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On solvability of the cohomology equation in function spaces

by

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Abstract. Let T be an endomorphism of a probability measure space $(\Omega, \mathcal{A}, \mu)$, and f be a real-valued measurable function on Ω. We consider the cohomology equation $f = h \circ T - h$. Conditions for the existence of real-valued measurable solutions h in some function spaces are deduced. The results obtained generalize and improve a recent result of Alonso, Hong and Obaya.

1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space and $T : \Omega \to \Omega$ be an endomorphism of $(\Omega, \mathcal{A}, \mu)$. Thus, if $A \in \mathcal{A}$ then $T^{-1}A \in \mathcal{A}$ and $\mu(T^{-1}A) = \mu(A)$. T is called an *automorphism* of $(\Omega, \mathcal{A}, \mu)$ if T is one-to-one and onto, and T^{-1} is again an endomorphism of $(\Omega, \mathcal{A}, \mu)$. If there does not exist a set A in $\mathcal{A}$ with $T^{-1}A = A$ and $0 < \mu(A) < 1$, then T is called *ergodic*. Let f be a real-valued measurable function on Ω. Then we define

$$S_0 f(\omega) = 0 \quad \text{and} \quad S_j f(\omega) = \sum_{k=0}^{j-1} f(T^k \omega) \quad \text{for } j \ge 1,$$

so that the cocycle identity $S_{j+k} f(\omega) = S_j f(\omega) + S_k f(T^j \omega)$ holds for each $j, k \ge 0$. The function f is called a *coboundary cocycle* if there exists a real-valued measurable function h on Ω such that

$$f(\omega) = h(T\omega) - h(\omega) \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

In this case we have

$$S_j f(\omega) = h(T^j \omega) - h(\omega) \quad \text{for } \mu\text{-a.e. } \omega \in \Omega$$

for all $j \ge 1$.

Recently Alonso, Hong and Obaya [1] considered the case where T is an ergodic automorphism and f is a function in $L_r(\Omega, \mu)$ with $0 < r < \infty$.

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They proved that if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \int_A |S_j f|^r \, d\mu < \infty$$

for some $A \in \mathcal{A}$ with $\mu(A) > 0$, then there exists a function h in $L_r(\Omega, \mu)$ such that

$$f(\omega) = h(T\omega) - h(\omega) \quad \text{ for } \mu\text{-a.e. } \omega \in \Omega.$$

For related results we refer the reader to [2] and [11]. In the present paper we intend to generalize and improve the result of [1]. We prove in Section 3 that if T is an ergodic endomorphism, and $\varphi : \mathbb{R} \to [0, \infty)$ is a Borel measurable function on the real line $\mathbb{R}$ such that $\lim_{|x|\to\infty} \varphi(x) = \infty$, then the condition

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \int_A \varphi(S_j f(\omega)) \, d\mu < \infty$$

for some $A \in \mathcal{A}$ with $\mu(A) > 0$ implies the existence of a real-valued measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$. If the function φ satisfies the additional hypotheses that $\sup\{\varphi(x) : |x| \le N\} < \infty$ for every $N \ge 1$ and $\limsup_{|x|\to\infty} \varphi(x + a)/\varphi(x) < \infty$ for every $a \in \mathbb{R}$, then $\int_\Omega \varphi(h(\omega)) \, d\mu < \infty$. Secondly we consider a Banach lattice $(L, \|\cdot\|_L)$ of equivalence classes of real-valued measurable functions on Ω. Under suitable conditions on L, we prove that if T is an ergodic endomorphism, and there exists a set A in $\mathcal{A}$ with $\mu(A) > 0$ such that $(S_j f)\chi_A \in L$ for $j \ge 1$ and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \|(S_j f)\chi_A\|_L < \infty,$$

then $f \in L$ and $f = h \circ T - h$ for some h in L. This extends a recent result of [13]. We note that Orlicz spaces and Lorentz spaces are typical examples of such Banach lattices (see e.g. [8], [9]). In the next section we prove some auxiliary results.

2. Preliminaries. Let $(\Omega, \mathcal{A}, \mu)$, T and f be as in the introduction. Let ∂D be the boundary of the open unit disc D in the complex plane, i.e., $\partial D = \{e^{ix} : 0 \le x < 2\pi\}$, and $\mathcal{B}(\partial D)$ be the σ-field of all Borel subsets of ∂D. Denote by dx the Lebesgue measure on ∂D. We consider the product measure space

$$(\Omega \times \partial D, \mathcal{A} \otimes \mathcal{B}(\partial D), \mu \otimes dx).$$

For $s \in \mathbb{R}$, define an endomorphism τ_s of $(\Omega \times \partial D, \mathcal{A} \otimes \mathcal{B}(\partial D), \mu \otimes dx)$ by

$$\tau_s(\omega, e^{ix}) = (T\omega, e^{ix} e^{-isS_1 f(\omega)}).$$

Since the function $g(\omega, e^{ix}) = e^{ix}$ is integrable on $(\Omega\times\partial D, \mathcal{A}\otimes\mathcal{B}(\partial D), \mu\otimes dx)$ and since $\tau_s^j(\omega, e^{ix}) = (T^j\omega, e^{ix}e^{-isS_jf(\omega)})$ for each $j \geq 1$, it follows from the Birkhoff ergodic theorem that the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} e^{ix}e^{-isS_jf(\omega)}$$

exists for $\mu\otimes dx$-a.e. $(\omega, e^{ix}) \in \Omega\times\partial D$. Then by Fubini's theorem the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} e^{-isS_jf(\omega)} \tag{1}$$

exists for μ-a.e. $\omega\in\Omega$. We now define

$$H(\omega, s) = \begin{cases} \lim\limits_{n\to\infty}\dfrac{1}{n}\displaystyle\sum_{j=1}^{n} e^{-isS_jf(\omega)} & \text{if the limit exists,} \\ 2 & \text{otherwise.}\end{cases} \tag{2}$$

Then $H(\omega, s)$ is a real-valued measurable function on $\Omega\times\mathbb{R}$ with respect to the σ-field $\mathcal{A}\otimes\mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the σ-field of all Borel subsets of $\mathbb{R}$. Since

$$e^{-isS_1f(\omega)}e^{-isS_jf(T\omega)} = e^{-isS_{j+1}f(\omega)}$$

by the cocycle identity, it then follows that

$$H(T\omega, s) = e^{isS_1f(\omega)}H(\omega, s) \quad \text{whenever } H(\omega, s)\neq 2. \tag{3}$$

