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On Banach spaces $C(K)$ isomorphic to $c_0(\Gamma)$

by

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Abstract. We give a characterization of compact spaces K such that the Banach space $C(K)$ is isomorphic to the space $c_0(\Gamma)$ for some set Γ. As an application we show that there exists an Eberlein compact space K of weight ω_ω and with the third derived set $K^{(3)}$ empty such that the space $C(K)$ is not isomorphic to any $c_0(\Gamma)$. For this compactum K, the spaces $C(K)$ and $c_0(\omega_\omega)$ are examples of weakly compactly generated (WCG) Banach spaces which are Lipschitz isomorphic but not isomorphic.

1. Introduction. In this paper we characterize compact spaces K such that the Banach space $C(K)$ of real-valued continuous functions on K is isomorphic to the space $c_0(\Gamma)$ for some set Γ.

We will first establish the notation and terminology we need for our characterization.

Given a set X and an $n \in \omega$, by $\sigma_n(2^X)$ we denote the subspace of the product 2^X consisting of all characteristic functions of sets of cardinality $\leq n$.

We say that a family $\mathcal{U}$ of sets has *finite order* if there is an $n \in \omega$ such that every subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality n has an empty intersection (in other terminology, the family $\mathcal{U}$ is point-$(n-1)$). The family $\mathcal{U}$ of subsets of a space X is T_0-*separating* if, for every pair of distinct points x, y of X, there is $U \in \mathcal{U}$ containing exactly one of the points x, y.

The space $C_{\mathrm{p}}(K)$ is the space of all continuous real-valued functions on a space K, equipped with the pointwise convergence topology. $A(\lambda)$ denotes the Aleksandrov compactification of a discrete space of cardinality λ. If the Cantor–Bendixson derivative $K^{(\omega)}$ of the space K is empty, we say that K has *finite height*. Finally, let us recall that a space K is an *Eberlein compact space* if K is homeomorphic to a weakly compact subset of a Banach space.

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Theorem 1.1. *For a compact space K the following conditions are equivalent*:

(i) *K has a T_0-separating family of clopen subsets of finite order*,
(ii) *K can be embedded in the space $\sigma_n(2^X)$ for some set X and $n \in \omega$*,
(iii) *$C_p(K)$ is linearly homeomorphic to the space $C_p(A(\kappa))$ for some cardinal κ*,
(iv) *$C(K)$ is isomorphic to $c_0(\Gamma)$ for some set Γ*,
(v) *$C(K)$ is isomorphic to a subspace of $c_0(\Gamma)$ for some set Γ.*

Our characterization was motivated by the following theorem [5, Thm. 4.8]:

Result 1.2 (Godefroy, Kalton and Lancien). *For a compact space K of weight $< \omega_\omega$, the space $C(K)$ is isomorphic to some $c_0(\Gamma)$ if and only if K is an Eberlein compactum of finite height.*

Godefroy, Kalton and Lancien conjectured in [5] that the above result may hold true without the assumption on the weight of K. Theorem 1.1, together with an example from [2], shows however that the cardinal restriction in 1.2 is necessary and cannot be improved.

It is easy to observe that if K embeds in some $\sigma_n(2^X)$ then K is an Eberlein compact space of finite height. Argyros and Godefroy have proved that this implication can be reversed under some restrictions on the weight of K. Namely, if K is an Eberlein compactum of weight $< \omega_\omega$ and of finite height, then K can be embedded in $\sigma_n(2^X)$ for some set X and $n \in \omega$. This (unpublished) result can also be derived from the above theorem of Godefroy, Kalton and Lancien and Theorem 1.1. However, we have the following example:

Result 1.3 (Bell and Marciszewski [2]). *There exists an Eberlein compactum K of weight ω_ω and finite height ($K^{(3)} = \emptyset$) which cannot be embedded into any $\sigma_n(2^X)$.*

Combining this example with Theorem 1.1 and a result from [3] (see also [4, Thm. 8.9]) we obtain

Corollary 1.4. *There exists an Eberlein compactum K of weight ω_ω and finite height such that the space $C(K)$ is not isomorphic to any $c_0(\Gamma)$. The spaces $C(K)$ and $c_0(\omega_\omega)$ are Lipschitz isomorphic WCG spaces which are not isomorphic.*

We prove Theorem 1.1 in Section 3. Section 2 contains some auxiliary results. Some additional remarks are included in Section 4.

