Lemma 2.1. $\mathfrak{H}(E)$ is the set of all $a \in A$ for which there is a positive integer $r = r(a)$ such that $[a,x^r] \in E$ for all $a \in A$.

Proof. Let $a \in \mathfrak{H}(E)$. For each positive integer $n$, let $F_n = \{ x \in A : [a,x^n] \in E \}$. Then $A$ is the union of the closed sets $F_n$ so that at least one of them, say $F_r$, contains a non-empty open set $\Omega$. Let $b \in \Omega$ and $y$ be any element of $A$. There is some $\varepsilon > 0$ such that $[a,(b+ty)^r] \in E$ for all real $t$, $0 \leq t \leq \varepsilon$. Hence $[a,y^r]$ is in $E$. $\blacksquare$

Lemma 2.2. For a positive integer $n$, either $[a,x^n] \in E$ for all $x \in A$ or the set $G_n$ of $x \in A$ with both $[a,x^n] \notin E$ and $[a,x^*n] \notin E$ is a dense open set in $A$.

Proof. Clearly $G_n$ is open. Suppose that $G_n$ is not dense in $A$. Then there is a non-empty open set $\Omega$ in $A$ such that, for each $x \in \Omega$, either $[a,x^n] \in E$ or $[a,x^*n] \in E$. Let $b \in \Omega$ and $y \in A$. There is some $\varepsilon > 0$ so that $b + ty \in \Omega$ for all real $t$, $0 \leq t \leq \varepsilon$. For each such $t$, either $[a,(b+ty)^n] \in E$ or $[a,(b+ty)^*n] \in E$. At least one of these possibilities holds for infinitely many values of $t$. Thus either $[a,y^n] \in E$ or $[a,y^*n] \in E$. Then $A$ is the union of two closed sets, $F_1 = \{ y \in A : y^n \in E \}$ and $F_2 = \{ y \in A : y^*n \in E \}$. At least one of the sets $F_1,F_2$ must contain a non-empty open subset $\Gamma$ of $A$. Say, $F_1 \supset \Gamma$. Let $w \in \Gamma$ and $y \in A$. There is an interval of reals of positive
length with \([a, (w + ty)^n] \in E\) for each such real \(t\). Hence \([a, y^n] \in E\) for all \(y \in A\). Likewise, if \(F_2 \supseteq \Gamma\), then \([a, y^n] \in E\) for all \(y \in A\).

**Theorem 2.3.** Either \(a \in \mathfrak{H}(E)\) or the set \(S(a, E)\), of \(x \in A\) such that both \([a, x^n] \notin E\) and \([a, x^*n] \notin E\) for all positive integers \(n\), is dense in \(A\).

**Proof.** If \(S(a, E)\) is dense, then \(a \notin \mathfrak{H}(E)\) by Lemma 2.1. Suppose that \(a \notin \mathfrak{H}(E)\). Then, by Lemma 2.1, for each positive integer \(n\) there is some \(x \in A\) where \([a, x^n] \notin E\). By Lemma 2.2, each of the sets \(G_n\) of that lemma is dense and open. By the Baire category theorem, their intersection 
\(\bigcap G_n = S(a, E)\) is dense in \(A\).

**Notation.** We will continue to use \(S(a, E)\) to denote the set of \(x \in A\) such that both \([a, x^n] \notin E\) and \([a, x^*n] \notin E\) for all positive integers \(n\).

We treat the case \(E = (0)\). Let \(\mathfrak{Z}\) denote the center of \(A\).

**Theorem 2.4.** Let \(A\) be a semi-prime Banach algebra and \(E = (0)\). Either \(a \in \mathfrak{Z}\) or \(S(a, E)\) is dense in \(A\).

**Proof.** Suppose \(S(a, E)\) is not dense in \(A\). Then by Theorem 2.3, \(a\) is in the hypercenter \(\mathfrak{H}\) of \(A\). Herstein [10, Theorem 2] has shown that if a ring \(R\) has no nil ideals, then \(\mathfrak{H} = \mathfrak{Z}\). For a Banach algebra \(A\), Dixon [6] showed that the condition for \(A\) to have no nil ideals is equivalent to \(A\) being semi-prime.

For the notion of a left or right approximate identity, see [7, p. 2].