On the other hand, Fubini's theorem shows that the set

$$\Omega_1 = \{\omega\in\Omega : H(\omega, s)\neq 2 \text{ for } ds\text{-a.e. } s\in\mathbb{R}\} \tag{4}$$

is in $\mathcal{A}$. Furthermore,

$$\mu(\Omega_1) = 1 \quad\text{and}\quad T^{-1}\Omega_1 = \Omega_1. \tag{5}$$

Now, fix $\omega\in\Omega_1$. As a function of $s\in\mathbb{R}$, $H(\omega, s)$ is the ds-a.e. limit of the continuous positive definite functions $(1/n)\sum_{j=1}^{n}\exp(-isS_jf(\omega))$, so there is a nonnegative finite Borel measure μ_ω on $\mathbb{R}$ such that

$$H(\omega, s) = \int_{\mathbb{R}} e^{ist}\,d\mu_\omega(t) \quad \text{for } ds\text{-a.e. } s\in\mathbb{R} \tag{6}$$

(cf. e.g. §32 and §33 of [7]). (This argument is due to Helson [5].) By (3) and the continuity of the mapping $\mathbb{R}\ni s\mapsto\int_{\mathbb{R}} e^{ist}\,d\mu_\omega(t)$, we see that

$$\int_{\mathbb{R}} e^{is(S_1f(\omega)+t)}\,d\mu_\omega(t) = e^{isS_1f(\omega)}\int_{\mathbb{R}} e^{ist}\,d\mu_\omega(t) = \int_{\mathbb{R}} e^{ist}\,d\mu_{T\omega}(t)$$

for all $s\in\mathbb{R}$. Therefore

$$\mu_{T\omega}(E) = \mu_\omega(E - S_1f(\omega)) \quad \text{for every } \omega\in\Omega_1 \text{ and } E\in\mathcal{B}(\mathbb{R}). \tag{7}$$

Next, let $N \geq 1$. For $\omega \in \Omega$ we put

$$g_N(\omega) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[-N,N]}(S_j f(\omega)), \tag{8}$$

$$g_\infty(\omega) = \lim_{N\to\infty} g_N(\omega). \tag{9}$$

Clearly, g_∞ is measurable on $(\Omega, \mathcal{A}, \mu)$. Since $|S_j f(T\omega) - S_{j+1} f(\omega)| = |f(\omega)| < \infty$, we observe that $g_\infty(\omega) > 0$ if and only if $g_\infty(T\omega) > 0$. Consequently, the set

$$P = \{\omega \in \Omega : g_\infty(\omega) > 0\} \cap \Omega_1 \tag{10}$$

belongs to $\mathcal{A}$, and $T^{-1}P = P$.

Suppose $\omega \in \Omega_1$. Since $\mu_\omega = 0$ is equivalent to

$$\int_{\mathbb{R}} \widehat{v}(t)\, d\mu_\omega(t) = 0 \quad \text{for every } v \in L_1(\mathbb{R}, ds), \tag{11}$$

and since

$$\begin{aligned}
\int_{\mathbb{R}} \widehat{v}(t)\, d\mu_\omega(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} v(s) e^{-ist}\, ds\, d\mu_\omega(t) \\
&= \int_{\mathbb{R}} \Big(v(s) \int_{\mathbb{R}} e^{-ist}\, d\mu_\omega(t) \Big)\, ds \qquad \text{(by Fubini's theorem)} \\
&= \int_{\mathbb{R}} v(s) H(\omega, -s)\, ds \qquad \text{(by (6))} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \widehat{v}(-S_j f(\omega)) \qquad \text{(by (2))},
\end{aligned}$$

it follows that $\mu_\omega = 0$ is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \widehat{v}(-S_j f(\omega)) = 0 \quad \text{for every } v \in L_1(\mathbb{R}, ds), \tag{12}$$

which is clearly equivalent to $g_\infty(\omega) = 0$. Thus

$$P = \{\omega \in \Omega_1 : \mu_\omega > 0\}. \tag{13}$$

The above argument shows that the function $P \ni \omega \mapsto \mu_\omega(\mathbb{R})$ is measurable with respect to the σ-field $\mathcal{A}$, and so is the function

$$h(\omega) = \sup\{t \in \mathbb{R} : \mu_\omega((-\infty, t]) \leq \mu_\omega(\mathbb{R})/2\} \quad (\omega \in P).$$

The relation $h(T\omega) = f(\omega) + h(\omega)$ holds for every $\omega \in P$, by (7).

We similarly see that the set

$$P_1 = \{\omega \in P : \mu_\omega(\mathbb{R}) = 1\} \tag{14}$$

belongs to $\mathcal{A}$, and $T^{-1}P_1 = P_1$ by (7).

Lastly, let $A \in \mathcal{A}$ be such that $A \subset \Omega_1$, $T^{-1}A = A$, and there exists a real-valued measurable function h_A on A satisfying

$$f(\omega) = h_A(T\omega) - h_A(\omega) \quad \text{for every } \omega \in A.$$

If we put

$$A_N = \{\omega \in A : |h_A(\omega)| \le N\} \qquad (N \ge 1),$$

$$\widetilde{g}_N(\omega) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{A_N}(T^j\omega) \quad (\omega \in A),$$

then the set

$$\widetilde{A} = \{\omega \in A : \lim_{N\to\infty} \widetilde{g}_N(\omega) = 1\}$$

satisfies $\mu(\widetilde{A}) = \mu(A)$ and $T^{-1}\widetilde{A} = \widetilde{A}$, by the Birkhoff ergodic theorem.

Suppose $\omega \in \widetilde{A}$. If $N \ge |h_A(\omega)|$, then

$$|S_j f(\omega)| = |h_A(T^j\omega) - h_A(\omega)| \le |h_A(T^j\omega)| + N,$$

so that $T^j\omega \in A_N$ implies $|S_j f(\omega)| \le 2N$, and hence

$$\frac{1}{n} \sum_{j=1}^{n} \chi_{A_N}(T^j\omega) \le \frac{1}{n} \sum_{j=1}^{n} \chi_{[-2N,2N]}(S_j f(\omega)).$$

On the other hand,

$$\int_{\mathbb{R}} e^{ist}\, d\mu_\omega(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{-isS_j f(\omega)} \quad \text{for } ds\text{-a.e. } s \in \mathbb{R}$$

by (2) and (6), since $\omega \in \widetilde{A} \subset A \subset \Omega_1$. Therefore we find that for ds-a.e. $s \in \mathbb{R}$ with $s = \theta/2N$ and $0 < \theta \le \pi/4$,

$$\begin{aligned} \Big|\int_{\mathbb{R}} e^{ist}\, d\mu_\omega(t)\Big| &= \lim_{n\to\infty} \frac{1}{n} \Big|\sum_{j=1}^{n} e^{-isS_j f(\omega)}\Big| \\ &\ge \cos\theta \cdot \Big(\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[-2N,2N]}(S_j f(\omega))\Big) - (1 - \widetilde{g}_N(\omega)) \\ &\ge \cos\theta \cdot \widetilde{g}_N(\omega) - (1 - \widetilde{g}_N(\omega)). \end{aligned}$$

Since $\widetilde{g}_N(\omega) \to 1$ as $N \to \infty$ and $\cos\theta \to 1$ as $\theta \to 0+0$, we must have $\mu_\omega(\mathbb{R}) \ge 1$. But this implies $\mu_\omega(\mathbb{R}) = 1$, because $\mu_\omega(\mathbb{R}) \le 1$ is a direct consequence of (6) and the fact that $|H(\omega, s)| \le 1$ for ds-a.e. $s \in \mathbb{R}$. We have proved that $\widetilde{A} \subset P_1$.

