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Weak type radial convolution operators on free groups

by

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Abstract. Radial convolution operators on free groups with nonnegative kernel of weak type $(2,2)$ and of restricted weak type $(2,2)$ are characterized. Estimates of weak type (p,p) are obtained as well for $1 < p < 2$.

1. Introduction. A discrete group G is called *amenable* if there exists a linear functional m on $\ell^\infty_{\mathbb{R}}(G)$ such that

(1) $\inf_{x\in G} f(x) \le m(f) \le \sup_{x\in G} f(x)$,

(2) $m({}_xf) = m(f)$, where ${}_xf(y) = f(x^{-1}y)$.

m is called a *left invariant mean*. Then the functional $M(f) = m(m(f_x))$ satisfies (1), (2) and is also right invariant, where $f_x(y) = f(yx)$.

Let G be a discrete group. Consider a symmetric probability measure μ on G, i.e.

$$\mu = \sum_{x\in G} \mu(x)\delta_x, \qquad \mu(x) \ge 0,\ \sum_{x\in G}\mu(x) = 1,\ \mu(x^{-1}) = \mu(x).$$

The left convolution operator $\lambda(\mu)$ with μ is bounded on $\ell^2(G)$ and

$$\|\lambda(\mu)(f)\|_2 = \|\mu * f\|_2 \le \|f\|_2, \quad f \in \ell^2(G).$$

Indeed,

$$\|\mu * f\|_2 = \Big\| \sum_{x\in G} \mu(x)[\delta_x * f]\Big\|_2 \le \sum_{x\in G}\mu(x)\|\delta_x * f\|_2 = \|f\|_2.$$

Thus $\|\lambda(\mu)\|_{2\to 2} \le 1$.

Kesten [5] showed that a discrete group G is amenable iff for any symmetric probability measure μ on G we have $\|\lambda(\mu)\|_{2\to 2} = 1$. He also showed

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that G is amenable if this condition is satisfied for one measure μ such that $\operatorname{supp}\mu$ generates G algebraically. In particular, let G be generated by $g_1, \ldots, g_k$ and $\mu = (2k)^{-1}\sum_{i=1}^{k}(\delta_{g_i} + \delta_{g_i^{-1}})$. Then G is amenable if and only if $\|\lambda(\mu)\|_{2\to 2} = 1$.

In [4] Følner came up with another property equivalent to amenability. We say that a discrete group G satisfies the *Følner condition* if for any number $\varepsilon > 0$ and any finite set $K \subset G$ there exists a finite set $N \subset G$ such that

$$|xN \triangle N| < \varepsilon|N|, \qquad x \in K, \tag{1}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. In other words, N is almost K-invariant. He showed that G is amenable if and only if the Følner condition holds.

Assume that G is amenable. Let μ be a probability measure with finite support K. For $\varepsilon = \eta^2 > 0$ choose N so as to satisfy (1). Then

$$\begin{aligned}\|\mu * \chi_N - \chi_N\|_2 &= \Big\|\sum_{x\in K}\mu(x)[\chi_{xN} - \chi_N]\Big\|_2 \le \sum_{x\in K}\mu(x)\|\chi_{xN} - \chi_N\|_2\\ &= \sum_{x\in K}\mu(x)\|\chi_{xN\triangle N}\|_2 = \sum_{x\in K}\mu(x)|xN \triangle N|^{1/2} \le \eta|N|^{1/2} = \eta\|\chi_N\|_2.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \mu * \chi_N, \chi_N\rangle_{\ell^2(G)} &= \langle \chi_N, \chi_N\rangle_{\ell^2(G)} + \langle \mu * \chi_N - \chi_N, \chi_N\rangle_{\ell^2(G)}\\ &\ge (1-\eta)\|\chi_N\|_2^2,\end{aligned}$$

which implies

$$\sup_{N,M \text{ finite}} \frac{\langle \mu * \chi_N,\, \chi_M\rangle}{\|\chi_N\|_2\|\chi_M\|_2} = 1 = \|\lambda(\mu)\|_{2\to 2}. \tag{2}$$

The same holds (with the same proof) for any $1 < p < \infty$, i.e.

$$\sup_{N,M \text{ finite}} \frac{\langle \mu * \chi_N,\, \chi_M\rangle}{\|\chi_N\|_p\|\chi_M\|_{p'}} = 1 = \|\lambda(\mu)\|_{p\to p}, \tag{3}$$

where $p' = p/(p-1)$.