**Theorem 2.5.** Suppose that \(A\) has a left or a right approximate identity \(\{e_\lambda\}\). Then \(\mathfrak{H}(E) = \mathfrak{Z}(E)\) so that either \(a \in \mathfrak{Z}(E)\) or \(S(a, E)\) is dense in \(A\).

**Proof.** Clearly \(\mathfrak{Z}(E) \subset \mathfrak{H}(E)\). Let \(a \in \mathfrak{H}(E)\). By Lemma 2.1, there is a fixed positive integer \(n\) such that \([a, x^n] \in E\) for all \(x \in A\). We show this holds for \(n = 1\) so that \(a \in \mathfrak{Z}(E)\).

Suppose that \(n > 1\) and \([a, x^n] \in E\) for all \(x \in A\). Then \([a, (x + te_\lambda)^n] \in E\) for each given \(x \in A\), each \(e_\lambda\) and all real values of \(t\). The coefficient of \(t\) in the polynomial \([a, (x + te_\lambda)^n]\) lies in \(E\), so that

\[
[a, \sum_{j=0}^{n-1} x^j e_\lambda x^{n-1-j}] \in E.
\]

Taking the limit on \(e_\lambda\), we see that \([a, x^{n-1}] \in E\) for all \(x \in A\). Continuing in this way, we see that \(a \in \mathfrak{Z}(E)\).

Theorem 2.5 need not hold if \(A\) has no approximate identity. For example, take any \(A\) such that, for some positive integer \(n\), \(x^n = 0\) for all \(x \in A\). However, when \(E = (0)\), we have seen in Theorem 2.4 that the conclusion holds for every semi-prime \(A\). We do not know if this is the case for all \(E\), but we show that such is the case if \(A\) has a dense socle.
Theorem 2.6. Let $A$ be a semi-prime Banach algebra with a dense socle $\Sigma$. Then $\mathcal{H}(E) = \mathcal{J}(E)$ for all $E$.

Proof. Let $a \in \mathcal{H}(E)$. By Lemma 2.1, there is a positive integer $n$ so that $[a, x^n] \in E$ for all $x \in A$. We show that the validity of this statement for some $n \geq 2$ implies its validity for $n = 1$.

Let $Ap$, where $p^2 = p$, be a minimal left ideal in $A$. Note that $[a, p] \in E$ as $p = p^n$. We have

$$t^{-1}[a, (p + ty)^n - p] \in E$$

for all real values of $t \neq 0$ and any $y \in A$. Also

$$(p + ty)^n = p + t[yp + (n - 2)py + py] + \cdots,$$

where we have omitted all terms in the expansion of $(p + ty)^n$ involving higher powers of $t$. Therefore, if we let $t \to 0$, we see that

$$[a, yp + (n - 2)py + py] \in E.$$  

However, $py + \lambda p$ for a scalar $\lambda$ so that $[a, yp + py] \in E$ for all $y \in A$. Replace $y$ by $yp$ to see that $[a, yp + py] \in E$ or $[a, y] \in E$ for all $y \in A$. Therefore $[a, w] \in E$ for all $w \in \Sigma$. As $\Sigma$ is dense in $A$, we have $a \in \mathcal{J}(E)$. \hfill $\blacksquare$

Note that $E$ contains the set $\mathcal{L}$ of all commutators if and only if $[x, y] \in E$ for all $x, y \in A$ or, equivalently, $\mathcal{J}(E) = A$. Thus if $E \nsubseteq \mathcal{L}$, then $\mathcal{J}(E)$ is a proper closed linear subspace of $A$, so that its complement is dense in $A$. We denote that complement by $\mathcal{R}(E)$.

Proposition 2.7. $\mathcal{R}(E)$ is an open subset of $A$, any two elements of which are connected by one or two line segments in $\mathcal{R}(E)$.

Proof. We assume that $\mathcal{R}(E)$ is not empty. For $a \in A$, $a \in \mathcal{R}(E)$ if and only if $[a, b] \notin E$ for some $b \in A$. Let $a, b \in A$ with $[a, b] \notin E$ so that $a$ and $b$ lie in $\mathcal{R}(E)$. For any scalars $\lambda$ and $\mu$ where $\lambda \neq 0$, we have $[\lambda a + \mu b, b] \notin E$. Thus the line segment from $a$ to $b$ lies in $\mathcal{R}(E)$.

