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Hypercyclic sequences of operators

by

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Abstract. A sequence (T_n) of bounded linear operators between Banach spaces X, Y is said to be hypercyclic if there exists a vector $x \in X$ such that the orbit $\{T_n x\}$ is dense in Y. The paper gives a survey of various conditions that imply the hypercyclicity of (T_n) and studies relations among them. The particular case of $X = Y$ and mutually commuting operators T_n is analyzed. This includes the most interesting cases (T^n) and $(\lambda_n T^n)$ where T is a fixed operator and λ_n are complex numbers. We also study when a sequence of operators has a large (either dense or closed infinite-dimensional) manifold consisting of hypercyclic vectors.

I. Introduction. Let X and Y be separable Banach spaces. Denote by $B(X,Y)$ the set of all bounded linear operators from X to Y. Let $(T_n) \subset B(X,Y)$ be a sequence of operators. A vector $x \in X$ is called *hypercyclic* for (T_n) if the set $\{T_n x\}$ is dense in Y. The sequence (T_n) is called *hypercyclic* if there is at least one vector hypercyclic for (T_n). We say that an operator $T : X \to X$ is *hypercyclic* if the sequence (T^n) of its iterates is hypercyclic.

Similarly, an operator T is said to be *supercyclic* if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense; the vector x with this property is called *supercyclic for T*.

Usually it is not easy to verify whether a sequence (T_n) is hypercyclic or not. There are many criteria that have been studied by a number of authors that imply the hypercyclicity of (T_n) (see e.g. [K], [GS], [BG], [BP]). In the second section we give a survey of various conditions implying the hypercyclicity and study relations among them. A number of illustrative examples are given.

The third section concentrates on the situation when $Y = X$ and the operators $T_n : X \to X$ are mutually commuting. The relations among vari-

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ous conditions are much simpler in this case. The following section studies the case when $T_n = S_1 \cdots S_n$ where $S_j : X \to X$ are mutually commuting. This includes the most interesting cases (T^n) and $(\lambda_n T^n)$ where T is a fixed operator and λ_n complex numbers.

Sequences of operators with "many hypercyclic vectors" are very important in hypercyclicity theory. The interest in them (especially in the cases of (T^n) and $(\lambda_n T^n)$) arises from the invariant subspace/subset problem.

There are two research lines in the literature. The first one, which was initiated by B. Beauzamy [Bea] and continued in [G], [GS], [He], [Bo], [B], [BC] and recently [Gri], studies the existence of dense manifolds consisting of hypercyclic vectors. The second, more recent, line studies the existence of closed infinite-dimensional subspaces all of whose nonzero elements are hypercyclic; see [Mo], [LMo], [GLM] and recently [BMP]. The questions of this type are studied in Section V below.

II. Hypercyclicity of sequences of operators. Let X, Y be separable Banach spaces and let $(T_n) \subset B(X,Y)$ be a sequence of operators. It is well known that the set of all hypercyclic vectors for (T_n) is a G_δ set. Indeed, $x \in X$ is hypercyclic for (T_n) if and only if $x \in \bigcap_U \bigcup_{n\in\mathbb{N}} T_n^{-1}U$, where U runs over a countable base of open subsets of Y; it is clear that $\bigcup_{n\in\mathbb{N}} T_n^{-1}U$ is open for each U.

Lemma 1 ([GS]). *Let $(T_n) \subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent*:

(i) *(T_n) has a dense subset of hypercyclic vectors*;
(ii) *the set of all hypercyclic vectors for (T_n) is residual* (*i.e., its complement is of the first category*);
(iii) *for all nonempty open subsets $U \subset X$, $V \subset Y$ there exists $n \in \mathbb{N}$ such that $T_nU \cap V \neq \emptyset$*;
(iv) *for all $x \in X$, $y \in Y$ and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $u \in X$ such that $\|u - x\| < \varepsilon$ and $\|T_n u - y\| < \varepsilon$.*

Denote by B_X the closed unit ball in a Banach space X. The most practical criteria of hypercyclicity are the following two:

Definition 2. We say that a sequence $(T_n) \subset B(X,Y)$ satisfies *condition* (C) if there exist an increasing sequence (n_k) of positive integers and a dense subset $X_0 \subset X$ such that

(i) $\lim_{k\to\infty} T_{n_k} x = 0$ $(x \in X_0)$;
(ii) $\bigcup_k T_{n_k} B_X$ is dense in Y.

The second condition is similar:

DEFINITION 3. We say that a sequence $(T_n) \subset B(X,Y)$ satisfies *condition* $(\mathrm{C_{fin}})$ if there exist an increasing sequence (n_k) of positive integers and a dense subset $X_0 \subset X$ such that

(i) $\lim_{k\to\infty} T_{n_k}x = 0$ $(x \in X_0)$;
(ii) $\bigcup_k(\underbrace{T_{n_k}B_X \oplus \cdots \oplus T_{n_k}B_X}_{j})$ is dense in $\underbrace{Y \oplus \cdots \oplus Y}_{j}$ for all $j \in \mathbb{N}$.