We will use the notion of Lorentz $L^{p,q}$ spaces (see [1]). Consider a general σ-finite measure space (Ω, ω) and $1 < p < \infty$. For $f \in L^p(\Omega, \omega)$ and $t > 0$ we have

$$t^p\,\omega\{x : |f(x)| > t\} \le \int_{\Omega} |f(x)|^p\, d\omega(x).$$

The functions for which the left hand side is bounded form a linear space

$$L^{p,\infty}(\Omega, \omega) = \{f : \sup_{t>0} t^p\omega\{x : |f(x)| > t\} < \infty\},$$

called the *weak L^p space*. This space contains $L^p(\Omega, \omega)$.

For $p' = p/(p-1)$ the predual of $L^{p',\infty}(\Omega,\omega)$ with respect to the standard inner product is denoted by $L^{p,1}(\Omega,\omega)$. We have

$$L^{p,1}(\Omega,\omega) \subset L^p(\Omega,\omega) \subset L^{p,\infty}(\Omega,\omega).$$

For $p > 1$ these spaces are normed.

Any linear operator mapping L^p into itself is called of *strong type* (p,p). Linear operators T mapping $L^p(\Omega,\omega)$ into $L^{p,\infty}(\Omega,\omega)$ are called *of weak type* (p,p), while those which map $L^{p,1}(\Omega,\omega)$ into $L^{p,\infty}(\Omega,\omega)$ are called *of restricted weak type* (p,p).

We will use the following facts. A linear operator T is bounded from $L^{p,1}$ into a Banach space X if and only if

$$\|T\|_{L^{(p,1)}\to X} = \sup_{E\subset\Omega} \frac{\|T\chi_E\|_X}{\|\chi_E\|_p} < \infty. \tag{4}$$

A linear operator T is bounded from $L^{p,1}$ into $L^{p,\infty}$ if and only if

$$\|T\|_{(p,1)\to(p,\infty)} = \sup_{E,F\subset\Omega} \frac{|\langle T\chi_E,\, \chi_F\rangle|}{\|\chi_E\|_p\|\chi_F\|_{p'}} < \infty. \tag{5}$$

Using this and duality between $L^{(p',1)}$ and $L^{(p,\infty)}$ we obtain

$$\|T\|_{p\to(p,\infty)} = \|T^*\|_{(p',1)\to p'} = \sup_{E\subset\Omega} \frac{\|T^*\chi_E\|_{p'}}{\|\chi_E\|_{p'}}. \tag{6}$$

The equalities (2) and (3) can be interpreted as follows. If the group G is discrete and amenable and μ is a symmetric probability measure on G, then

$$\begin{aligned} \|\lambda(\mu)\|_{p\to p} &= \|\lambda(\mu)\|_{(p',1)\to p'} = \|\lambda(\mu)\|_{p\to(p,\infty)} \\ &= \|\lambda(\mu)\|_{(p,1)\to(p,\infty)} = 1. \end{aligned} \tag{7}$$

Hence for these groups convolution operators with nonnegative functions of strong type (p,p), of weak type (p,p) and of restricted weak type (p,p) coincide for any $1 < p < \infty$.