We can summarize the above as follows.

FACT 1 (cf. Theorem 3 of [5]). *Let T be an endomorphism and f be a real-valued measurable function on Ω. Then the following hold*:

(I) *The set* $P_1 = \{\omega \in \Omega_1 : \mu_\omega(\mathbb{R}) = 1\}$ *is a* T*-invariant measurable subset of* Ω, *and there exists a real-valued measurable function* h *on* P_1 *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for every* $\omega \in P_1$.

(II) $\mu_\omega = 0$ *for* μ*-a.e.* $\omega \in \Omega \setminus P_1$.

(III) *If* A *is a* T*-invariant measurable subset of* Ω *for which there exists a real-valued measurable function* h_A *on* A *such that* $f(\omega) = h_A(T\omega) - h_A(\omega)$ *for every* $\omega \in A$, *then* $A \subset P_1 \pmod{\mu}$, *i.e.*, $\mu(A \setminus P_1) = 0$.

FACT 2. *Assume that* T *is an ergodic endomorphism. Then there exists a real-valued measurable function* h *on* Ω *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for* μ*-a.e.* $\omega \in \Omega$ *if and only if* $\mu(\{\omega : g_\infty(\omega) > 0\}) > 0$.

Next we consider the probability measure space

$$(K_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}}, \mu_{\mathbb{R}}) = \left(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \mu \otimes \frac{dx}{\pi(1+x^2)}\right), \tag{15}$$

and the null-preserving transformation ϑ of $(K_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}}, \mu_{\mathbb{R}})$ defined by

$$\vartheta(\omega, x) = (T\omega, x + S_1 f(\omega)) \quad \text{for } (\omega, x) \in K_{\mathbb{R}}. \tag{16}$$

It follows that $\vartheta^j(\omega, x) = (T^j\omega, x + S_j f(\omega))$ for $j \geq 0$.

Let $M : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{x\to\infty} M(x) = \infty$ and there exist constants $k > 0$ and $x_0 \geq 0$ satisfying

$$M(2x) \leq kM(x) \quad \text{for } x \geq x_0. \tag{17}$$

(See [9] for the properties of functions satisfying (17).)

Suppose h is a real-valued measurable function on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$. Then define a function w_h on $K_{\mathbb{R}}$ by

$$w_h(\omega, x) = \frac{1 + x^2}{1 + (x - h(\omega))^2} \quad \text{for } (\omega, x) \in K_{\mathbb{R}}. \tag{18}$$

It follows that $\int_{K_{\mathbb{R}}} w_h \, d\mu_{\mathbb{R}} = 1$, and thus we can define a $\mu_{\mathbb{R}}$-equivalent probability measure λ_h on $(K_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}})$ by

$$\lambda_h = w_h \, d\mu_{\mathbb{R}}. \tag{19}$$

We will prove that λ_h is invariant with respect to ϑ. To do this it may be assumed from the start that $f(\omega) = h(T\omega) - h(\omega)$ for every $\omega \in \Omega$. For $A \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we introduce a set E in $\mathcal{A}_{\mathbb{R}}$ by

$$E = \{(\omega, x) : \omega \in A,\ h(\omega) + \alpha \leq x < h(\omega) + \beta\}.$$

Then

$$\begin{aligned}
\vartheta^{-1}E &= \{(\omega, x) : (T\omega, x + f(\omega)) \in E\} \\
&= \{(\omega, x) : \omega \in T^{-1}A,\ h(T\omega) + \alpha \leq x + f(\omega) < h(T\omega) + \beta\} \\
&= \{(\omega, x) : \omega \in T^{-1}A,\ h(\omega) + \alpha \leq x < h(\omega) + \beta\},
\end{aligned}$$

and thus the definition of λ_h yields

$$\lambda_h(\vartheta^{-1}E) = \lambda_h(E), \tag{20}$$

since T is an endomorphism of $(\Omega, \mathcal{A}, \mu)$. It follows from a standard approximation argument that λ_h is invariant with respect to ϑ.

On the other hand, an elementary calculation shows that

$$\frac{1+x^2}{1+(x-t)^2} < 2 + t^2 \quad \text{for } x, t \in \mathbb{R}, \tag{21}$$

whence

$$0 < w_h(\omega, x) < 2 + h^2(\omega) \quad \text{for all } (\omega, x) \in K_{\mathbb{R}}. \tag{22}$$

Therefore, if h is a function in $L_\infty(\Omega, \mu)$, then w_h is a function in $L_\infty(K_{\mathbb{R}}, \mu_{\mathbb{R}})$ such that $\|w_h\|_\infty \le 2 + \|h\|^2$.

Next, assume that $h \in L_M(\Omega, \mu)$, where $L_M(\Omega, \mu)$ denotes the space of all real-valued measurable functions u on $(\Omega, \mathcal{A}, \mu)$ such that

$$\int_\Omega M(|u(\omega)|)\, d\mu < \infty.$$

Then we have, using Fubini's theorem,

$$\begin{aligned}\int_{K_{\mathbb{R}}} w_h M(\sqrt{w_h})\, d\mu_{\mathbb{R}} &\le \int_\Omega M(\sqrt{2 + h^2(\omega)}) \Big(\int_{\mathbb{R}} w_h(\omega, x) \frac{dx}{\pi(1+x^2)} \Big) d\mu(\omega) \\ &= \int_\Omega M(\sqrt{2 + h^2(\omega)})\, d\mu(\omega) \le \int_\Omega M(2 + |h(\omega)|)\, d\mu < \infty,\end{aligned}$$

where the last inequality comes from (17). Consequently, $h \in L_M(\Omega, \mu)$ implies that the function $w_h M(\sqrt{w_h})$ belongs to $L_1(K_{\mathbb{R}}, \mu_{\mathbb{R}})$.