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2. Auxiliary results

LEMMA 2.1. *Let $\{C_t : t \in T\}$ be a point-finite family of countable subsets of a set X. Then, for every $t \in T$, there exists a finite set $F_t \subset (T \setminus \{t\})$ such that all sets $C_t \setminus \bigcup\{C_s : s \in F_t\}$ are pairwise disjoint.*

Proof. We may assume that all sets C_t are nonempty. For $s, u \in T$ write $s \sim u$ if there exist $t_1, \ldots, t_n \in T$ such that $t_1 = s$, $t_n = u$ and $C_{t_i} \cap C_{t_{i+1}} \neq \emptyset$ for $i = 1, \ldots, n-1$. The relation $\sim$ is an equivalence relation. Since the family $\{C_t : t \in T\}$ is point-finite, every C_t intersects only countably many C_s. Therefore the equivalence classes of $\sim$ are countable. Let $\{T_a : a \in A\}$ be the partition of T into equivalence classes. It is clear that $C_s \cap C_u = \emptyset$ for every $s \in T_a$ and $u \in T_b$, $a \neq b$. Enumerate each T_a as $\{t_i^a : i < n\}$, where $n \leq \omega$. For $t = t_i^a$, define $F_t = \{t_j^a : j < i\}$. One can easily verify that the sets F_t have the required properties. ∎

LEMMA 2.2. *Let K be an Eberlein compactum of finite height. Then there exists a family $\{U_a : a \in K\}$ such that*

(a) *for every $a \in K$, U_a is a clopen neighborhood of a,*
(b) *for every $n \in \omega$ and $a \in K^{(n)} \setminus K^{(n+1)}$, $U_a \cap K^{(n)} = \{a\}$,*
(c) *for every $a, b \in K$, if $a \in U_b$ then $U_a \subset U_b$,*
(d) *$\{U_a : a \in K\}$ is point-finite.*

Proof. Since K is a scattered Eberlein compactum, we can assume that K is a subspace of $\{\chi_A \in 2^X : |A| < \omega\}$ for some set X (see [1]). For every $a = \chi_A \in K$ we put $V_a = \{\chi_B \in K : A \subset B\}$. It is clear that the family $\{V_a : a \in K\}$ of clopen neighborhoods is point-finite. Every point $a \in K^{(n)} \setminus K^{(n+1)}$ is isolated in $K^{(n)}$, therefore we can find a clopen neighborhood W_a of a such that $W_a \cap K^{(n)} = \{a\}$. Obviously, we can require that $W_a \subset V_a$, hence the family $\{W_a : a \in K\}$ is point-finite. Take a minimal m such that $K^{(m)} = \emptyset$. Put $U_a = W_a$ for $a \in K^{(m-1)}$. Then define inductively, for $n = m-2, m-3, \ldots, 0$ and $a \in K^{(n)} \setminus K^{(n+1)}$, $U_a = W_a \cap \bigcap\{U_b : a \in U_b,\ b \in K^{(n+1)}\}$. It can be easily verified that the sets U_a have the required properties (a)–(d). ∎

For a subset $A \subset \Gamma$ the map $p_A : c_0(\Gamma) \to c_0(\Gamma)$ is defined by

$$p_A(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in A, \\ 0 & \text{if } \gamma \notin A, \end{cases}$$

for $x \in c_0(\Gamma)$ and $\gamma \in \Gamma$.

We will need the following standard fact.