The situation is entirely different for nonamenable groups. Only special examples have been studied. It has been shown [9] that for $p = 2$ and $G = \mathbb{F}_k$, the free group on k generators, $k \geq 2$, there exist nonnegative functions f on G such that $\|\lambda(f)\|_{2\to(2,\infty)}$ is finite while $\|\lambda(f)\|_{2\to2}$ is infinite, i.e. there exist convolution operators with nonnegative functions of weak type $(2,2)$ which are not of strong type $(2,2)$. The same has been shown for $1 < p < 2$ [10]. These functions f can be chosen to be radial, i.e. constant on elements of the group G of the same length. It is an open problem if these results remain true for any discrete nonamenable group.

In this work will focus on $G = \mathbb{F}_k$. We are going to determine all nonnegative radial functions f on G such that $\lambda(f)$ is of weak type $(2,2)$, as well those f for which $\lambda(f)$ is of restricted weak type $(2,2)$. In particular, we prove that these spaces are different. Next we will turn our attention to the case $1 < p < 2$. By using interpolation machinery, duality and the results for $p = 2$ we will be able to determine the nonnegative radial functions f for which $\lambda(f)$ is of weak type (p, p). In this way we obtain a simpler proof of the upper estimate of $\|\lambda(f)\|_{p\to(p,\infty)}$ obtained in [3]. Our method does not rely on any deep theorems of representation theory.

2. Radial convolution operators of weak type $(2, 2)$**.** Let $\mathbb{F}_k = \mathrm{gp}\{g_1, \ldots, g_k\}$ be a free group on $k \geq 2$ generators. The group consists of reduced words in generators and their inverses. The reduced representation of a word is unique. The number of letters in it defines a length function on $\mathbb{F}_k$. Let χ_n denote the indicator function of the words of length n. There are $2k(2k-1)^{n-1}$ such words, as we have $2k$ choices for the first letter and $2k-1$ choices for every consecutive one. Let $q = 2k - 1$. The next theorem generalizes the estimate for $\|\lambda(\chi_n)\|_{2\to(2,\infty)}$ given in [9].

THEOREM 1. *Let* $f = \sum_{n=0}^{\infty} f_n \chi_n$. *The operator* $\lambda(f)$ *is of weak type* $(2, 2)$ *if*

$$A(f) := \sum_{n,m=0}^{\infty} |f_n|\,|f_m| q^{-(n+m)/2}\{1 + \min(n, m)\} < \infty.$$

Moreover, *if* $f_n \geq 0$ *the condition is necessary and*

$$\frac{1}{6} A(f) \leq \|\lambda(f)\|^2_{2\to(2,\infty)} \leq 4A(f).$$

Proof. By (7), instead of estimating $\|\lambda(f)\|_{2\to(2,\infty)}$ we may estimate $\|\lambda(f)\|_{(2,1)\to 2}$, which (see (4)) is equivalent to

$$\sup_{E\subset\mathbb{F}_r} \frac{\|f * \chi_E\|_2}{|E|^{1/2}}.$$

We have

$$\|f * \chi_E\|_2^2 = \langle f * f * \chi_E, \chi_E\rangle = \sum_{n,m=0}^{\infty} f_n f_m \langle \chi_n * \chi_m * \chi_E, \chi_E\rangle.$$

Simple calculation shows that for $n \geq 1$ we have

$$\chi_n * \chi_m = q^{n-1}\delta_n^m \chi_0 + \sum_{\substack{k=|n-m| \\ k\equiv n+m \bmod 2}}^{n+m} q^{(n+m-k)/2}\,\chi_k.$$

Clearly $\chi_0 * \chi_0 = \chi_0$. Therefore

$$\chi_n * \chi_m \leq 2 \sum_{\substack{k=|n-m| \\ k\equiv n+m \bmod 2}}^{n+m} q^{(n+m-k)/2}\chi_k.$$

Hence

$$\|f * \chi_E\|_2^2 \leq 2 \sum_{n,m=0}^{\infty} f_n f_m\, q^{(n+m)/2} \sum_{\substack{k=|n-m| \\ k\equiv n+m \bmod 2}}^{n+m} q^{-k/2}\,\langle \chi_k * \chi_E, \chi_E\rangle.$$