Conversely, assume that $\lambda = w d\mu_{\mathbb{R}}$, where $0 \le w \in L_1(K_{\mathbb{R}}, \mu_{\mathbb{R}})$, is a $\mu_{\mathbb{R}}$-absolutely continuous probability measure invariant with respect to ϑ. Then we introduce a function w_Ω on Ω by

$$w_\Omega(\omega) = \int_{\mathbb{R}} w(\omega, x) \frac{dx}{\pi(1+x^2)} \quad (\omega \in \Omega). \tag{23}$$

Fubini's theorem implies that $w_\Omega \in L_1(\Omega, \mu)$, and thus $\lambda_\Omega = w_\Omega\, d\mu$ is a μ-absolutely continuous probability measure invariant with respect to T. Notice that if T is assumed to be ergodic, then $w_\Omega(\omega) = 1$ for μ-a.e. $\omega \in \Omega$. On the other hand, if λ is a $\mu_{\mathbb{R}}$-equivalent probability measure, then $w_\Omega(\omega) > 0$ for μ-a.e. $\omega \in \Omega$, without assuming the ergodicity of T.

Thus, in the following, we will assume that $w_\Omega(\omega) > 0$ *for* μ*-a.e.* $\omega \in \Omega$. Then we can define a Borel probability measure λ_ω on $\mathbb{R}$, for μ-a.e. $\omega \in \Omega$, by

$$\lambda_\omega(B) = \frac{1}{w_\Omega(\omega)} \int_B w(\omega, x) \frac{dx}{\pi(1+x^2)} \quad (B \in \mathcal{B}(\mathbb{R})). \tag{24}$$

To prove the existence of a real-valued measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$, we must assume below that T is an *automorphism*. By this assumption, both ϑ and ϑ^{-1} are null-preserving transformations of $(K_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}}, \mu_{\mathbb{R}})$; and if $A \in \mathcal{A}$ and $B \in \mathcal{B}(\mathbb{R})$, then

$$\lambda(A \times B) = \int_A \lambda_\omega(B) w_\Omega(\omega)\, d\mu(\omega),$$

and

$$\begin{aligned}\lambda(\vartheta(A \times B)) &= \int_{TA} \lambda_\omega(B + f(T^{-1}\omega)) w_\Omega(\omega)\, d\mu(\omega) \\ &= \int_A \lambda_{T\omega}(B + f(\omega)) w_\Omega(\omega)\, d\mu(\omega),\end{aligned}$$

where the last equality comes from the invariance of the measure $\lambda_\Omega = w_\Omega\, d\mu$ with respect to T. Since $\lambda(A \times B) = \lambda(\vartheta(A \times B))$, it follows that $\lambda_\omega(B) = \lambda_{T\omega}(B + f(\omega))$ for μ-a.e. $\omega \in \Omega$, and since $\mathcal{B}(\mathbb{R})$ is separable, this shows that $\lambda_\omega(B) = \lambda_{T\omega}(B + f(\omega))$ for all $B \in \mathcal{B}(\mathbb{R})$ and for μ-a.e. $\omega \in \Omega$. Consequently, the function

$$h(\omega) = \sup\{t \in \mathbb{R} : \lambda_\omega((-\infty, t]) = 1/2\} \quad (\omega \in \Omega) \tag{25}$$

satisfies $h(T\omega) = h(\omega) + f(\omega)$ for μ-a.e. $\omega \in \Omega$, and is measurable with respect to $\mathcal{A}$ by an easy approximation argument.

Now, assume that $w = d\lambda/d\mu_{\mathbb{R}} \in L_\infty(K_{\mathbb{R}}, \mu_{\mathbb{R}})$. Since λ is invariant with respect to ϑ by hypothesis, it follows that the set

$$E_\lambda = \{(\omega, x) \in K_{\mathbb{R}} : w(\omega, x) > 0\} \tag{26}$$

satisfies $\vartheta^{-1}E_\lambda = E_\lambda \pmod{\mu_{\mathbb{R}}}$. On the other hand, the measure $\lambda_h = w_h\, d\mu_{\mathbb{R}}$ is also invariant with respect to ϑ (cf. (18)–(20)). Thus it follows from the Neveu–Chacon identification theorem (see e.g. Theorem 3.3.4 of [10]) for the Chacon–Ornstein ratio ergodic limit that the function w/w_h is measurable with respect to the σ-field

$$\mathcal{I}_{\mathbb{R}} = \{A \in \mathcal{A}_{\mathbb{R}} : \vartheta^{-1}A = A \pmod{\mu_{\mathbb{R}}}\}. \tag{27}$$

Thus there exists a real-valued positive function G on E_λ, measurable with respect to $\mathcal{I}_{\mathbb{R}}$, such that

$$w_h = G \cdot w \pmod{\mu_{\mathbb{R}}} \quad \text{on } E_\lambda. \tag{28}$$

Since $G \circ \vartheta = G \pmod{\mu_{\mathbb{R}}}$ on E_λ, it follows that for $\mu_{\mathbb{R}}$-a.e. $(\omega, x) \in E_\lambda$,

$$\begin{aligned}\sup\{w_h(\vartheta^j(\omega, x)) : j \ge 0\} &= G(\omega, x) \cdot \sup\{w(\vartheta^j(\omega, x)) : j \ge 0\} \\ &\le G(\omega, x) \cdot \|w\|_\infty.\end{aligned}$$

The definition of w_h (cf. (18)) implies that for $\mu_{\mathbb{R}}$-a.e. $(\omega, x) \in K_{\mathbb{R}}$,

$$w_h(\vartheta^j(\omega, x)) = w_h(T^j\omega, x + S_j f(\omega)) = w_h(T^j\omega, x + h(T^j\omega) - h(\omega))$$
$$= \frac{1 + (x + h(T^j\omega) - h(\omega))^2}{1 + (x - h(\omega))^2}.$$

Hence

$$\sup_{j\ge 0} \frac{1 + (x + h(T^j\omega) - h(\omega))^2}{1 + (x - h(\omega))^2} < \infty \quad \text{for } \mu_{\mathbb{R}}\text{-a.e. } (\omega, x) \in E_\lambda,$$

and therefore the function

$$g(\omega) = \sup_{j\ge 0} |h(T^j\omega)| \quad (\omega \in \Omega)$$

satisfies $g(\omega) < \infty$ for $\mu_{\mathbb{R}}$-a.e. $(\omega, x) \in E_\lambda$. Now, if λ is a $\mu_{\mathbb{R}}$-equivalent measure, then, since $E_\lambda = K_{\mathbb{R}} \pmod{\mu_{\mathbb{R}}}$, it follows that $g(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$. If T is ergodic, and λ is a $\mu_{\mathbb{R}}$-absolutely continuous probability measure, then, since $\mu_{\mathbb{R}}(E_\lambda) > 0$, the set $\{\omega \in \Omega : g(\omega) < \infty\}$ is of positive μ-measure, and the ergodicity of T implies that $g(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$. Thus, in either case, $0 \le g(T\omega) \le g(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$. Since μ is an invariant measure with respect to T, it then follows that $g(T\omega) = g(\omega)$ for μ-a.e. $\omega \in \Omega$, and hence the sets

$$A_n = \{\omega \in \Omega : n - 1 \le g(\omega) < n\}, \quad n \ge 1,$$

satisfy $T^{-1}A_n = A_n \pmod{\mu}$ and $\Omega = \bigcup_{n=1}^{\infty} A_n \pmod{\mu}$. In particular, if T is ergodic, then g is a constant function in $L_\infty(\Omega, \mu)$.