Clearly, condition $(\mathrm{C_{fin}})$ implies (C). Condition (C) is the weakest known property which can be practically used to show the hypercyclicity of a sequence (T_n). Moreover, it implies the existence of a dense (and hence residual) set of hypercyclic vectors. Furthermore, under a reasonable additional condition it implies that there is a closed infinite-dimensional subspace of hypercyclic vectors.

Condition $(\mathrm{C_{fin}})$ has a number of equivalent formulations and it implies that there is a dense linear subspace consisting of hypercyclic vectors. The existence of subspaces consisting of hypercyclic vectors will be studied in Section V.

THEOREM 4. *Let $(T_n) \subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent*:

(i) *(T_n) satisfies condition* (C);
(ii) *for all $j \in \mathbb{N}$ and nonempty open subsets $U_0, U_1, \dots, U_j \subset X$ and $V_0, V \subset Y$ such that U_0 and V_0 contain the origins of X and Y, respectively, there exists $n \in \mathbb{N}$ such that $T_nU_i \cap V_0 \neq \emptyset$ $(i = 1, \dots, j)$ and $T_nU_0 \cap V \neq \emptyset$.*

In particular, if (T_n) satisfies (C) *then there is a dense set of hypercyclic vectors for (T_n).*

Proof. (i)$\Rightarrow$(ii). Clear.

(ii)$\Rightarrow$(i). Let $(x_n) \subset X$ and $(y_n) \subset Y$ be dense sequences. Set $u_{i,i} = x_i$ $(i \in \mathbb{N})$. By induction on k we construct an increasing sequence (n_k) and vectors $u_{i,k} \in X$ $(i = 1, \dots, k-1)$ and $v_k \in B_X$ such that

$$\|T_{n_k}v_k - y_k\| < 2^{-k}, \qquad \|T_{n_k}u_{i,k}\| < 2^{-k},$$
$$\|u_{i,k} - u_{i,k-1}\| < \frac{1}{2^k \max\{1, \|T_{n_1}\|, \dots, \|T_{n_{k-1}}\|\}},$$

for all i, k with $1 \le i < k$. For each i the sequence $(u_{i,k})_k$ is Cauchy. Let u_i be its limit. Then

$$\|x_i - u_i\| \le \sum_{k=i+1}^{\infty} \|u_{i,k} - u_{i,k-1}\| \le \sum_{k=i+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^i}.$$

Therefore (u_i) is dense in X.

Clearly the sequence $(T_{n_k}v_k)$ is dense, and so $\overline{\bigcup_k T_{n_k}B_X} = Y$. Further,

$$\lim_{k\to\infty}\|T_{n_k}u_i\| \le \lim_{k\to\infty}\Big(\|T_{n_k}u_{i,k}\| + \sum_{j=k}^{\infty}\|T_{n_k}\|\cdot\|u_{i,j+1}-u_{i,j}\|\Big)$$
$$\le \lim_{k\to\infty}\Big(\frac{1}{2^k}+\sum_{j=k}^{\infty}\frac{1}{2^{j+1}}\Big) = \lim_{k\to\infty}\frac{1}{2^{k-1}} = 0$$

for each i. Thus (T_n) satisfies (C).

To show that condition (C) implies the existence of a dense subset of hypercyclic vectors we use Lemma 1. Let $x\in X$, $y\in Y$ and $\varepsilon>0$. By (ii), there are $x_0,x_1\in X$ and $n\in\mathbb{N}$ such that $\|x_0\|<\varepsilon/2$, $\|T_nx_0-y\|<\varepsilon/2$, $\|x-x_1\|<\varepsilon/2$ and $\|T_nx_1\|<\varepsilon/2$. Then $\|(x_0+x_1)-x\|<\varepsilon$ and $\|T_n(x_0+x_1)-y\|<\varepsilon$. By Lemma 1, (T_n) has a dense subset of hypercyclic vectors. ∎

THEOREM 5. *Let $(T_n)\subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent*:

(i) *(T_n) satisfies condition* ($\mathrm{C_{fin}}$);
(ii) *$(\underbrace{T_n\oplus\cdots\oplus T_n}_{j})$ satisfies condition* (C) *for all $j\in\mathbb{N}$;*
(iii) *$(\underbrace{T_n\oplus\cdots\oplus T_n}_{j})$ has a dense subset of hypercyclic vectors for all $j\in\mathbb{N}$;*
(iv) *for all $j\in\mathbb{N}$ and all nonempty open subsets $U_1,\dots,U_j\subset X$ and $V_1,\dots,V_j\subset Y$ there is an $n\in\mathbb{N}$ such that $T_nU_i\cap V_i\ne\emptyset$ $(i=1,\dots,j)$;*
(v) *there is a subsequence (T_{n_k}) such that each of its subsubsequences has a dense set of hypercyclic vectors*;
(vi) *there are dense subsets $X_0\subset X$ and $Y_0\subset Y$, an increasing sequence $(n_k)\subset\mathbb{N}$ and mappings $S_i: Y_0\to X$ $(i\in\mathbb{N})$ such that*

$$\begin{aligned} T_{n_k}x&\to 0 \quad (x\in X_0),\\ S_ky&\to 0 \quad (y\in Y_0),\\ T_{n_k}S_ky&\to y \quad (y\in Y_0);\end{aligned}$$

(vii) *for each Banach space Z the sequence of operators $L_{T_n}:\overline{F(Z,X)}\to\overline{F(Z,Y)}$ defined by $L_{T_n}S=T_nS$ $(S\in\overline{F(Z,X)})$ has a dense set of hypercyclic vectors*; *here $F(Z,X)$ denotes the set of all finite rank operators from Z to X*;
(viii) *for each Banach space Z the sequence (L_{T_n}) satisfies condition* (C).

Proof. The equivalences (vi)⇔(v)⇔(iii) were proved in [BG]. The implications (i)⇒(ii) and (vi)⇒(i) are obvious. The equivalence (iii)⇔(iv) follows from Lemma 1 and the implications (viii)⇒(vii) and (ii)⇒(iii) follow from Theorem 4.

(i)⇒(viii). Let X_0 be a dense subset of X and let (n_k) satisfy $T_{n_k}x \to 0$ $(x \in X_0)$ and $\overline{\bigcup(T_{n_k}B_X \oplus \cdots \oplus T_{n_k}B_X)} = Y \oplus \cdots \oplus Y$.

Let $\mathcal{M} \subset B(Z,X)$ be the set of all finite rank operators with range included in the linear space generated by X_0. Clearly $\mathcal{M}$ is dense in $\overline{F(Z,X)}$. For $G \in \mathcal{M}$ we have $\lim_k L_{T_{n_k}} G = \lim_k T_{n_k} G = 0$.

Let $F \in F(Z,Y)$ and $\varepsilon > 0$. We can express $F = \sum_{i=1}^{j} z_i^* \otimes y_i$ for some $y_i \in Y$ and $z_i^* \in Z^*$. Since (T_n) satisfies condition ($\mathrm{C_{fin}}$), there are vectors $u_i \in X$ $(i = 1, \ldots, j)$ and $k \in \mathbb{N}$ such that

$$\|T_{n_k}u_i - y_i\| < \frac{\varepsilon}{j\max\{\|z_1^*\|, \ldots \|z_j^*\|\}}, \qquad \|u_i\| \le \frac{1}{j\max\{\|z_1^*\|, \ldots, \|z_j^*\|\}}.$$

Set $F_0 = \sum_{i=1}^{j} z_i^* \otimes u_i \in F(Z,X)$. Then $\|F_0\| \le \sum_{i=1}^{j} \|z_i^*\| \cdot \|u_i\| \le 1$ and

$$\begin{aligned}\|L_{T_{n_k}}F_0 - F\| = \|T_{n_k}F_0 - F\| &= \Big\|\sum_{i=1}^{j} z_i^* \otimes T_{n_k}u_i - \sum_{i=1}^{j} z_i^* \otimes y_i\Big\| \\ &= \Big\|\sum_{i=1}^{j} z_i^* \otimes (T_{n_k}u_i - y_i)\Big\| \le \sum_{i=1}^{j} \|z_i^*\| \cdot \|T_{n_k}u_i - y_i\| < \varepsilon.\end{aligned}$$

(vii)⇒(iii). Let $j \in \mathbb{N}$ and let Z be a j-dimensional Banach space. Then $\overline{F(Z,X)}$ is isomorphic to $\underbrace{X \oplus \cdots \oplus X}_{j}$ and $\overline{F(Z,Y)}$ to $\underbrace{Y \oplus \cdots \oplus Y}_{j}$. In the same way L_{T_n} can be identified with $T_n \oplus \cdots \oplus T_n$. ∎

For completeness we also mention other conditions that have been studied in the literature, for instance, in [BG, Theorem 2.2] (see also [BP, Remark 2.6(3)] in the case of commuting operators with dense range). In the diagram below we show the relations among them. The abbreviations there mean:

(HC) (Hypercyclicity criterion) There exist dense subsets $X_0 \subset X$ and $Y_0 \subset Y$, an increasing sequence $(n_k) \subset \mathbb{N}$ and mappings $S_k : Y_0 \to X$ such that

$$\begin{aligned} T_{n_k}x &\to 0 \qquad (x \in X_0), \\ S_k y &\to 0 \qquad (y \in Y_0), \\ T_{n_k}S_k y &= y \qquad (y \in Y_0, k \in \mathbb{N}). \end{aligned}$$

(hc) (T_n) is hypercyclic.