Let $v, w \in \mathcal{R}(E)$. We show that there is $y \in A$ with $[v, y] \notin E$ and $[w, y] \notin E$. For suppose otherwise. Let $F_1 = \{x \in A : [v, x] \notin E\}$ and $F_2 = \{x \in A : [w, x] \in E\}$. Then $A$ is the union of the closed sets $F_1$ and $F_2$ so that at least one of them, say $F_1$, contains a non-empty open subset. Arguing as in the proof of Lemma 2.2, we see that $[v, x] \in E$ for all $x \in A$, so that $v \notin \mathcal{R}(E)$.

Now let $y \in A$ with $[v, y] \notin E$ and $[w, y] \notin E$. The line segments joining $v$ to $y$ and $y$ to $w$ lie in $\mathcal{R}(E)$. \hfill $\blacksquare$

We say that a subset $S$ of $A$ is power-closed if $x^n \in S$ for all positive integers $n$ whenever $x \in S$.

Theorem 2.8. Suppose that $A$ has an identity $e$ and that $E \nsubseteq \mathcal{L}$. Then $\mathcal{R}(E)$ contains a dense power-closed $^*$-subset of $A$. 

Proof. Let $S_1$ be the set of all $x \in A$ for which both $x^n \in \mathcal{R}(E)$ and $x^{*n} \in \mathcal{R}(E)$ for all positive integers $n$. We show $S_1$ to be dense in $A$. Suppose otherwise. Then there exists a non-empty open set $G$ where, for each $x \in G$, either there is a positive integer $n$ with $x^n \in \mathcal{Z}(E)$ or a positive integer $m$ with $x^{*m} \in \mathcal{Z}(E)$. For positive integers $p$ and $q$, let

$$W_{p,q} = \{ x \in A : x^p \notin \mathcal{Z}(E) \text{ and } x^{*q} \notin \mathcal{Z}(E) \}.$$ 

If every $W_{p,q}$ was dense in $A$, then so also would be their intersection by the Baire category theorem. But this would contradict the existence of $G$. Then we have the existence of a non-empty open set $\Omega$ in the complement of $W_{r,s}$, say. Let $a \in \Omega$ and $y \in A$. For some $\varepsilon > 0$ either $(a + ty)^r \in \mathcal{Z}(E)$ or $(a + ty^*)^s \in \mathcal{Z}(E)$ for each $t$, $0 \leq t \leq \varepsilon$. Arguing as in the proof of Lemma 2.2, we see that there is a positive integer $n$ so that $y^n \in \mathcal{Z}(E)$ for all $y \in A$.

We employ notation used in [13, p. 204]. Let $B_r$ denote the sum of those terms in the expansion of $(a + b)^n$ for which the sum of the exponents of the $b^j$ factors is $r$. Thus $B_0 = a^n$ and $B_1 = \sum_{k=0}^{n-1} a^k ba^{n-1-k}$. For any $a$ and $b$ in $A$ and any real value of $t$, we see that $(a + tb)^n = \sum_{r=0}^{n} B_r t^r$ lies in $\mathcal{Z}(E)$. Therefore each $B_r$ is in $\mathcal{Z}(E)$ and hence $[B_0, B_1] \in E$. We use this for $a = e + tx$ and $b = y$ to see, as in [13, p. 208], that

$$[t^{-1}((e + tx)^n - e), \sum_{j=0}^{n-1} (e + tx)^j y (e + tx)^{n-1-j}] \in E$$

for every real $t \neq 0$. We let $t \to 0$ to see that $[x, y] \in E$ for all $x, y \in A$. This contradicts $E \not\supset \mathcal{C}$ so that $S_1$ is dense in $A$.  

We let $\mathcal{D}(E)$ be the set of all $[a, b] \in \mathcal{C}$ where $[a^k, b^r] \notin E$ for all positive integers $k$ and $r$.

**Theorem 2.9.** Suppose that $A$ has an identity and that $E \not\supset \mathcal{C}$. Then $\mathcal{D}(E)$ is dense in $\mathcal{C}$.