Next, assume that the function $w = d\lambda/d\mu_{\mathbb{R}}$ is such that $wM(\sqrt{w}) \in L_1(K_{\mathbb{R}}, \mu_{\mathbb{R}})$. Let v_j denote the Radon–Nikodym derivative $d(\mu_{\mathbb{R}} \circ \vartheta^j)/d\mu_{\mathbb{R}}$, where $\mu_{\mathbb{R}} \circ \vartheta^j$ denotes the measure defined by $(\mu_{\mathbb{R}} \circ \vartheta^j)(A) = \mu_{\mathbb{R}}(\vartheta^j A)$ for $A \in \mathcal{A}_{\mathbb{R}}$. Since $\vartheta^j(\omega, x) = (T^j\omega, x + S_j f(\omega))$ for $(\omega, x) \in K_{\mathbb{R}}$, it is easy to see that v_j has the form

$$v_j(\omega, x) = \frac{1 + x^2}{1 + (x + S_j f(\omega))^2}$$
$$= \frac{1 + x^2}{1 + (x + h(T^j\omega) - h(\omega))^2} \quad \text{for } (\omega, x) \in K_{\mathbb{R}}.$$

Since $\lambda = w\, d\mu_{\mathbb{R}}$ is invariant with respect to ϑ by hypothesis, it follows that if $A \in \mathcal{A}_{\mathbb{R}}$, then

$$\int_A w\, d\mu_{\mathbb{R}} = \int_{\vartheta^j A} w\, d\mu_{\mathbb{R}} = \int (w \cdot \chi_{\vartheta^j A}) \circ \vartheta^j\, d(\mu_{\mathbb{R}} \circ \vartheta^j) = \int_A (w \circ \vartheta^j) v_j\, d\mu_{\mathbb{R}},$$

whence $w = (w \circ \vartheta^j) v_j \pmod{\mu_{\mathbb{R}}}$ on $K_{\mathbb{R}}$. Therefore,

$$w^{1/2}(\omega, x)\, \frac{|x + h(T^j\omega) - h(\omega)|}{\sqrt{1 + x^2}} \le w^{1/2} \circ \vartheta^j(\omega, x)$$

for $\mu_{\mathbb{R}}$-a.e. $(\omega, x) \in K_{\mathbb{R}}$. Now, if $C \cdot w^{1/2}(\omega, x) \geq \sqrt{1+x^2}$ for some constant $C > 1$, then

$$|x + h(T^j\omega) - h(\omega)| \leq C \cdot w^{1/2} \circ \vartheta^j(\omega, x);$$

and by (17) there exist constants $k(C) > 0$ and $x_1 \geq 0$ such that

$$M(|x + h(T^j\omega) - h(\omega)|) \leq k(C) \cdot M(w^{1/2} \circ \vartheta^j(\omega, x))$$

whenever $w^{1/2} \circ \vartheta^j(\omega, x) \geq x_1$; consequently,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} & M(|x + h(T^j\omega) - h(\omega)|) \\ &\leq M(Cx_1) + k(C) \cdot \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} M(w^{1/2} \circ \vartheta^j(\omega, x)) \\ &= M(Cx_1) + k(C) \cdot E\{M(w^{1/2}) \,|\, (K_{\mathbb{R}}, \mathcal{I}_{\mathbb{R}}, w d\mu_{\mathbb{R}})\}(\omega, x) < \infty\end{aligned}$$

by the Birkhoff ergodic theorem, where $E\{M(w^{1/2}) \,|\, (K_{\mathbb{R}}, \mathcal{I}_{\mathbb{R}}, w d\mu_{\mathbb{R}})\}$ denotes the conditional expectation of the function $M(w^{1/2})$ with respect to the σ-field $\mathcal{I}_{\mathbb{R}}$ and the measure $w d\mu_{\mathbb{R}}$. Since the constant $C > 0$ can be arbitrarily large, this proves that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} M(|x + h(T^j\omega) - h(\omega)|) < \infty \quad \text{for } \mu_{\mathbb{R}}\text{-a.e. } (\omega, x) \in E_\lambda. \tag{29}$$

Hence, if we define a function $\widetilde{g}$ on Ω by

$$\widetilde{g}(\omega) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} M(|h(T^j\omega)|) \quad \text{for } \omega \in \Omega,$$

then (29) and (17) show that $\widetilde{g}(\omega) < \infty$ for $\mu_{\mathbb{R}}$-a.e. $(\omega, x) \in E_\lambda$. Therefore, as before, if λ is a $\mu_{\mathbb{R}}$-equivalent measure, or if T is ergodic and λ is a $\mu_{\mathbb{R}}$-absolutely continuous probability measure, then $0 \leq \widetilde{g}(T\omega) = \widetilde{g}(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$; and the sets $B_n = \{\omega \in \Omega : n - 1 \leq \widetilde{g}(\omega) < n\}$, $n \geq 1$, satisfy $T^{-1}B_n = B_n$, $\int_{B_n} M(|h|)\, d\mu = \int_{B_n} \widetilde{g}\, d\mu < \infty$ (by the Birkhoff ergodic theorem), and $\Omega = \bigcup_{n=1}^{\infty} B_n \pmod{\mu}$. In particular, if T is ergodic, then $\widetilde{g}(\omega) = \int_\Omega M(|h|)\, d\mu < \infty$ for μ-a.e. $\omega \in \Omega$.

We can summarize the above as follows.