(dense hc) (T_n) has a dense set of hypercyclic vectors.

(4 nbhd) (4 neighbourhoods condition) for all nonempty open subsets $U, U_0 \subset X$ and $V, V_0 \subset Y$ such that U_0 and V_0 contain the origins in X and Y, respectively, there exists $n \in \mathbb{N}$ such

that $T_nU \cap V_0 \neq \emptyset$ and $T_nU_0 \cap V_n \neq \emptyset$ (when $X = Y$ this condition reduces to "the three open sets condition", which was introduced in [GS, Section III]).

(her hc) (Hereditarily hypercyclic) There is a subsequence (T_{n_k}) such that each of its subsubsequences is hypercyclic.

The relations among these conditions are given in the following diagram:

$$\begin{array}{ccccccc} (\mathrm{HC}) & \longrightarrow & (\mathrm{C_{fin}}) & \longrightarrow & (\text{her hc}) & \longrightarrow & (\mathrm{hc}) \\ & & \big\downarrow & & & & \big\uparrow \\ & & (\mathrm{C}) & \longrightarrow & (\text{4 nbhd}) & \longrightarrow & (\text{dense hc}) \end{array}$$

Moreover, there are no other implications among the conditions in question.

The implications (HC) $\to$ ($\mathrm{C_{fin}}$) and ($\mathrm{C_{fin}}$) $\to$ (her hc) were proved in Theorem 5, the implication (C) $\to$ (4 nbhd) in Theorem 4. For the implication (4 nbhd) $\to$ (dense hc) see Proposition 6 below.

The remaining implications are trivial.

The negative results follow from the examples below. Note that it is sufficient to show ($\mathrm{C_{fin}}$) $\not\to$ (HC), (C) $\not\to$ (her hc), (her hc) $\not\to$ (dense hc) and (dense hc) $\not\to$ (C).

Proposition 6 (cf. [GS]). *If $(T_n) \subset B(X,Y)$ satisfies* (4 nbhd) *then there is a dense subset of hypercyclic vectors for (T_n).*

Proof. The result was essentially proved already in the proof of Theorem 4. Let $x \in X$, $y \in Y$ and $\varepsilon > 0$. Then there are $n \in \mathbb{N}$, $u \in X$ and $v \in Y$ such that $\|u - x\| < \varepsilon/2$, $\|T_nu\| < \varepsilon/2$, $\|v\| < \varepsilon/2$ and $\|T_nu - y\| < \varepsilon/2$. Set $x' = u + v$. Then $\|x - x'\| \le \|x - u\| + \|v\| < \varepsilon$ and $\|T_nx' - y\| < \|T_nv - y\| + \|T_nu\| < \varepsilon$. By Lemma 1, this implies that (T_n) has a dense subset of hypercyclic vectors. ∎

Example 7. Let X be a Hilbert space with an orthonormal basis $\{e_F : F \subset \mathbb{N},\ \operatorname{card} F < \infty\}$. Let $Y = X$ and let the operators $T_n : X \to X$ be defined by

$$T_ne_F = \begin{cases} ne_{F\setminus\{n\}} & (n \in F), \\ 0 & (n \notin F). \end{cases}$$

It is easy to verify that the sequence (T_n) satisfies condition ($\mathrm{C_{fin}}$) for the dense subspace $X_0 \subset X$ generated by the vectors $\{e_F : F \subset \mathbb{N}\}$. However, (T_n) does not satisfy (HC) since the operators T_n have nondense ranges, $T_nX = \bigvee\{e_F : n \notin F\}$.

Note that the operators T_n are even commuting.

Example 8. Let X and $T_n : X \to X$ be as in the previous example. Note that $e_{\{n\}} \perp T_nX$ for each n. Consider the operators $S_n : X \oplus \mathbb{C} \to X$

defined by

$$S_n(x \oplus \lambda) = T_n x + \lambda e_{\{n\}} \qquad (x \in X,\ \lambda \in \mathbb{C}).$$

Since the operators (T_n) satisfy $(\mathrm{C}_{\mathrm{fin}})$ and hence are hereditarily hypercyclic, it is easy to see that the sequence (S_n) is also hereditarily hypercyclic. On the other hand, the set of all vectors hypercyclic for (S_n) is not dense. Indeed, let $x \in X$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then

$$\|S_n(x \oplus \lambda)\| = \|T_n x + \lambda e_{\{n\}}\| \geq |\lambda| > 0,$$

and so $x \oplus \lambda$ is not hypercyclic. This shows that (her hc) $\not\to$ (dense hc).

