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Invertibility preserving linear mappings into $M_2(\mathbb{C})$

by

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Abstract. We study discontinuous invertibility preserving linear mappings from a Banach algebra into the algebra of $n \times n$ matrices and give an explicit representation of such a mapping when $n = 2$.

1. Introduction. An invertibility preserving linear functional on a unital Banach algebra is automatically continuous. In fact the Gleason–Kahane–Żelazko theorem ([3] and [4]) states that such a mapping is multiplicative if it is unital, or equivalently its kernel is (contained in) a maximal ideal.

Constructing discontinuous unital invertibility preserving linear mappings from a Banach algebra $\mathcal{A}$ with unit I into $M_2(\mathbb{C})$ is very simple. For example let δ and δ' be nonzero multiplicative linear functionals on $\mathcal{A}$ and μ be any discontinuous functional on $\mathcal{A}$ with $\mu(I) = 0$. Then the mapping

$$T = \begin{bmatrix} \delta & \mu \\ 0 & \delta' \end{bmatrix}$$

is a unital invertibility preserving linear mapping from $\mathcal{A}$ into $M_2(\mathbb{C})$.

In this paper we show that every discontinuous unital invertibility preserving linear mapping T from a Banach algebra into $M_2(\mathbb{C})$ is of the above form, up to similarity; i.e. there exists a nonsingular matrix $S \in M_2(\mathbb{C})$ such that

$$S^{-1}TS = \begin{bmatrix} \delta & \mu \\ 0 & \delta' \end{bmatrix},$$

with δ, δ' and μ as described above.

We present our main theorem and consequences in the following section. Let us now give some history of the subject.

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Following the work of Aupetit [1], Christensen [2] gives two generalizations of the Gleason–Kahane–Żelazko theorem for matrix-valued invertibility preserving linear mappings:

Theorem 1.1 ([2, Theorem 3.2]). *If T is a continuous unital invertibility preserving linear mapping from a unital Banach algebra $\mathcal{A}$ into $M_n(\mathbb{C})$, then* $\ker T$ *is contained in a two-sided maximal ideal.*

Theorem 1.2 ([2, Theorem 3.5]). *Let T be a continuous unital linear mapping from a unital Banach algebra $\mathcal{A}$ into $M_n(\mathbb{C})$. Then T is invertibility preserving if and only if for all positive integers k and any element a in $\mathcal{A}$,*

$$\operatorname{tr}(T(a^k)) = \operatorname{tr}((T(a))^k).$$

Since the Gleason–Kahane–Żelazko theorem needs no assumption of continuity, one can ask if the continuity assumption is removable from the above two theorems. This is in fact the case at least for $n = 2$ (Corollaries 2.2 and 2.3 below).

Aupetit ([1, p. 13]) proves that the continuity is automatic if T is surjective, and even a stronger result:

Theorem 1.3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras and $T : \mathcal{A} \to \mathcal{B}$ is a linear mapping such that $\varrho(Tx) \le \varrho(x)$ for all $x \in \mathcal{A}$. If $\mathcal{B}$ is semisimple and the spectral radius ϱ is continuous on $\mathcal{B}$, then T is continuous provided that the quasi-nilpotent elements of $T(\mathcal{A})$ are dense in the set of all quasi-nilpotent elements of $\mathcal{B}$.*

After finding an explicit form for discontinuous unital invertibility preserving linear mappings from a unital Banach algebra into $M_2(\mathbb{C})$, we easily deduce Theorems 1.1 and 1.2 for $n = 2$, whenever T is discontinuous. More related results can be found in [5].

2. Main theorem

Main Theorem 2.1. *Let T be a unital invertibility preserving linear mapping from a Banach algebra $\mathcal{A}$ into $M_2(\mathbb{C})$. If T is discontinuous, then T is similar to a mapping of the form*

$$\begin{bmatrix} \delta & \mu \\ 0 & \delta' \end{bmatrix},$$

where δ and δ' are multiplicative functionals on $\mathcal{A}$ and μ is a discontinuous functional on $\mathcal{A}$ with $\mu(I) = 0$.

Before starting the proof let us give some corollaries.

Corollary 2.2. *Let T be a unital invertibility preserving linear mapping from a unital Banach algebra into $M_2(\mathbb{C})$. Then* $\ker T$ *is contained in a two-sided maximal ideal.*

Proof. If T is continuous, the result is a special case of Theorem 1.1. If T is not continuous, then $\ker T \subset \ker\delta$ and $\ker\delta$ is a maximal ideal. ∎

COROLLARY 2.3. *Let T be a unital linear mapping from a unital Banach algebra into $M_2(\mathbb{C})$. Then T preserves invertibility if and only if for every natural number k and every element $a \in \mathcal{A}$,*

$$\mathrm{tr}(T(a^k)) = \mathrm{tr}((T(a))^k).$$

Proof. First suppose T preserves invertibility. If T is continuous, the result is a special case of Theorem 1.2. If T is not continuous, since T is similar to the stated form in Theorem 2.1, we can easily check the equality by noting that δ and δ' are multiplicative.

For the converse, we refer to the proof of Theorem 3.5 in [2]; there is no assumption of continuity for proving the converse there. ∎

COROLLARY 2.4. *Let T be a unital invertibility preserving linear mapping from a Banach algebra into $M_2(\mathbb{C})$. Then the set-valued function $\sigma\circ T$ is continuous on $\mathcal{A}$, where σ denotes the spectrum function and the metric on the sets in the range of $\sigma\circ T$ is the Hausdorff metric.*

Proof. If T is continuous, the result follows from the fact that the spectrum function is continuous on $M_n(\mathbb{C})$ for every natural number n. If T is not continuous then the continuity of $\sigma\circ T$ follows from the continuity of δ and δ'. ∎

To prove the main theorem, we need the following three lemmas.