FACT 3 (cf. Proposition 3.1 of [1]). *Let $M : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{x\to\infty} M(x) = \infty$ and there exist constants $k > 0$ and $x_0 \geq 0$ satisfying* (17). *Then the following hold*:

(I) *If T is an endomorphism and h is a real-valued measurable function on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$, then the measure $\lambda_h = w_h\, d\mu_{\mathbb{R}}$ in* (19) *is a $\mu_{\mathbb{R}}$-equivalent probability measure invariant with* *respect to* ϑ*. If moreover* $h \in L_\infty(\Omega,\mu)$ *then* $w_h \in L_\infty(K_{\mathbb{R}},\mu_{\mathbb{R}})$*; and if* $h \in L_M(\Omega,\mu)$ *then* $w_h M(\sqrt{w_h}) \in L_1(K_{\mathbb{R}},\mu_{\mathbb{R}})$*.*

(II) *If* T *is an automorphism and* $\lambda = w\,d\mu_{\mathbb{R}}$*, where* $0 < w \in L_1(K_{\mathbb{R}},\mu_{\mathbb{R}})$*, is a* $\mu_{\mathbb{R}}$*-equivalent probability measure invariant with respect to* ϑ*, then there exists a real-valued measurable function* h *on* Ω *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for* μ*-a.e.* $\omega \in \Omega$*. If moreover* $w \in L_\infty(K_{\mathbb{R}},\mu_{\mathbb{R}})$ *[resp.* $wM(\sqrt{w}) \in L_1(K_{\mathbb{R}},\mu_{\mathbb{R}})$*], then there exists a countable measurable decomposition* $\{A_n : n \ge 1\}$ *of* Ω *with* $T^{-1}A_n = A_n$ *for* $n \ge 1$ *such that* $h\chi_{A_n} \in L_\infty(\Omega,\mu)$ *[resp.* $h\chi_{A_n} \in L_M(\Omega,\mu)$*] for every* $n \ge 1$*.*

(III) *If* T *is an ergodic automorphism and* $\lambda = w d\mu_{\mathbb{R}}$*, where* $0 \le w \in L_1(K_{\mathbb{R}},\mu_{\mathbb{R}})$*, is a* $\mu_{\mathbb{R}}$*-absolutely continuous probability measure invariant with respect to* ϑ*, then there exists a real-valued measurable function* h *on* Ω *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for* μ*-a.e.* $\omega \in \Omega$*. If moreover* $w \in L_\infty(K_{\mathbb{R}},\mu_{\mathbb{R}})$ *[resp.* $wM(\sqrt{w}) \in L_1(K_{\mathbb{R}},\mu_{\mathbb{R}})$*], then* $h \in L_\infty(\Omega,\mu)$ *[resp.* $h \in L_M(\Omega,\mu)$*].*

3. Main results

Theorem 1 (cf. Theorem 1 on p. 62 of [6]). *Let* $\varphi : \mathbb{R} \to [0,\infty)$ *be a Borel measurable function on* $\mathbb{R}$ *such that* $\lim_{|x|\to\infty} \varphi(x) = \infty$*. Assume that* T *is an ergodic endomorphism. If there exists a set* $A \in \mathcal{A}$ *with* $\mu(A) > 0$ *such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \int_A \varphi(S_j f(\omega))\,d\mu < \infty, \tag{30}$$

then there exists a real-valued measurable function h *on* Ω *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for* μ*-a.e.* $\omega \in \Omega$*. If, in addition,* $\sup\{\varphi(x) : |x| \le N\} < \infty$ *for every* $N \ge 1$*, and* $\limsup_{|x|\to\infty} \varphi(x+a)/\varphi(x) < \infty$ *for every* $a \in \mathbb{R}$*, then* $\int_\Omega \varphi(h(\omega))\,d\mu < \infty$*.*

Corollary (cf. Theorem 3.2 of [1]). *Let* T *be an ergodic endomorphism, and* $0 < r < \infty$*. If*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \int_A |S_j f(\omega)|^r\,d\mu < \infty$$

for some $A \in \mathcal{A}$ *with* $\mu(A) > 0$*, then there exists a real-valued measurable function* h *on* Ω *with* $\int_\Omega |h(\omega)|^r\,d\mu < \infty$ *such that* $f(\omega) = h(T\omega) - h(\omega)$ *for* μ*-a.e.* $\omega \in \Omega$*.*

Proof of Theorem 1. By Fatou's lemma, the function

$$\varphi(f)_*(\omega) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(S_j f(\omega))$$

satisfies

$$\int_A \varphi(f)_*(\omega)\,d\mu \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \int_A \varphi(S_j f(\omega))\,d\mu < \infty,$$

whence $\varphi(f)_*(\omega) < \infty$ for μ-a.e. $\omega \in A$. Since $\lim_{|x|\to\infty} \varphi(x) = \infty$, the inequality $\varphi(f)_*(\omega) < \infty$ implies the existence of an $N \ge 1$ such that

$$g_N(\omega) = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[-N,N]}(S_j f(\omega)) > 0.$$

It follows that $g_\infty(\omega) = \lim_{N\to\infty} g_N(\omega) > 0$ for μ-a.e. $\omega \in A$, and therefore by Fact 2 there exists a real-valued measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$. Assume that φ satisfies $\sup\{\varphi(x) : |x| \le N\} < \infty$ for every $N \ge 1$, and $\limsup_{|x|\to\infty} \varphi(x+a)/\varphi(x) < \infty$ for every $a \in \mathbb{R}$. Then, since $S_j f(\omega) = h(T^j\omega) - h(\omega)$ on Ω, it follows that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(h(T^j\omega)) = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(S_j f(\omega) + h(\omega)) < \infty$$

for μ-a.e. $\omega \in A$ with $\varphi(f)_*(\omega) < \infty$, so that the function

$$h_*(\omega) = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(h(T^j\omega))$$

satisfies $h_*(\omega) < \infty$ for μ-a.e. $\omega \in A$. Since h_* is an invariant function with respect to T, the Birkhoff ergodic theorem and the ergodicity of T imply that $h_*(\omega) = \int_\Omega \varphi \circ h\,d\mu < \infty$ for μ-a.e. $\omega \in \Omega$, and hence the desired result has been established.

REMARK 1. If the ergodicity of T is not assumed in Theorem 1, then the above argument shows that inequality (30) with $A = \Omega$ implies the existence of a real-valued measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$; if moreover $\sup\{\varphi(x) : |x| \le N\} < \infty$ for every $N \ge 1$, and $\limsup_{|x|\to\infty} \varphi(x+a)/\varphi(x) < \infty$ for every $a \in \mathbb{R}$, then there exists a countable measurable decomposition $\{A_n : n \ge 1\}$ of Ω with $T^{-1}A_n = A_n$ for $n \ge 1$ such that $\int_{A_n} \varphi(h(\omega))\,d\mu < \infty$ for every $n \ge 1$.