Example 9. Let X be a separable Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Let $Y = \mathbb{C}^2$ and (y_n) be a dense sequence of elements of Y. Define $T_n : X \to Y$ by $T_n e_n = y_n$ and $T_n e_i = 0$ $(i \neq n)$. Clearly $T_n x \to 0$ for each x that is a finite linear combination of the vectors e_i $(i \in \mathbb{N})$. Further $\overline{\bigcup_n T_n B_X} \supset \{y_n : n \in \mathbb{N}\}^- = Y$. Thus (T_n) satisfies condition (C).

On the other hand, let (n_k) be any increasing sequence of positive integers such that (T_{n_k}) is hypercyclic. Let $U \subset Y$ be a nonempty open set such that $\overline{U} \neq Y$ and $\mathbb{C} \cdot U \subset U$. Choose a subsequence of those indices n_k for which $y_{n_k} \in U$. For such an n_k we have $T_{n_k} X = \mathbb{C} \cdot y_{n_k} \subset U$, and so (T_n) is not hereditarily hypercyclic. Consequently, (C) $\not\to$ (her hc).

Example 10. Let $\dim X = 1$ (i.e., $X = \mathbb{C}$) and let Y be a separable Hilbert space. Let (y_n) be a dense sequence in Y. Define $T_n : X \to Y$ by $T_n(\lambda) = \lambda y_n$. Clearly each nonzero $\lambda \in X$ is hypercyclic for (T_n). It is easy to see that (T_n) does not satisfy condition (4 nbhd). Indeed, consider the neighbourhoods $U = \{z \in \mathbb{C} : |z| > 2\}$, $V_0 = \{y \in Y : \|y\| < 1\}$, $U_0 = \{z \in \mathbb{C} : |z| < 1\}$ and $V = \{y \in Y : \|y\| > 2\}$. Thus (dense hc) $\not\to$(4 nbhd).

Example 11. Let $X = \mathbb{C}^2$ and $\dim Y = \infty$. Let (y_n) be a dense sequence in Y and (x_n) dense in X. For each n find $u_n \in X$ linearly independent of x_n such that $\|u_n\| = 1/n$. For $m, n \in \mathbb{N}$ define $T_{m,n} : X \to Y$ by $T_{m,n} x_n = 0$ and $T_{m,n} u_n = y_m$. Then $(T_{m,n})$ is a countable set of operators satisfying condition (4 nbhd).

Let (T_{m_k,n_k}) be any subsequence such that $T_{m_k,n_k} x \to 0$ for all x in a dense subset of X. Then $T_{m_k,n_k} \to 0$ in the strong operator topology, and therefore this subsequence is bounded by the Banach–Steinhaus theorem. Thus $(T_{m,n})$ does not satisfy (C). Hence (4 nbhd) $\not\to$ (C).

III. Sequences of commuting operators. In this section we assume that $Y = X$ and $(T_n) \subset B(X)$ is a sequence of mutually commuting operators. The situation is much simpler in this case.

Theorem 12. *Let $(T_n) \subset B(X)$ be a sequence of mutually commuting operators. The following conditions are equivalent*:

(i) (T_n) *satisfies condition* (C);
(ii) (T_n) *satisfies condition* ($\mathrm{C}_{\mathrm{fin}}$);
(iii) (T_n) *is hereditarily hypercyclic*;
(iv) (T_n) *satisfies* (4 nbhd); *in fact in this case the* 4 *neighbourhoods condition reduces to the "*3 *neighbourhoods condition": for all nonempty open subsets* $U, V, W \subset X$ *with* $0 \in W$ *there exists* n *such that* $T_n U \cap W \neq \emptyset$ *and* $T_n W \cap V \neq \emptyset$.

Proof. (i)$\Rightarrow$(ii). Let $X_0 \subset X$ be a dense subset and $(n_k) \subset \mathbb{N}$ an increasing sequence such that $T_{n_k} x \to 0$ $(x \in X_0)$ and $\overline{\bigcup_k T_{n_k} B_x} = X$.

By Theorem 4, (T_{n_k}) is hypercyclic. Let $x \in X$ be a hypercyclic vector for (T_{n_k}).

Let $y_1, \dots, y_r \in X$ and $\varepsilon > 0$. Since x is hypercyclic, there are $k_1, \dots, k_r$ such that $\|T_{n_{k_i}} x - y_i\| < \varepsilon/2$ $(i = 1, \dots, r)$. Further, there are $u \in X$, $\|u\| \le \max\{\|T_{n_{k_i}}\| : i = 1, \dots, r\}^{-1}$ and $s \in \mathbb{N}$ such that

$$\|T_{n_{k_s}} u - x\| < \frac{\varepsilon}{2 \max\{\|T_{n_{k_1}}\|, \dots, \|T_{n_{k_r}}\|\}}.$$

Set $x_i = T_{n_{k_i}} u$ $(i = 1, \dots, r)$. Then $x_i \in B_X$ and

$$\|T_{n_{k_s}} x_i - y_i\| = \|T_{n_{k_s}} T_{n_{k_i}} u - y_i\| \le \|T_{n_{k_i}} (T_{n_{k_s}} u - x)\| + \|T_{n_{k_i}} x - y_i\| < \varepsilon$$

for all $i = 1, \dots, r$.