Lemma 1.

$$\langle \chi_k * \chi_E, \chi_E \rangle \le 2q^{[k/2]}|E|.$$

Proof. Define an operator P_k by the rule

$$\langle P_k\delta_x, \delta_y\rangle = \begin{cases} \langle \chi_k * \delta_x, \delta_y\rangle & \text{if } |x| \ge |y|, \\ 0 & \text{if } |x| < |y|. \end{cases}$$

Then

$$\langle \chi_k * \delta_x, \delta_y\rangle \le \langle P_k\delta_x, \delta_y\rangle + \langle \delta_x, P_k\delta_y\rangle.$$

This implies

$$\langle \chi_k * \chi_E, \chi_E\rangle \le 2\langle P_k\chi_E, \chi_E\rangle \le 2\|P_k\chi_E\|_1 \le 2|E| \sup_x \|P_k\delta_x\|_1.$$

Next

$$P_k\delta_x = \sum_{\substack{|w|=k \\ |wx|\le|x|}} \delta_{wx}.$$

Let $w = w_1w_2$ where $|w_1| \le |w_2| \le (k+1)/2$. The conditions $|w| = k$ and $|wx| \le |x|$ imply that w_2 is determined by the first $[(k+1)/2]$ letters of x. Hence we have as many terms in the sum as choices for w_1, i.e. at most $q^{[k/2]}$. Thus

$$\|P_k\delta_x\|_1 \le q^{[k/2]}.$$

Therefore

$$\langle \chi_k * \chi_E, \chi_E\rangle \le 2q^{[k/2]}|E|. \ ■$$

Lemma 1 implies that

$$\frac{\|f * \chi_E\|_2^2}{|E|} \le 4\sum_{n,m=0}^{\infty} |f_n|\,|f_m| q^{(n+m)/2} \sum_{\substack{k=|n-m| \\ k\equiv n+m \bmod 2}}^{n+m} 1 = 4\sum_{n,m=0}^{\infty} |f_n|\,|f_m| q^{(n+m)/2}\{1+\min(m,n)\}.$$

We obtain the upper estimate

$$\|\lambda(f)\|_{2\to(2,\infty)}^2 \le 4\sum_{n,m=0}^{\infty} |f_n|\,|f_m| q^{(n+m)/2}\{1+\min(m,n)\}.$$

On the other hand, if $f_n \ge 0$ we have

$$\|\lambda(f)\|_{2\to(2,\infty)}^2 \ge \frac{q}{q+1}\, q^{-2k}\|f * \chi_{2k}\|_2^2 \ge \frac{2}{3}\, q^{-2k}\Big\|\sum_{n=0}^{\infty} f_n(\chi_n * \chi_{2k})\Big\|_2^2$$

$$\ge \frac{2}{3}\, q^{-2k}\Big\|\sum_{n=0}^{\infty} f_n \sum_{\substack{l=|n-2k| \\ l\equiv n \bmod 2}}^{n+2k} q^{(n+2k-l)/2}\,\chi_l\Big\|_2^2$$

$$= \frac{2}{3}\Big\| \sum_{l=0}^{\infty} q^{-l/2}\chi_l\Big(\sum_{\substack{n=|2k-l|\\ n\equiv l \bmod 2}}^{2k+l} f_n q^{n/2}\Big)\Big\|_2^2 \ge \frac{2}{3}\sum_{l=0}^{\infty}\Big(\sum_{\substack{n=|2k-l|\\ n\equiv l \bmod 2}}^{2k+l} f_n q^{n/2}\Big)^2$$

$$\ge \frac{2}{3}\sum_{l=0}^{2k}\Big(\sum_{\substack{n=2k-l\\ n\equiv l \bmod 2}}^{2k+l} f_n q^{n/2}\Big)^2 \ge \frac{2}{3}\sum_{n,m=0}^{2k} f_n f_m q^{(n+m)/2} \sum_{\substack{l=\max(2k-n,2k-m)\\ l\equiv n\equiv m \bmod 2}}^{2k} 1.$$