PROPOSITION 2.3. *Let X be a closed linear subspace of $c_0(\Gamma)$. For every countable subset $A \subset \Gamma$ there exists a countable $B \subset \Gamma$ such that $A \subset B$ and $p_B(X) \subset X$.*

Proof. The space X, being WCG, is the closure of $\operatorname{span} K$ for some weakly compact set $K \subset X$. Clearly, K is compact in the pointwise topology in $c_0(\Gamma)$, therefore there exists a countable set $B \subset \Gamma$ such that $A \subset B$ and $p_B(K) \subset K$ (see [6, Lemma 1] or [4, p. 254]). A routine verification shows that also $p_B(X) \subset X$. ∎

3. Proof of Theorem 1.1

PROPOSITION 3.1. *Let K and L be nonempty closed subsets of $\sigma_n(2^X)$ with $K \subset L$. Then there exists a continuous linear extension operator $e : C_{\mathrm{p}}(K) \to C_{\mathrm{p}}(L)$, i.e., $e(f)|K = f$ for every $f \in C_{\mathrm{p}}(K)$.*

Proof. It is enough to prove the statement for $L = \sigma_n(2^X)$ (for other L we can use the restriction operator $g \mapsto g|L$). We will prove this by induction on n. For $n = 0$ this is trivial. Suppose that our assertion holds true for $n \geq 0$. Let K be a nonempty closed subset of $\sigma_{n+1}(2^X)$. Put $M = K \cap \sigma_n(2^X)$ and let $e' : C_{\mathrm{p}}(M) \to C_{\mathrm{p}}(\sigma_n(2^X))$ be a continuous linear extension operator. Then we can define the operator $e : C_{\mathrm{p}}(K) \to C_{\mathrm{p}}(\sigma_{n+1}(2^X))$ by the formula (recall that elements of $\sigma_{n+1}(2^X)$ are the characteristic functions χ_A of sets A of cardinality $\leq n+1$)

$$e(f)(\chi_a) = \begin{cases} f(\chi_A) & \text{for } \chi_A \in K, \\ e'(f|M)(\chi_A) & \text{for } \chi_A \in \sigma_n(2^X), \\ \sum_{B \subsetneq A} (-1)^{n-|B|} e'(f|M)(\chi_B) & \\ & \text{for } \chi_A \in \sigma_{n+1}(2^X) \setminus (K \cup \sigma_n(2^X)). \end{cases}$$

It is clear that the operator e is linear and pointwise continuous. It remains to verify that, for every $f \in C_{\mathrm{p}}(K)$, the function $e(f)$ is continuous on $\sigma_{n+1}(2^X)$. From the definition of $e(f)$, it easily follows that $e(f)|K \cup \sigma_n(2^X)$ is continuous. Observe that all points in $\sigma_{n+1}(2^X) \setminus \sigma_n(2^X)$ are the characteristic functions of sets of cardinality $n+1$ and are isolated in $\sigma_{n+1}(2^X)$. The space $\sigma_{n+1}(2^X)$, being Eberlein compact, is a Fréchet topological space. Therefore, it is enough to show that, for every sequence (χ_{A_k}) of points of $\sigma_{n+1}(2^X) \setminus (K \cup \sigma_n(2^X))$ converging to a point $\chi_B \in \sigma_n(2^X)$, we have $e(f)(\chi_{A_k}) \to e(f)(\chi_B)$. Without loss of generality we may assume that $B \subset A_k$ for all k. One can easily verify that, for every $D \subset B$ and $C_k \subset A_k \setminus B$, we have $\chi_{C_k \cup D} \to \chi_D$. Hence, we obtain

$$\begin{aligned} e(f)(\chi_{A_k}) &= \sum_{B_k \subsetneq A_k} (-1)^{n-|B_k|} e'(f|M)(\chi_{B_k}) \\ &= \sum_{C_k \subsetneq A_k \setminus B} (-1)^{n-|C_k|-|B|} e'(f|M)(\chi_{C_k \cup B}) \\ &\quad + \sum_{D \subsetneq B} \sum_{E_k \subset A_k \setminus B} (-1)^{n-|E_k|-|D|} e'(f|M)(\chi_{E_k \cup D}) \end{aligned}$$