**Proof.** By Theorem 2.5 the set $S(w, E)$ is dense in $A$ for each $w \notin \mathcal{Z}(E)$. Recall that $S(w, E)$ is the intersection of countably many open dense subsets of $A$. We employ the set $S_1$ of Theorem 2.8.

Let $[a, b] \in \mathcal{C}$. Fix attention on the positive integer $n$. There is $a_n \in S_1$ where $\|a - a_n\| < n^{-1}$. As $S_1$ is power-closed, we have $a_n^k \in S_1$ for each positive integer $k$, so that $S(a_n^k, E)$ is a dense *-subset of $A$. By the Baire category theorem, the set $Q_n = \bigcap_k S(a_n^k, E)$ is dense in $A$. By its definition, every $S(a_n^k, E)$ is power-closed. Therefore, so is $Q_n$. We select $b_n \in Q_n$ with $\|b - b_n\| < n^{-1}$. Then $[a_n^k, b_n^r] \notin E$ for all positive integers $k$ and $r$. Also $[a_n, b_n] \to [a, b]$.  

Corollary 2.10. Let \( \mathfrak{A} \) be the algebra of all bounded linear operators on \( \ell_2 \). Let \( E \) be a proper closed linear subspace of \( \mathfrak{A} \). Then \( \mathfrak{D}(E) \) is dense in \( \mathfrak{A} \).

Proof. In [3, Corollary 5.2] it is pointed out that \( \mathfrak{C} \) is dense in \( \mathfrak{A} \). Therefore \( E \not\supseteq \mathfrak{C} \). We apply Theorem 2.9 to see that \( \mathfrak{D}(E) \) is dense in \( \mathfrak{A} \) as well as in \( \mathfrak{C} \).

3. Sets related to the center \( \mathfrak{Z} \). We examine the sets \( \mathfrak{Z}, \mathfrak{Z}(\mathfrak{Z}) = \{ a \in A : [a, A] \subset \mathfrak{Z} \} \) and \( \Gamma(\mathfrak{C}) \), the centralizer of \( \mathfrak{C} \), i.e., the set of \( x \in A \) such that \( [x, y] = 0 \) for all \( y \in \mathfrak{C} \).

First we show that properties of \( \mathfrak{Z} \) alone can affect the nature of \( \mathfrak{Z}(\mathfrak{Z}), \Gamma(\mathfrak{C}) \) and \( \mathfrak{C} \).

Theorem 3.1. If \( \mathfrak{Z} \) is a semi-prime algebra, then \( \mathfrak{Z}(\mathfrak{Z}) = \mathfrak{Z} \).

Proof. Let \( a \in \mathfrak{Z}(\mathfrak{Z}) \). Since \( [a, [a, x]] = 0 \) for all \( x \in A \), arguments in [9, p. 4] show that \( [a, x][a, y] = 0 \) for all \( x, y \in A \). Let \( z \in \mathfrak{Z} \). Then \( z[a, x] = [a, xz] = [a, x]z \) for all \( x, y \in A \). Hence \( [a, A] \) is an ideal in \( \mathfrak{Z} \) with \( uv = 0 \) for all \( u, v \in [a, A] \) so that \( [a, A]^2 = (0) \). As \( \mathfrak{Z} \) is semi-prime, we have \( [a, A] = (0) \) or \( a \in \mathfrak{Z} \).

Let \( J \) denote the radical of \( A \) and \( r(x) \) the spectral radius of \( x \in A \).

Lemma 3.2. Let \( x, y \in A \). If \( [x, y] \in \mathfrak{Z} \), then \( [x, y] \in J \).

Proof. Since \( x \) permutes with \( [x, y] \), by the Kleinecke–Shirokov theorem [1, p. 91], we have \( r([x, y]) = 0 \). Let \( v \in A \). As \( v \) also permutes with \( [x, y] \) we have, by [12, Theorem 1.4.1],

\[
r([x, y]v) \leq r([x, y])r(v) = 0.
\]

Therefore \( [x, y]v \) is quasi-regular for each \( v \in A \), so that \( [x, y] \in J \).

Theorem 3.3. No non-zero idempotent lies in \( \mathfrak{C} \cap \mathfrak{Z} \). If \( \mathfrak{Z} \) is semi-simple, then \( \mathfrak{C} \cap \mathfrak{Z} = (0) \).