LEMMA 2.5. *If T is a unital invertibility preserving linear mapping from a Banach algebra $\mathcal{A}$ into $M_n(\mathbb{C})$, then the functions $\mathrm{tr}\circ T$ and $\det\circ T$ are continuous at 0. Consequently, since $\mathrm{tr}\circ T$ is linear, it is continuous everywhere on $\mathcal{A}$.*

Proof. Note that for any $a \in \mathcal{A}$, $|\mathrm{tr}\circ T(a)| \le n\|a\|$, and $|\det\circ T(a)| \le \|a\|^n$. ∎

LEMMA 2.6. *If $T = \left[\begin{smallmatrix}\nu & \mu\\ \mu' & \nu'\end{smallmatrix}\right]$ is a discontinuous unital invertibility preserving linear mapping from a Banach algebra $\mathcal{A}$ into $M_2(\mathbb{C})$, then either μ or μ' is discontinuous.*

Proof. Suppose that, on the contrary, both μ and μ' are continuous. The discontinuity of T implies that either ν or ν' is discontinuous. Let ν be discontinuous. All the elements of $\ker\nu\cap\ker\mu$ are noninvertible, since they are mapped to noninvertible matrices. Therefore, all elements of its closure, $\ker\mu$, including I, are noninvertible, which is a contradiction. If ν' is discontinuous, we do the same. ∎

LEMMA 2.7. *If f and g are linear functionals on a Banach algebra $\mathcal{A}$ such that f is discontinuous and fg is continuous at* 0, *then g is identically zero on $\mathcal{A}$.*

Proof. Let x_n be a sequence in $\mathcal{A}$ and y_n be a sequence in $\ker f$ such that $x_n \to 0$ and $y_n \to 0$. Then

$$f(x_n+y_n)g(x_n+y_n) = f(x_n)g(x_n) + f(x_n)g(y_n) \to 0,$$

by the continuity of fg at 0. This gives, again by continuity,

$$f(x_n)g(y_n) \to 0. \tag{1}$$

Suppose $y \in \ker f$. If we choose $x_n \to 0$ such that $f(x_n) \to \infty$ and put $y_n = y/f(x_n)$, then (1) shows that $g(y) = 0$. Since $y \in \ker f$ was arbitrary, we conclude that $\ker f \subseteq \ker g$. Hence there exists a scalar r such that $g = rf$. If $r \neq 0$, it follows that $f^2 = (1/r)fg$ is continuous at 0, and this yields the continuity of f at 0, which is a contradiction. So, $r = 0$ and g is identically zero on $\mathcal{A}$. ∎

REMARK. Note that the lemma above holds for all topological vector spaces $\mathcal{A}$.

Proof of Theorem 2.1. Let $T = \left[\begin{smallmatrix} \nu & \mu \\ \mu' & \nu' \end{smallmatrix}\right]$ be as stated in the theorem. By Lemma 2.6, either μ or μ' is discontinuous. Without loss of generality we can assume that μ is discontinuous: otherwise replace T by the similar mapping

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \nu' & \mu' \\ \mu & \nu \end{bmatrix}.$$

If we consider the restriction of T to the dense subspace $M = \ker \mu$, we observe that $\nu|_M$ and $\nu'|_M$ are invertibility preserving linear functionals on M, so they have respective continuous extensions δ and δ', to the whole $\mathcal{A}$. By the Gleason–Kahane–Żelazko theorem, δ and δ' are multiplicative.

Now set $\alpha = \nu - \delta$ and $\beta = \nu' - \delta'$. Clearly the functionals α and β vanish on $M = \ker \mu$, hence there exist scalars s and t such that $\alpha = s\mu$ and $\beta = t\mu$. This gives

$$\operatorname{tr} \circ T = \delta + \delta' + (s+t)\mu.$$

The continuity of $\operatorname{tr} \circ T$ together with the discontinuity of μ give $s = -t$. We now have

$$T = \begin{bmatrix} \delta + s\mu & \mu \\ \mu' & \delta' - s\mu \end{bmatrix}$$

and

$$\det \circ T = \delta\delta' - \mu[s(\delta - \delta') + s^2\mu + \mu'].$$

The continuity of $\det \circ T$ and $\delta\delta'$ at 0 implies that the function

$$\mu[s(\delta - \delta') + s^2\mu + \mu']$$

is continuous at 0. Since μ is discontinuous, Lemma 2.7 shows that the functional $s(\delta-\delta')+s^2\mu+\mu'$ is identically zero. Now $\mu'=s(\delta'-\delta)-s^2\mu$ everywhere on $\mathcal{A}$, and

$$T=\begin{bmatrix}\delta+s\mu & \mu\\ s(\delta'-\delta)-s^2\mu & \delta'-s\mu\end{bmatrix}.$$

To end the proof note that

$$\begin{bmatrix}1 & 0\\ s & 1\end{bmatrix}T\begin{bmatrix}1 & 0\\ s & 1\end{bmatrix}^{-1}=\begin{bmatrix}\delta & \mu\\ 0 & \delta'\end{bmatrix}$$

as desired. ■

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