(ii)$\Rightarrow$(iii). Clear.

(iii)$\Rightarrow$(iv). Let $U, V, W \subset X$ be nonempty open sets, $0 \in W$. Let (n_k) be a sequence of positive integers such that each subsequence of (T_{n_k}) is hypercyclic. Let x be a hypercyclic vector for (T_{n_k}). Since each nonzero multiple of x is also hypercyclic, we can assume that $x \in W$. Consider the subsequence $(T_{n_k})_{k \in F}$ where $F = \{k : T_{n_k} x \in V\}$. Consequently, each $k \in F$ satisfies $T_{n_k} W \cap V \neq \emptyset$.

Let y be a vector hypercyclic for this subsequence. Thus there exists $k_0 \in F$ such that $T_{n_{k_0}} y \in U$. Moreover, we can choose increasing numbers $k_i \in F$ such that $T_{n_{k_i}} y \to 0$ $(i \to \infty)$. Thus

$$\lim_{i \to \infty} T_{n_{k_i}} T_{n_{k_0}} y = \lim_{i \to \infty} T_{n_{k_0}} T_{n_{k_i}} y = 0$$

and there is an i with $T_{n_{k_i}} T_{n_{k_0}} y \in W$. Hence $T_{n_{k_i}} U \cap W \neq \emptyset$.

(iv)$\Rightarrow$(i). By Proposition 6, the sequence (T_n) has a dense subset of hypercyclic vectors.

Let $U_1, \dots, U_r, V, W \subset X$ be nonempty open subsets, $0 \in W$. Let x be a hypercyclic vector for the sequence (T_n). Find $n_1, \dots, n_r \in \mathbb{N}$ such that $T_{n_i} x \in U_i$ $(i = 1, \dots, r)$. Let $\varepsilon > 0$ satisfy $\{y : \|y - T_{n_i} x\| < \varepsilon\} \subset U_i$ $(i = 1, \dots, r)$ and $\{y : \|y\| < \varepsilon\} \subset W$. By assumption, there are $x' \in X$ and $n_0 \in \mathbb{N}$ such that $\|x' - x\| < \varepsilon \max\{\|T_{n_1}\|, \dots, \|T_{n_r}\|\}^{-1}$, $\|T_{n_0} x'\| < \varepsilon \max\{\|T_{n_1}\|, \dots, \|T_{n_r}\|\}^{-1}$ and $T_{n_0} W \cap V \neq \emptyset$. Then $\|T_{n_i} x' - T_{n_i} x\| \le$

$\|T_{n_i}\| \cdot \|x' - x\| < \varepsilon$, and so $T_{n_i}x' \in U_i$ $(i = 1, \ldots, r)$. Further $\|T_{n_0}T_{n_i}x'\| = \|T_{n_i}T_{n_0}x'\| \le \|T_{n_i}\| \cdot \|T_{n_0}x'\| < \varepsilon$, and so $T_{n_0}T_{n_i}x' \in W$. ■

Thus for commuting operators $T_n : X \to X$ we have the following situation:

$$\text{(HC)} \quad \underset{\not\leftarrow}{\rightarrow} \quad \text{(C)} \quad \underset{\not\leftarrow}{\rightarrow} \quad \text{(dense hc)} \quad \underset{\not\leftarrow}{\rightarrow} \quad \text{(hc)}$$

A sequence (T_n) of commuting operators satisfying condition (C) but not (HC) was given in Example 7.

An example of commuting operators with a dense set of hypercyclic vectors but not satisfying condition (C) is the space $X = \mathbb{C}$ and $T_n(\lambda) = r_n\lambda$ $(\lambda \in \mathbb{C})$ where (r_n) is a dense sequence in $\mathbb{C}$.

The existence of a hypercyclic sequence of commuting operators with a nondense set of hypercyclic vectors is an open problem:

Problem 13. Let (T_n) be a hypercyclic sequence of mutually commuting operators acting on a Banach space X. Is the set of all vectors hypercyclic for (T_n) dense in X?