From now on we consider a Banach lattice $(L, \|\cdot\|_L)$ of equivalence classes of real-valued measurable functions on Ω. Thus, two functions u and v in L are not distinguished provided that $u(\omega) = v(\omega)$ for μ-a.e. $\omega \in \Omega$. By definition, the norm $\|\cdot\|_L$ has the property:

(A) If $u, v \in L$ and $|u(\omega)| \le |v(\omega)|$ for μ-a.e. $\omega \in \Omega$, then $\|u\|_L \le \|v\|_L$.

In this paper we assume the additional properties:

(B) If v is a real-valued measurable function on Ω and there exists a function $u \in L$ such that $|v(\omega)| \le |u(\omega)|$ for μ-a.e. $\omega \in \Omega$, then $v \in L$.

(C) If (u_n) is a sequence of functions in L such that $|u_1(\omega)| \le |u_2(\omega)| \le \ldots$ for μ-a.e. $\omega \in \Omega$, and $\sup_{n\ge1}\|u_n\|_L < \infty$, then there exists a function $u \in L$ such that $|u_n(\omega)| \le |u(\omega)|$ for μ-a.e. $\omega \in \Omega$ and all $n \ge 1$.

(D) If v is a real-valued measurable function on Ω and $u \in L$ is such that

$$\mu(\{\omega : |v(\omega)| > a\}) = \mu(\{\omega : |u(\omega)| > a\})$$

for every $0 < a \in \mathbb{R}$, then $v \in L$ and $\|v\|_L = \|u\|_L$.

It should be noted that, besides the usual $L_p(\Omega,\mu)$-spaces with $1 \le p \le \infty$, there are many interesting Banach lattices of functions which share these additional properties (B), (C) and (D). Examples are Orlicz spaces, Lorentz spaces, etc. By Property (D), the mapping $u \mapsto u \circ T$ is a linear isometry of $(L, \|\cdot\|_L)$ for every endomorphism T.

THEOREM 2 (cf. [13]). *Assume that T is an ergodic endomorphism. If there exists a set $A \in \mathcal{A}$ with $\mu(A) > 0$ such that $(S_jf)\chi_A \in L$ for $j \ge 1$ and*

$$K := \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|(S_jf)\chi_A\|_L < \infty, \tag{31}$$

then there exists a function h in L such that $f = h \circ T - h$ on Ω.

Proof. Define

$$f_n(\omega) = \inf_{m\ge n} \frac{1}{m}\sum_{j=1}^{m} |S_jf(\omega)| \quad \text{for } n \ge 1, \tag{32}$$

$$f_\infty(\omega) = \lim_{n\to\infty} f_n(\omega). \tag{33}$$

By Property (B) and the hypothesis $(S_jf)\chi_A \in L$ for $j \ge 1$ it follows that $f_n\chi_A \in L$ for $n \ge 1$; and by Property (A) we have $\|f_n\chi_A\|_L \le (1/m)\sum_{j=1}^{m}\|(S_jf)\chi_A\|_L$ for $m \ge n$. Thus

$$\|f_n\chi_A\|_L \le \liminf_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} \|(S_jf)\chi_A\|_L = K \quad \text{for every } n \ge 1, \tag{34}$$

and hence

$$f_\infty\chi_A \in L \tag{35}$$

by Properties (C) and (B). In particular, $f_\infty(\omega) < \infty$ for μ-a.e. $\omega \in A$, and thus we can apply the proof of Theorem 1 with $\varphi(x) = |x|$ to infer that there exists a real-valued measurable function h on Ω with $\int_\Omega |h(\omega)|\,d\mu < \infty$ such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$.

To prove that $h \in L$, we may assume below that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. (Indeed, if $(\Omega, \mathcal{A}, \mu)$ is atomic, then the ergodicity of T implies that Ω is essentially a finite set, and then $h \in L$ is obvious.) Further we may assume that $(\Omega, \mathcal{A}, \mu)$ is separable, because there is a countable family $\{A_i : i \geq 1\}$ in $\mathcal{A}$ such that

(i) the measure space $(\Omega, \mathcal{A}_1, \mu)$, where $\mathcal{A}_1$ denotes the σ-field generated by $\{A_i : i \geq 1\}$, is nonatomic,

(ii) T is an ergodic endomorphism of $(\Omega, \mathcal{A}_1, \mu)$,

(iii) f is measurable with respect to $\mathcal{A}_1$, and

(iv) the space $L_1 = \{g \in L : g \text{ is } \mathcal{A}_1\text{-measurable}\}$, with norm $\|\cdot\|_{L_1} =$ the restriction of the norm $\|\cdot\|_L$ to L_1, is a Banach lattice having Properties (B)–(D) with L_1 and $\|\cdot\|_{L_1}$ in place of L and $\|\cdot\|_L$, respectively.

Next, we use the isomorphism theorem for measure algebras (cf. e.g. Theorem 41.C of [4]) to infer that there exists an isomorphism Φ from the measure algebra $(\mathcal{A}(\mu), \mu)$ onto the measure algebra $(\mathcal{B}([0,1]), dx)$, where $\mathcal{B}([0,1])$ denotes the σ-field of all Borel subsets of $[0,1]$ and dx is the Lebesgue measure. Thus we may assume that $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{B}([0,1]), dx)$. Then T can be regarded as an isomorphism from $(\mathcal{B}([0,1]), dx)$ *into* itself. Since $[0,1]$ is an uncountable complete separable metric space, it then follows (see e.g. Proposition 15.19 of [12]) that T can be considered to be an (ergodic) endomorphism of the measure space $([0,1], \mathcal{B}([0,1]), dx)$.

We then use the natural extension of the ergodic endomorphism T of $([0,1], \mathcal{B}([0,1]), dx)$ (cf. e.g. §4 of Chapter 10 of [3]). That is, it is known that there exists an ergodic automorphism $\widetilde{T}$ of a separable nonatomic probability measure space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$, and a measure preserving transformation S from $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ to $(\Omega, \mathcal{A}, \mu)$ such that

$$(S \circ \widetilde{T})(\widetilde{\omega}) = (T \circ S)(\widetilde{\omega}) \quad \text{for every } \widetilde{\omega} \in \widetilde{\Omega}. \tag{36}$$