Proof. Let \( p \) be a non-zero idempotent. Since \( p \notin J \), we see by Lemma 3.2 that \( p \notin \mathfrak{C} \cap \mathfrak{Z} \). If \( \mathfrak{Z} \) is semi-simple, then \( \mathfrak{C} \cap \mathfrak{Z} = (0) \) by Lemma 3.2.

This is an extension of the classical result [11, p. 2] that the identity in a Banach algebra cannot be a commutator. We note also that if \( A \) is semi-simple, then so is \( \mathfrak{Z} \). For, let \( x_0 \) be in the radical of \( \mathfrak{Z} \) and \( y \in A \). By [12, Theorem 1.4.1], \( r(x_0y) \leq r(x_0)r(y) = 0 \). Thus \( x_0y \) is quasi-regular for each \( y \in A \), so that \( x_0 \in J \).

Corollary 3.4. If \( \mathfrak{Z} \) is semi-prime, then \( \Gamma(\mathfrak{Z}) \) is commutative.

Proof. Let \( a, b \in \Gamma(\mathfrak{C}) \). By the Jacobi identity \( [a, [b, x]] + [b, [x, a]] + [x, [a, b]] = 0 \) for all \( x \in A \). Hence \( [x, [a, b]] = 0 \) for all \( x \in A \), so that
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As $\mathfrak{z} Z$ is semi-simple, $[a,b] = 0$ by Theorem 3.3. Thus $\Gamma(\mathfrak{c})$ is commutative. ■

**Theorem 3.5.** $\Gamma(\mathfrak{c}) \supseteq \mathfrak{z}(\mathfrak{z})$. If $J \subset \mathfrak{z}$, then $\Gamma(\mathfrak{c}) = \mathfrak{z}(\mathfrak{z})$.

**Proof.** Let $a \in \mathfrak{z}(\mathfrak{z})$. By the Jacobi identity $[a, [x, y]] + [x, [y, a]] + [y, [a, x]] = 0$ for all $x, y \in A$. But as $[y, a] \in \mathfrak{z} Z$ and $[a, x] \in \mathfrak{z} Z$, we see that $[a, [x, y]] = 0$ for all $x, y \in A$, or $a \in \Gamma(\mathfrak{c})$.

Next let $a \in \Gamma(\mathfrak{c})$. Then $[a, [a, x]] = 0$ for all $x \in A$. Arguments of Herstein [9, p. 4] show that $[a, x] A[a, x] = (0)$ for all $x \in A$. Then $[a, x] \in J \subset \mathfrak{z}$.

There are interesting examples of $A$ where $J \neq (0)$ is the set of $x \in A$ where $xA = Ax = (0)$. Of course, in that case $J \subset \mathfrak{z}$. The prototype of such instances is an example of C. Feldman [12, p. 297]. That example is commutative. More elaborate examples, where $J \neq (0)$, $J \subset \mathfrak{z}$, which are not commutative, are given in [14]; the Feldman example is a special case.

**Theorem 3.6.** If $A$ is a semi-prime algebra, then $\mathfrak{z}(\mathfrak{z}) = \Gamma(\mathfrak{c}) = \mathfrak{z}$.

**Proof.** This is valid for any algebra, not just a Banach algebra. By [9, Lemma 1.5, p. 11] we see that $\Gamma(\mathfrak{c}) = \mathfrak{z}$.

4. On the center modulo an ideal. All ideals considered here are two-sided unless otherwise specified. Henceforth $K$ will denote a closed ideal in $A$, and $\pi$ will denote the natural homomorphism of $A$ onto $A/K$. We recall the notation $\mathfrak{z}(K)$ and its complement $\mathfrak{r}(K)$ of Section 2. We examine properties of $\mathfrak{r}(K)$, motivated by the example of $A = B(\ell_2)$, the algebra of all bounded linear operators on $\ell_2$. Let $K$ be the subset of its compact operators. It follows from [3] that $\mathfrak{r}(K)$ is the set of all elements of $\mathfrak{c}$ which are not compact.

**Theorem 4.1.** Suppose that $A/K$ is semi-simple and $a \in A$. Either $[a, A] \subset K$, or the set of $x \in A$ such that $[a, x^n] \notin K$ and $[a, x^{*n}] \notin K$ for all positive integers $n$ is dense in $A$.