Considering even or odd values of m and n gives

$$\|\lambda(f)\|^2_{2\to(2,\infty)} \ge \frac{2}{3}\sum_{n,m=0}^{k} f_{2n}f_{2m}q^{n+m}\{1+\min(n,m)\},$$

$$\|\lambda(f)\|^2_{2\to(2,\infty)} \ge \frac{2}{3}\sum_{n,m=0}^{k-1} f_{2n+1}f_{2m+1}q^{n+m+1}\{1+\min(n,m)\}.$$

Since k is arbitrary,

$$\|\lambda(f)\|^2_{2\to(2,\infty)} \ge \frac{1}{3}\sum_{\substack{n,m=0\\ n\equiv m \bmod 2}}^{\infty} f_n f_m q^{(n+m)/2}\{1+\min(n,m)\}.$$

This implies

$$\|\lambda(f)\|^2_{2\to(2,\infty)} \ge \frac{1}{6}\sum_{n,m=0}^{\infty} f_n f_m q^{(n+m)/2}\{1+\min(n,m)\},$$

because the matrix $a(n,m) = 1+\min(n,m)$ is positive definite. ∎

THEOREM 2. *For $n \ge 0$ we have*

$$\|\lambda(\chi_n)\|_{(2,1)\to(2,\infty)} \le cq^{n/2}.$$

Proof. We have

$$\|\lambda(\chi_n)\|_{(2,1)\to(2,\infty)} = \sup_{E,F\subset\mathbb{F}_r} \frac{\langle \chi_n * \chi_E, \chi_F\rangle}{|E|^{1/2}|F|^{1/2}}.$$

The proof will be completed if we show

$$\langle \chi_n * \chi_E, \chi_F\rangle \le cq^{n/2}|E|^{1/2}|F|^{1/2}. \tag{8}$$

We will prove (8) by modifying the argument used in the proof of Lemma 1. Fix $\alpha\in\mathbb{R}$. Let Q_n^α denote the operator defined by the rule

$$\langle Q_n^\alpha \delta_x, \delta_y\rangle = \begin{cases} \langle \chi_n * \delta_x, \delta_y\rangle & \text{if } |x| \ge q^\alpha |y|,\\ 0 & \text{if } |x| < q^\alpha|y|.\end{cases}$$

Then

$$\langle \chi_n * \delta_x, \delta_y\rangle \le \langle Q_n^\alpha \delta_x, \delta_y\rangle + \langle \delta_x, Q_n^{-\alpha}\delta_y\rangle.$$

This implies

$$\langle \chi_n * \chi_E, \chi_F\rangle \le \|Q_n^{\alpha}\chi_E\|_1 + \|Q_n^{-\alpha}\chi_F\|_1 \le |E| \sup_x \|Q_n^{\alpha}\delta_x\|_1 + |F| \sup_x \|Q_n^{-\alpha}\delta_x\|_1 \tag{9}$$

Next

$$Q_n^{\alpha}\delta_x = \sum_{\substack{|w|=n\\ |wx|\le q^{-\alpha}|x|}} \delta_{wx}.$$

Let $w = w_2 w_1$ where $|w_1| = [n/2] + [\alpha]$ and $|w_2| = n - [n/2] - [\alpha]$. The conditions $|w| = n$ and $|wx| \le q^{-\alpha}|x|$ imply that w_1 is determined by the first $[n/2] + [\alpha]$ letters of x. Hence we have as many terms in the sum as choices for w_2, i.e. at most $q^{n-[n/2]-[\alpha]}$. Thus

$$\|Q_n^{\alpha}\delta_x\|_1 \le q^{3/2} q^{-\alpha} q^{n/2}. \tag{10}$$

Similarly

$$\|Q_n^{-\alpha}\delta_x\|_1 \le q^{3/2} q^{\alpha} q^{n/2}.$$

Hence by (9) we get

$$\langle \chi_n * \chi_E, \chi_F\rangle \le q^{3/2} q^{n/2} \{q^{-\alpha}\,|E| + q^{\alpha}\,|F|\}.$$