Denote by $\widetilde{L}$ the space of all real-valued $\widetilde{\mathcal{A}}$-measurable functions $\widetilde{u}$ on $\widetilde{\Omega}$ to which there corresponds a function $u \in L$ such that $\widetilde{\mu}(\{\widetilde{\omega} : |\widetilde{u}(\widetilde{\omega})| > a\}) = \mu(\{\omega : |u(\omega)| > a\})$ for every $0 < a \in \mathbb{R}$, and define $\|\widetilde{u}\|_{\widetilde{L}} = \|u\|_L$. By using Property (D) it is easy to check that if $\widetilde{u}$ and $\widetilde{v}$ are in $\widetilde{L}$, then $\widetilde{u} + \widetilde{v} \in \widetilde{L}$ and $\|\widetilde{u} + \widetilde{v}\|_{\widetilde{L}} \leq \|\widetilde{u}\|_{\widetilde{L}} + \|\widetilde{v}\|_{\widetilde{L}}$. (In fact, by the isomorphism theorem, there exists an isomorphism Ψ from the measure algebra $(\widetilde{\mathcal{A}}(\widetilde{\mu}), \widetilde{\mu})$ onto $(\mathcal{A}(\mu), \mu)$. Then Ψ can be uniquely extended, in an obvious manner, to an invertible linear operator (denoted also by Ψ) from $L_0(\widetilde{\Omega}, \widetilde{\mu})$ onto $L_0(\Omega, \mu)$, where $L_0(\widetilde{\Omega}, \widetilde{\mu})$ is the space of all real-valued measurable functions on $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ and, by definition, two functions $\widetilde{u}$ and $\widetilde{v}$ in $L_0(\widetilde{\Omega}, \widetilde{\mu})$ are not distinguished provided that $\widetilde{u}(\widetilde{\omega}) = \widetilde{v}(\widetilde{\omega})$ for $\widetilde{\mu}$-a.e. $\widetilde{\omega} \in \widetilde{\Omega}$; $L_0(\Omega, \mu)$ is defined similarly for $(\Omega, \mathcal{A}, \mu)$. Then $\Psi\widetilde{L} = L$ and $\|\widetilde{u}\|_{\widetilde{L}} = \|\Psi\widetilde{u}\|_L$ for all $\widetilde{u} \in \widetilde{L}$.) It follows that $(\widetilde{L}, \|\cdot\|_{\widetilde{L}})$ is a Banach lattice having Properties (B)–(D) with $\widetilde{L}$ and $\|\cdot\|_{\widetilde{L}}$ replaced by L and $\|\cdot\|_L$, respectively. In order to prove that $h \in L$, it is now sufficient to prove that $h \circ S \in \widetilde{L}$.

Consequently, we have reached the conclusion that T may be assumed to be an ergodic *automorphism* of $(\Omega, \mathcal{A}, \mu)$. With this assumption we continue the proof as follows.

First, as is easily seen, it is enough to consider the case where $\mu(\{\omega : f(\omega) \neq 0\}) > 0$. Then, since T is ergodic, we have

$$\bigcup_{n=0}^{\infty} T^{-n}\{\omega : f(\omega) \neq 0\} = \Omega \ (\mathrm{mod}\, \mu),$$

and thus by Property (B) there exists a set $A_1 \in \mathcal{A}$ with $A_1 \subset A$ and $\mu(A_1) > 0$ for which $h\chi_{A_1} \in L$. This shows that we may assume from the start that $h\chi_A \in L$.

Now, since $(h \circ T^j)\chi_A = (S_j f)\chi_A + h\chi_A$ for $j \geq 1$,

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} \|(S_j f)\chi_A\|_L + \|h\chi_A\|_L &\geq \frac{1}{n}\sum_{j=1}^{n} \|(h \circ T^j)\chi_A\|_L \\ &= \frac{1}{n}\sum_{j=1}^{n} \|h(\chi_A \circ T^{-j})\|_L \qquad \text{(by Property (D))} \\ &\geq \left\| h\left(\frac{1}{n}\sum_{j=1}^{n} \chi_A \circ T^{-j}\right)\right\|_L.\end{aligned}$$

If we let, for $n \geq 1$,

$$d_n(\omega) = \inf_{m \geq n} \frac{1}{m}\sum_{j=1}^{m} \chi_A(T^{-j}\omega),$$

then, by the Birkhoff ergodic theorem together with the ergodicity of T,

$$0 \leq d_1(\omega) \leq d_2(\omega) \leq \ldots \to \mu(A) > 0 \qquad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

Hence, $\mu(A)|h|(\omega) = \lim_{n\to\infty} d_n(\omega)|h|(\omega)$ for μ-a.e. $\omega \in \Omega$. Furthermore,

$$\|d_n h\|_L \leq \left\| h\left(\frac{1}{n}\sum_{j=1}^{n} \chi_A \circ T^{-j}\right)\right\|_L \leq \frac{1}{n}\sum_{j=1}^{n} \|(S_j f)\chi_A\|_L + \|h\chi_A\|_L.$$

Thus, by (31), there exists a subsequence (n') of (n) such that $\|d_{n'}h\|_L \leq K + \|h\chi_A\|_L + 1$ for all n'. By Properties (C) and (B) it follows that $\mu(A)h \in L$, and hence the proof is complete.

Remark 2. If the ergodicity of T is not assumed in Theorem 2, then inequality (31) with $A = \Omega$ implies the same conclusion of the theorem.

To see this, we first notice that the above proof, together with Remark 1, shows that there exists a real-valued measurable function h on Ω such that $f(\omega) = h(T\omega) - h(\omega)$ for μ-a.e. $\omega \in \Omega$. Since $|S_j f| = |h \circ T^j - h| \geq |h \circ T^j| - |h|$

for $j \geq 1$, it follows that

$$\frac{1}{n}\sum_{j=1}^{n}|h|\circ T^j \leq \frac{1}{n}\sum_{j=1}^{n}|S_j f| + |h|.$$

Therefore, the Birkhoff ergodic theorem implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|h|(T^j\omega) \leq \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|S_j f(\omega)| + |h(\omega)| < \infty$$

for μ-a.e. $\omega \in \Omega$, and hence the limit

$$h_\infty(\omega) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}h(T^j\omega)$$

exists and is finite for μ-a.e. $\omega \in \Omega$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}S_j f(\omega) = h_\infty(\omega) - h(\omega)$$

for μ-a.e. $\omega \in \Omega$. Since $h_\infty(\omega) = h_\infty(T\omega)$ for μ-a.e. $\omega \in \Omega$, it follows that the function $h_0(\omega) := h(\omega) - h_\infty(\omega)$ satisfies $f(\omega) = h_0(T\omega) - h_0(\omega)$ for μ-a.e. $\omega \in \Omega$. Furthermore,

$$|h_0(\omega)| \leq \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|S_j f(\omega)| < \infty$$

for μ-a.e. $\omega \in \Omega$. Since inequality (31) with $A = \Omega$ shows that the function $f_\infty(\omega) = \liminf_{n\to\infty}(1/n)\sum_{j=1}^{n}|S_j f(\omega)|$ belongs to L (see (35)), we get $h_0 \in L$ by Property (B). This establishes the desired conclusion.

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