**Proof.** An equivalent statement is that $\mathfrak{r}(K)$ is the set of all $a \in A$ with the stated properties. First, note that $\pi(\mathfrak{s}(K))$ is the hypercenter $\mathfrak{s}^\#$ of $A/K$. As $A/K$ is semi-simple, $\mathfrak{s}^\#$ is the center $\mathfrak{z}^\#$ of $A/K$ by [10, Lemma 2]. Now $\pi^{-1}(\mathfrak{z}^\#) = \{y \in A : [y, A] \subset K\} = \mathfrak{z}(K)$. Thus $\mathfrak{z}(K) \subset \mathfrak{s}(K) \subset \pi^{-1}(\mathfrak{z}^\#) = \mathfrak{z}(K)$.

Therefore $\mathfrak{z}(K) = \mathfrak{s}(K)$. We now apply Theorem 2.3 to see that if $a \notin \mathfrak{z}(K)$, then $a$ has the required properties. ■

**Theorem 4.2.** Suppose that $A/K$ is semi-simple. If $a \in \mathfrak{z}(K) \cap \mathfrak{c}$, then $a \in K$. 
Proof. In other words, if \( a \in C \) is not in \( K \), then \( a \in R(K) \). Note that \( \pi(a) \) is a commutator lying in the center \( 3^# \) of \( A/K \) (see the preceding proof). Since \( A/K \) is semi-simple, \( \pi(a) = 0 \) by Theorem 3.3. ■

If the ideal \( K \neq A \) of Theorem 4.2 is modular and \( j \) is an identity for \( A \) modulo \( K \), then, by Theorem 4.2, \( j \notin C \) since \( j \in 3(K) \) and \( j \notin K \).

Henceforth, let \( P \) denote a modular primitive ideal where \( j \) is an identity for \( A \) modulo \( P \).

**Lemma 4.3.** \( 3(P) \) is the set of elements in \( A \) of the form \( \lambda j + y \) where \( \lambda \) is a scalar and \( y \in P \). Also, \( A = 3(P) \) if and only if \( P \supseteq C \).

**Proof.** Let \( \pi \) be the canonical homomorphism of \( A \) onto \( A/P \). \( A/P \) is a primitive algebra with \( \pi(j) \) as its identity. By [12, Corollary 2.4.5], the center \( 3^# \) of \( A/P \) is the set of scalar multiples of its identity \( \pi(j) \). As in the proof of Theorem 4.1, \( 3(P) = \pi^{-1}(3^#) \) so that \( 3(P) \) is the set of elements \( \lambda j + y \) where \( \lambda \) is a scalar and \( y \in P \).

Suppose \( A = 3(P) \). By Theorem 4.2, we have \( P \supseteq C \). Conversely, suppose that \( P \supseteq C \), so that \( [\pi(x), \pi(y)] = 0 \) for all \( x, y \in A \). Thus \( A/P \) is commutative and is the set of all \( \lambda \pi(j) \) elements. Then \( 3(P) = A \) by the description above of \( 3(P) \). ■

Thus, if \( P \not\supseteq C \) then \( R(P) \) is the set of elements not of the form \( \lambda j + y \), \( \lambda \neq 0 \), \( y \in P \) and where \( \lambda j + y \notin P \).

**Theorem 4.4.** Suppose that \( P \) is a modular maximal ideal in \( A \) and that \( P \not\supseteq C \). Then any \( a \in A \) is of the form \( x + y \) where \( x \) is in the subalgebra \( Q \) generated by \( C \), and \( y \in P \).

**Proof.** Let \( \Gamma_0 \) be the subalgebra of \( A/P \) generated by its commutators. Here \( A/P \) is a simple algebra which, as \( P \not\supseteq C \), is not commutative. By a corollary of Herstein [9, p. 6], we have \( \Gamma_0 = A/P \). Now \( \Gamma_0 \) is the set of all \( [\pi(x), \pi(y)] \), \( x, y \in A \), where \( \pi \) is the natural homomorphism of \( A \) onto \( A/P \). Then any \( a \in A \) is of the form \( x + y \), \( x \in Q \) and \( y \in P \). ■

Henceforth, we confine attention to the algebra of all bounded linear operators on \( \ell_2 \), which we denote by \( \mathfrak{A} \). Let \( \mathfrak{R} \) denote the subset of all compact operators.