Choosing $\alpha = (\log|E| - \log|F|)/(2\log q)$ gives

$$\langle \chi_n * \chi_E, \chi_F\rangle \le 2q^{3/2} q^{n/2} |E|^{1/2}|F|^{1/2}. \ ■$$

THEOREM 3. *Let $f = \sum_{n=0}^{\infty} f_n\chi_n$ and $f_n \ge 0$. The operator $\lambda(f)$ is of restricted weak type $(2, 2)$ if and only if $f \in L^{2,1}$.*

Proof. By Theorem 2 we have

$$\|\lambda(\chi_n)\|_{(2,1)\to(2,\infty)} \le C q^{n/2}$$

for some constant $C > 0$. Let $f = \sum_{n=0}^{\infty} f_n\chi_n$. Then the triangle inequality yields

$$\|f\|_{(2,1)\to(2,\infty)} \le C \sum_{n=0}^{\infty} f_n q^{n/2}.$$

By [8, Lemma 1],

$$\sum_{n=0}^{\infty} f_n q^{n/2} \approx \|f\|_{(2,1)}. \tag{11}$$

On the other hand, for $f_n \ge 0$ we have

$$\|f\|_{(2,1)\to(2,\infty)} \ge C \sup_{n,m} q^{-(n+m)/2}\,\langle f * \chi_n, \chi_m\rangle = C \sup_{n,m} q^{-(n+m)/2}\,\langle f, \chi_m * \chi_n\rangle \ge C \sum_{\substack{k=|n-m|\\ k\equiv n+m \bmod 2}}^{n+m} q^{k/2} f_k.$$

Taking $m = n$ or $m = n + 1$ and letting n tend to infinity gives

$$\|f\|_{(2,1)\to(2,\infty)} \geq C \sum_{k=0}^{\infty} q^{2k/2} f_{2k},$$

$$\|f\|_{(2,1)\to(2,\infty)} \geq C \sum_{k=0}^{\infty} q^{(2k+1)/2} f_{2k+1}.$$

Therefore $\sum_{k=0}^{\infty} q^{k/2} f_k < \infty$, i.e. $f \in L^{2,1}$ by (11). ∎

3. Weak type (p,p) for $1 < p < 2$. Part of the next theorem, namely the first inequality, is known from [3]. Actually, it has been simply observed there that the inequality follows by applying a multilinear interpolation theorem to Pytlik's estimate for $\|\sum f_n \lambda(\chi_n)\|_{p\to p}$ given in [8]. We will reprove the second inequality by applying the same interpolation theorem to restricted weak type estimates given in the previous section. In this way we skip the $p \to p$ estimates whose proof in [8] is tricky, and the later proof in [3] makes use of advanced representation theory.

Theorem 4. *For $1 < p < 2$ and $f = \sum_{n=0}^{\infty} f_n \chi_n$ we have*

$$\|\lambda(f)\|_{p\to(p,\infty)} \leq C\|f\|_{(p,p')}.$$

Moreover, *if $f \geq 0$ then*

$$c\|f\|_{(p,p')} \leq \|\lambda(f)\|_{p\to(p,\infty)}.$$

Proof. The subscript r will denote the subspace of radial functions, i.e. functions of the form $\sum_{n=0}^{\infty} f_n \chi_n$, where f_n are complex coefficients. By Theorem 3 we have $L_r^{2,1} * L^{2,1} \subset L^{2,\infty}$. On the other hand, $L_r^1 * L^1 \subset L^1$. By the multilinear interpolation theorem [1, 3.13.5, p. 76] we get $L_r^{p,s} * L^{p,t} \subset L^{p,u}$ where $1 \leq p < 2$ and $1 + 1/u = 1/s + 1/t$. Taking $u = \infty$, $t = p$ and $s = p'$ gives $L_r^{p,p'} * L^p \subset L^{p,\infty}$. This gives the first inequality.