IV. Commuting chains of operators. The most important case of a sequence of operators is the sequence (T^n) of powers of a fixed operator $T \in B(X)$. Of importance are also sequences of the form $(\lambda_n T^n)$ where $T \in B(X)$ and λ_n are nonzero complex numbers. Hypercyclicity of these sequences is closely connected with the supercyclicity of the operator T. Indeed, an operator $T \in B(X)$ is supercyclic (i.e., there exists $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \ge 0\}$ is dense) if and only if there are complex numbers λ_n such that the sequence $(\lambda_n T^n)$ is hypercyclic. In this way problems concerning the supercyclicity of operators reduce to problems concerning hypercyclicity of sequences of operators.

It turns out that the most important property of sequences (T^n) or $(\lambda_n T^n)$ is that they form a chain of commuting operators. We call a sequence $(T_n) \subset B(X)$ a *chain of commuting operators* if there are mutually commuting operators $S_j \in B(X)$ such that $T_n = S_1 \cdots S_n$ for all n.

For chains of commuting operators the situation is even simpler. A hypercyclic chain always has a dense subset of hypercyclic vectors and condition (C) is equivalent to (HC).

Proposition 14. *Let $S_j \in B(X)$ $(j \in \mathbb{N})$ be mutually commuting operators and $T_n = S_1 \cdots S_n$. Suppose that the sequence (T_n) is hypercyclic. Then there exists a dense subset of vectors hypercyclic for (T_n).*

Proof. Let $x \in X$ be a vector hypercyclic for (T_n). Clearly $T_1 X \supset T_2 X \supset \cdots$, and so T_n has dense range for all n. We show that $T_j x$ is hypercyclic for all j. We have $\{T_n T_j x : n \in \mathbb{N}\}^- \supset T_j\{T_n x : n \in \mathbb{N}\}^- = T_j X$, which is a dense subset of X. Hence $T_j x$ is hypercyclic and the sequence (T_n) has a dense subset of hypercyclic vectors. ■

THEOREM 15. *Let $S_j : X \to X$ $(j \in \mathbb{N})$ be mutually commuting operators and let $T_n = S_1 \cdots S_n$. Suppose that the sequence (T_n) satisfies condition* $(\mathrm{C_{fin}})$*. Then it satisfies* (HC).

Proof. Since any subsequence of (T_n) is again a chain of commuting operators, without loss of generality we can assume that (T_n) satisfies condition $(\mathrm{C_{fin}})$ for the whole sequence (T_n), i.e., $T^n x \to 0$ for all x in a dense subset of X.

Note first that for all $k, j \in \mathbb{N}$ we have

$$\overline{\bigcup_{n>k} \big(S_{k+1} \cdots S_n B_X\big)^j} = Y^j. \tag{1}$$

Indeed, we have $T_k B_X \subset \|T_k\| B_X$, and so

$$\bigcup_{n>k} \big(S_{k+1} \cdots S_n B_X\big)^j \supset \|T_k\|^{-1} \bigcup_{n>k} \big(S_1 \cdots S_n B_X\big)^j = \|T_k\|^{-1} \bigcup_{n>k} (T_n B_X)^j,$$

which is dense in Y^j.

Let (x_k) be a sequence dense in X. By induction on j we construct an increasing sequence n_j and vectors $u_{k,j} \in X$ $(k, j \in \mathbb{N},\ j \ge k)$. Set formally $n_0 = 0$ and $u_{k,k} = x_k$.

Let $j \ge 2$ and suppose that n_{j-1} and $u_{k,j-1} \in X$ $(k \in \mathbb{N})$ have already been constructed. By (1), we can find $n_j > n_{j-1}$ and vectors $u_{k,j} \in X$ $(k = 1, \ldots, j-1)$ such that

$$\|S_{n_{j-1}+1} \cdots S_{n_j} u_{k,j} - u_{k,j-1}\| < \frac{1}{2^{k+j} \prod_{i \le n_{j-1}} \max\{1, \|S_i\|\}}$$

and $\|u_{k,j}\| < 1/2^{k+j}$, which completes the construction.

Write for short $R_j = S_{n_{j-1}+1} \cdots S_{n_j}$. Then

$$\|R_j u_{k,j} - u_{k,j-1}\| < \frac{1}{2^{k+j} \prod_{i \le j-1} \max\{1, \|R_i\|\}}$$

for all k, j, and $\|u_{k,j}\| < 2^{-(k+j)}$ $(k < j)$.