**Corollary 4.5.** The subalgebra of \( \mathfrak{A} \) generated by \( C \) is all of \( \mathfrak{A} \).

**Proof.** Here \( \mathfrak{R} \) is a modular maximal ideal in \( \mathfrak{A} \). Also, \( C \supseteq \mathfrak{R} \) as shown in [2]. We apply Theorem 4.4. ■

**Theorem 4.6.** In \( \mathfrak{A} \), \( C = \mathfrak{R} \cup R(\mathfrak{R}) \).

**Proof.** Let \( I \) denote the identity of \( \mathfrak{A} \). By Lemma 4.3, \( R(\mathfrak{R}) \) is the set of elements of \( \mathfrak{A} \) not of the form \( \lambda I + T \) where \( \lambda \) is a scalar and \( T \in \mathfrak{R} \). Then
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\[ R(\mathcal{K}) \cup \mathcal{K} \] is the set of those elements not of the form \( \lambda I + T \) where \( \lambda \neq 0 \) and \( T \in \mathcal{K} \). This, however, is \( \mathcal{C} \), as shown in [3].

**Theorem 4.7.** \( R(\mathcal{K}) \) and hence \( \mathcal{C} \) contains a power-closed ∗-subset dense in \( \mathbb{A} \).

**Proof.** By Theorem 2.8, \( R(\mathcal{K}) \) possesses such a dense subset. By Theorem 4.6, \( R(\mathcal{K}) \subset \mathcal{C} \).

**Theorem 4.8.** The subset of \( \mathcal{C} \), consisting of all \( [T, U] \) with \( [T^k, U^r] \notin \mathcal{K} \) for all positive integers \( k \) and \( r \), is dense in \( \mathbb{A} \). Every \( T^k \) and \( U^r \) lies in \( \mathcal{C} \).

**Proof.** By Corollary 2.10, the subset in question is dense in \( \mathcal{C} \) and therefore dense in \( \mathbb{A} \). By Theorem 4.6, every \( T^k \) and \( U^r \) is in \( \mathcal{C} \) as \( T^k, U^r \notin \mathbb{J}(\mathcal{K}) \).

**Theorem 4.9.** \( \mathcal{C} \) is a connected subset of \( \mathbb{A} \), any two elements of which are connected by one or two line segments lying entirely in \( \mathcal{C} \).

**Proof.** By Theorem 4.6, we have \( \mathcal{C} = \mathcal{K} \cup R(\mathcal{K}) \). Any two elements of \( \mathcal{K} \) are connected by a line segment in \( \mathcal{K} \). By Proposition 2.7, any two elements of \( R(\mathcal{K}) \) are connected in \( R(\mathcal{K}) \) by one or two line segments. Now let \( T \in \mathcal{K} \) and \( U \in R(\mathcal{K}) \). We claim that \( \alpha T + \beta U \in \mathcal{C} \) for any scalars \( \alpha \) and \( \beta \). For otherwise, by [3], there exists a scalar \( \gamma \neq 0 \) and \( W \in \mathcal{K} \) so that \( \alpha T + \beta U = \gamma I + W \). Then \( \beta U = \gamma I + (W - \alpha T) \) where \( W - \alpha T \in \mathcal{K} \). This is impossible as \( U \in \mathcal{C} \).

**Theorem 4.10.** \( \mathcal{K} \) and \( (0) \) are the only closed Lie ideals of \( \mathcal{K} \).

**Proof.** \( \mathcal{K} \) is a primitive Banach algebra with a dense socle \( \mathcal{G} \). The center \( \mathcal{Z} \) of \( \mathcal{K} \) is \( (0) \). By [5, Theorem 6.1], any closed Lie ideal \( \mathcal{L} \) of \( \mathcal{K} \) must contain \( [T, U] \) for all \( T \in \mathcal{G} \) and \( U \in \mathcal{K} \). Thus \( \mathcal{L} \) contains all commutators of \( \mathcal{K} \). Then, as shown in [2], \( \mathcal{L} = \mathcal{K} \).


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