On the other hand, for $f = \sum_{n=0}^{\infty} f_n \chi_n$ by (4) and by duality (6) we have

$$\|\lambda(f)\|_{p\to(p,\infty)} = \|\lambda(f)\|_{(p',1)\to p'} \geq c \sup_n q^{-n/p'} \|f * \chi_n\|_{p'}.$$

Similarly to the proof of Theorem 1 we obtain

$$f * \chi_n \geq \sum_{l=0}^{\infty} q^{(n-l)/2} \Big[\sum_{\substack{m=|n-l| \\ m \equiv l+n \bmod 2}}^{l+n} q^{m/2} f_m \Big] \chi_l$$

Hence

$$q^{-n}\|f*\chi_n\|_{p'}^{p'} \ge \sum_{l=0}^{n} q^{p'(n-l)/2} q^{l-n}\Big[\sum_{\substack{m=n-l\\ m\equiv l+n \bmod 2}}^{l+n} q^{m/2} f_m\Big]^{p'}$$
$$\ge \sum_{l=0}^{n} q^{(n-l)(p'-1)} f_{n-l}^{p'} = \sum_{l=0}^{n} q^{lp'/p} f_l^{p'}.$$

Taking the supremum with respect to n and raising to the power $1/p'$ gives

$$\|\lambda(f)\|_{p\to(p,\infty)} \ge c\Big(\sum_{n=0}^{\infty} f_n^{p'} q^{np'/p}\Big)^{1/p'}.$$

Since the norm of $f=\sum_{n=0}^{\infty} f_n\chi_n$ in $L_r^{p,p'}$ is equivalent to $(\sum_{n=0}^{\infty} f_n^{p'} q^{np'/p})^{1/p'}$ the second inequality is proved. ■

4. Other estimates

THEOREM 5. *For* $1 \le s \le 2 \le t \le \infty$ *we have*

$$cn^{1-1/s+1/t} q^{n/2} \le \|\lambda(\chi_n)\|_{(2,s)\to(2,t)} \le Cn^{1-1/s+1/t} q^{n/2}.$$

Proof. In order to get the second inequality we use only interpolation. First observe that the inequality is valid for $s=2,\ t=\infty$ by Theorem 1 and for $s=t=2$ by [2, 7]. Hence by complex interpolation of the Lorentz spaces it is valid for $s=2$, $t\ge 2$.

Next it is valid for $s=1$, $t=\infty$ by Theorem 3 and for $s=t=2$. Hence by complex interpolation it is valid for $1\le s\le 2$, $t=s'$.

Now we can use again complex interpolation to get the conclusion for $1\le s\le 2\le t\le\infty$.

The estimate from below can be obtained from

$$\|\lambda(\chi_n)\|_{(2,s)\to(2,t)} \ge \frac{\|\chi_n * f\|_{(2,t)}}{\|f\|_{(2,s)}},$$

where $f=\sum_{k=0}^{2n} q^{-k/2}\chi_k$. ■

Theorems 1, 2 and 5 suggest the following.

CONJECTURE. Let $f=\sum_{n=0}^{\infty} f_n\chi_n \ge 0$. Then for $1\le s\le 2$ the operator $\lambda(f)$ maps $L^{2,s}$ into $L^{2,\infty}$ if and only if

$$\sum_{n,m=0}^{\infty} f_n f_m q^{-(n+m)/2}\{1+\min(n^{1/s'}, m^{1/s'})\} < \infty.$$

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