$$\xrightarrow{k\to\infty} e'(f|M)(\chi_B)\sum_{i=0}^{n-|B|}(-1)^{n-i-|B|}\binom{n+1-|B|}{i}$$
$$+\sum_{D\subsetneq B} e'(f|M)(\chi_D)\sum_{i=0}^{n+1-|B|}(-1)^{n-i-|D|}\binom{n+1-|B|}{i}$$
$$= e'(f|M)(\chi_B)\Big(1+(-1)^{n-|B|}\sum_{i=0}^{n+1-|B|}(-1)^i\binom{n+1-|B|}{i}\Big)$$
$$+\sum_{D\subsetneq B} e'(f|M)(\chi_D)(-1)^{n-|D|}\sum_{i=0}^{n+1-|B|}(-1)^i\binom{n+1-|B|}{i}$$
$$= e'(f|M)(\chi_B)(1+(-1)^{n-|B|}(1-1)^{n+1-|B|})$$
$$+\sum_{D\subsetneq B} e'(f|M)(\chi_D)(-1)^{n-|D|}(1-1)^{n+1-|B|}$$
$$= e'(f|M)(\chi_B) = e(f)(\chi_B). \ \blacksquare$$

LEMMA 3.2. *Let K be an Eberlein compact space of finite height and without a T_0-separating family of clopen subsets of finite order. Then every family $\{G_a : a \in K\}$ of G_δ-subsets of K with $a \in G_a$ for all $a \in K$ has infinite order.*

Proof. Suppose on the contrary that there exists a family $\{G_a : a \in K\}$ of G_δ-subsets of K with $a \in G_a$ for every $a \in K$, which has finite order. Let $\{U_a : a \in K\}$ be the family of clopen neighborhoods from Lemma 2.2. For every point $a \in K$, we will choose a clopen neighborhood $V_a \subset U_a$ of a such that the family $\{V_a : a \in K\}$ will have finite order. This will give the desired contradiction since condition (b) of Lemma 2.2 implies that the family $\{V_a : a \in K\}$ is T_0-separating. Let $m \in \omega$ be such that $K^{(m)} = \emptyset$. We will choose sets V_a, for $a \in K^{(n)} \setminus K^{(n+1)}$ and $n = 0, 1, \ldots, m-1$, by induction on n.

For $n = 0$, all points $a \in K \setminus K^{(1)}$ are isolated, so we can simply take $V_a = \{a\}$. Let $n > 0$ and suppose that we have defined the family of neighborhoods $\{V_a : a \in K \setminus K^{(n)}\}$ of finite order. Fix a point $a \in K^{(n)} \setminus K^{(n+1)}$. Observe that from Lemma 2.2(b) it follows that $\bigcap\{U_a \setminus V_b : b \in U_a \cap (K \setminus K^{(n)})\} = \{a\}$. Therefore, every neighborhood of a contains a set of the form $U_a \setminus \bigcup\{V_b : b \in F\}$ for some finite set $F \subset U_a \cap (K \setminus K^{(n)})$. Hence, we can assume that there exists a countable set $C_a \subset U_a \cap (K \setminus K^{(n)})$ such that $G_a = U_a \setminus \bigcup\{V_b : b \in C_a\}$. Condition (d) of Lemma 2.2 guarantees that the family $\{C_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ is point-finite. By Lemma 2.1 we can find finite sets $F_a \subset K^{(n)} \setminus (K^{(n+1)} \cup \{a\})$, for every $a \in K^{(n)} \setminus K^{(n+1)}$, such that the sets $D_a = C_a \setminus \bigcup\{C_c : c \in F_a\}$ are pairwise disjoint. We put

$V_a = U_a \setminus \bigcup\{U_c : c \in F_a\}$. Because $a \notin F_a$, we have $a \in V_a$ by Lemma 2.2(b).

Let $W_a = \bigcup\{V_b : b \in D_a\}$ for $a \in K^{(n)} \setminus K^{(n+1)}$. Since the sets D_a are pairwise disjoint, our inductive assumption on the sets V_b implies that the family $\{W_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ has finite order. By condition (c) of Lemma 2.2 we have $V_b \subset U_c$ for all $b \in C_c \subset U_c$, therefore $V_a \subset G_a \cup W_a$ for every $a \in K^{(n)} \setminus K^{(n+1)}$. It follows that the family $\{V_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ also has finite order. Clearly, the family $\{V_a : a \in K \setminus K^{(n+1)}\}$, being a finite union of families of finite order, has finite order. ∎

LEMMA 3.3. *Let K be a compact space such that each family $\{G_a : a \in K\}$ of G_δ-subsets of K with $a \in G_a$ for every $a \in K$ has infinite order. Then $C(K)$ is not isomorphic to any subspace of $c_0(\Gamma)$.*

Proof. Assume towards a contradiction that $T : C(K) \to X$ is an isomorphism of $C(K)$ onto a subspace X of some $c_0(\Gamma)$. We may assume that $\|T\| = 1$ and put $M = \|T^{-1}\|$. Let $n > M$ be a natural number. For every $a \in K$ we denote by δ_a the Dirac measure supported at a.

For every $a \in K$ we take $z_a \in \ell_1(\Gamma)$ such that $z_a|X = (T^{-1})^*(\delta_a)$ (we treat z_a as an element of the dual $(c_0(\Gamma))^*$). Let A_a be the support of z_a, i.e., $A_a = \{\gamma \in \Gamma : z_a(\gamma) \neq 0\}$. By Proposition 2.3 we can find a countable set $B_a \subset \Gamma$ containing A_a and such that $p_{B_a}(X) \subset X$.

Observe that for each $x \in X$ the set $G(a,x) = \{b \in K : (T^{-1})^*(\delta_b)(x) = (T^{-1})^*(\delta_a)(x)\} = \{b \in K : T^{-1}(x)(b) = T^{-1}(x)(a)\}$ is a G_δ-set in K. Let $\{x_k^a : k \in \omega\}$ be a dense subset of $p_{B_a}(X)$. Then $G_a = \bigcap\{G(a, x_k^a) : k \in \omega\}$ is a G_δ-set containing a and by density of $\{x_k^a : k \in \omega\}$ we have $(T^{-1})^*(\delta_b)(x) = (T^{-1})^*(\delta_a)(x)$ for every $b \in G_a$ and $x \in p_{B_a}(X)$. The family $\{G_a : a \in K\}$ has infinite order, hence we can find distinct $a_1, \ldots, a_n \in K$ such that $G_{a_1} \cap \ldots \cap G_{a_n} \neq \emptyset$. Take a point $b \in G_{a_1} \cap \ldots \cap G_{a_n}$. We can find continuous functions $f_i : K \to [0,1]$, for $i = 1, \ldots, n$, such that $f_i(a_i) = 1$ for every i and the sets $f_i^{-1}((0,1])$ are pairwise disjoint. For every sequence $(\varepsilon_i)_{i=1}^n$ with $|\varepsilon_i| = 1$ we have $\|\sum_{i=1}^n \varepsilon_i f_i\| = 1$, therefore $\|T(\sum_{i=1}^n \varepsilon_i f_i)\| \leq 1$. It follows that, for every $\gamma \in \Gamma$, we have $\sum_{i=1}^n |T(f_i)(\gamma)| \leq 1$.

Let $x_i = p_{B_{a_i}}(T(f_i)) \in X$ for $i = 1, \ldots, n$. By the above inequality we have $\|\sum_{i=1}^n x_i\| \leq 1$. For every i, we have $1 = \delta_{a_i}(f_i) = (T^{-1})^*(\delta_{a_i})(T(f_i)) = z_{a_i}(T(f_i))$. The fact that B_{a_i} contains the support of z_{a_i} and the equality $x_i|B_{a_i} = T(f_i)|B_{a_i}$ imply that also $1 = z_{a_i}(x_i) = (T^{-1})^*(\delta_{a_i})(x_i)$. Furthermore, since $b \in G_{a_i}$ we find that $(T^{-1})^*(\delta_b)(x_i) = 1$ for all i. Then $(T^{-1})^*(\delta_b)(\sum_{i=1}^n x_i) = n$, which shows that $\|(T^{-1})^*\| = \|T^{-1}\| \geq n > M$, a contradiction. ∎

Proof of Theorem 1.1. (i)⇒(ii). Suppose that $\{U_x : x \in X\}$ is a T_0-separating family of finite order consisting of clopen subsets of K. Let $g_x =$

$\chi_{U_x} : K \to 2$ for $x \in X$. Then the diagonal map $h = \triangle_{x\in X} g_x : K \to 2^X$ is an embedding with $h(K) \subset \sigma_n(2^X)$ for some $n \in \omega$.

(ii)⇒(iii). We will prove this implication by induction on n. The case $n = 1$ is obvious since every closed subset of $\sigma_1(2^X)$ is homeomorphic to some $A(\kappa)$. Suppose that the implication considered holds true for n, and K is a closed subset of $\sigma_{n+1}(2^X)$. Take $M = K \cap \sigma_n(2^X)$. Then $C_{\rm p}(M)$ is linearly homeomorphic to $C_{\rm p}(A(\kappa))$ for some κ. By Proposition 3.1 there exists a continuous linear extension operator $e : C_{\rm p}(M) \to C_{\rm p}(K)$. It is well known that this implies that the space $C_{\rm p}(K)$ is linearly homeomorphic to the product $C_{\rm p}(M) \times \{f \in C_{\rm p}(K) : f|M \equiv 0\}$ (see [8, Proposition 6.6.6]). Since all points of $K \setminus M$ are isolated, the second factor can be identified with $c_0(K\setminus M)$ equipped with the pointwise convergence topology. Standard verification shows that the product $C_{\rm p}(A(\kappa))\times c_0(\mathit{\Gamma})$ (the second factor with the pointwise topology) is linearly homeomorphic to $C_{\rm p}(A(\eta))$ for some η.

(iii)⇒(iv). This follows easily from the Closed Graph Theorem and the facts that $C_{\rm p}(A(\kappa))$ is linearly homeomorphic to $c_0(\kappa)$ (with the pointwise topology) and the pointwise topology is weaker than the norm topology.

(iv)⇒(v). Trivial.

(v)⇒(i). Let K be a compact space with $C(K)$ isomorphic to a subspace of $c_0(\mathit{\Gamma})$. It follows that $C(K)$ is a WCG space, hence K is an Eberlein compactum. We also have $K^{(\omega)} = \emptyset$ by [7, Thm. 3.8]. Then condition (i) follows immediately from Lemmas 3.2 and 3.3. ∎

4. Remarks

Remark 1. Note that the proof of the implication (v)⇒(i) of Theorem 1.1 can be simplified if we only want to prove the (weaker) implication (iv)⇒(i). The implication (v)⇒(iv) can be derived from Remark 5.4 in [5] stating that an $\mathcal{L}^\infty$ subspace of $c_0(\mathit{\Gamma})$ is isomorphic to $c_0(\mathit{\Gamma})$. Since the proof of that fact is not included in [5], we decided to give an independent proof of the implication (v)⇒(i).

Remark 2. Godefroy, Kalton and Lancien have proved that, for a WCG Banach space X of weight less than ω_ω, every subspace Y of X which is isomorphic to $c_0(\mathit{\Gamma})$ is complemented in X (see [5, proof of Thm. 4.8]). Again, the restriction on the weight of X cannot be omitted in this result. Let K be an Eberlein compactum (with $K^{(3)} = \emptyset$) described in Result 1.3. Take $Y = \{f \in C(K) : f|K' \equiv 0\}$. Then Y is isometric to $c_0(K \setminus K')$ and $C(K)/Y$ is isometric to $C(K')$, which, in turn, is isomorphic to $c_0(\mathit{\Gamma})$ since $K^{(3)} = \emptyset$. Hence Y cannot be complemented in $C(K)$ because $C(K)$ is not isomorphic to $c_0(\mathit{\Gamma})$ (cf. [5, proof of Thm. 4.8]).

Remark 3. Corollary 1.4 also shows that the cardinality restriction in the following result of Godefroy, Kalton and Lancien cannot be removed.

They proved [5, Corollary 5.2] that every WCG Banach space X of weight less than ω_ω which is Lipschitz isomorphic to $c_0(\Gamma)$ is isomorphic to $c_0(\Gamma)$.

Remark 4. In this paper we restricted ourselves to the spaces of real-valued functions. One can easily verify that this restriction is inessential; the proofs of our results work in the complex case as well.

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