For fixed $k, j \in \mathbb{N}$ consider the sequence $(R_{j+1} \cdots R_m u_{k,m})_{m=j}^{\infty}$. Since

$$\begin{aligned} &\|R_{j+1} \cdots R_{m+1} u_{k,m+1} - R_{j+1} \cdots R_m u_{k,m}\| \\ &\qquad \le \|R_{j+1} \cdots R_m\| \cdot \|R_{m+1} u_{k,m+1} - u_{k,m}\| \le 1/2^{k+m+1}, \end{aligned}$$

the sequence $(R_{j+1} \cdots R_m u_{k,m})_{m=j}^{\infty}$ is Cauchy. Denote by $v_{k,j}$ its limit. For all k, j we have

$$R_{j+1} v_{k,j+1} = \lim_{m \to \infty} R_{j+1} R_{j+2} \cdots R_m u_{k,m} = v_{k,j}.$$

In particular, $T_{n_j} v_{k,j} = R_1 \cdots R_j v_{k,j} = v_{k,0}$ for all k, j. Furthermore,

$$\begin{aligned}\|v_{k,0} - x_k\| &= \lim_{m\to\infty} \|R_1 \cdots R_m u_{k,m} - u_{k,k}\| \\ &\le \sum_{m=k}^{\infty} \|R_1 \cdots R_{m+1} u_{k,m+1} - R_1 \cdots R_m u_{k,m}\| \\ &\le \sum_{m=0}^{\infty} \frac{1}{2^{k+m+1}} = \frac{1}{2^k},\end{aligned}$$

and so the sequence $(v_{k,0})$ is dense in X.

Finally, for $j > k$ we have

$$\begin{aligned}\|v_{k,j}\| &= \lim_{m\to\infty} \|R_{j+1} \cdots R_m u_{k,m}\| \\ &\le \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_{j+1} \cdots R_{m+1} u_{k,m+1} - R_{j+1} \cdots R_m u_{k,m}\| \\ &\le \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_1 \cdots R_m\| \cdot \|R_{m+1} u_{k,m+1} - u_{k,m}\| \\ &\le \frac{1}{2^{k+j}} + \sum_{m=j}^{\infty} \frac{1}{2^{k+m+1}} = \frac{1}{2^{k+j-1}},\end{aligned}$$

and so $\lim_{j\to\infty} \|v_{k,j}\| = 0$. Hence the sequence (T_n) satisfies condition (HC) for the sequence (n_j) and the dense set $\{v_{k,0} : k \in \mathbb{N}\}$. ∎

Corollary 16. *Let $T \in B(X)$ and let (λ_n) be a sequence of complex numbers. Then all the conditions* (C), ($\mathrm{C_{fin}}$), (HC), (her hc) *and* (4 nbhd) *are equivalent for the sequence $(\lambda_n T^n)$.*

If $(\lambda_n T^n)$ is hypercyclic then there is a dense subset of hypercyclic vectors.

Problem 17. Is there a chain of commuting operators (and in particular a sequence of the form (T^n)) which is hypercyclic but does not satisfy the hypercyclicity criterion (or any equivalent conditions)?

V. Subspaces of hypercyclic vectors. In this section we study the existence of a dense (closed infinite-dimensional, respectively) subspace consisting of hypercyclic vectors.

In the case of a hypercyclic sequence (T^n) where $T \in B(X)$ is a fixed operator it is known that there is always a dense subspace consisting of hypercyclic vectors. The proof, however, uses special properties of the sequence (T^n).

Our first result gives the existence of a dense subspace consisting of hypercyclic vectors for any sequence $(T_n) \subset B(X,Y)$ satisfying $\mathrm{C_{fin}}$.

THEOREM 18. *Let $(T_n) \subset B(X,Y)$ be a sequence of operators satisfying condition* $(\mathrm{C_{fin}})$. *Then there exists a dense subspace $X_1 \subset X$ such that each nonzero vector in X_1 is hypercyclic for (T_n).*

Proof. Let Z be any separable infinite-dimensional Banach space. Let $x \in X$, $x \neq 0$ and $\varepsilon > 0$. Set $\mathcal{M} = \bigl\{V \in \overline{F(Z,X)} : \operatorname{dist}\{x, VZ\} < \varepsilon\bigr\}$. Clearly $\mathcal{M}$ is open. We show that it is dense in $\overline{F(Z,X)}$.

Let $W \in \overline{F(Z,X)}$ and $\delta > 0$. Then there exists a finite rank operator $W_1 : Z \to X$ such that $\|W - W_1\| < \delta/2$. Let $z \in \ker W_1$ and $z^* \in Z^*$ satisfy $\langle z, z^*\rangle = 1$. Set

$$W_2 = W_1 + \frac{\delta \cdot (z^* \otimes x)}{2\|x\| \cdot \|z^*\|}.$$

Then

$$\|W - W_2\| \le \|W - W_1\| + \|W_1 - W_2\| < \delta \quad \text{and} \quad W_2 z = \frac{\delta x}{2\|x\| \cdot \|z^*\|}.$$

Thus $W_2 \in \mathcal{M}$ and $\mathcal{M}$ is dense in $\overline{F(Z,X)}$.

Let $(x_k) \subset X$ be a dense sequence of nonzero vectors. Clearly $V \in \overline{F(Z,X)}$ has dense range if and only if $\operatorname{dist}\{x_k, VZ\} < 1/k$ for all k. By the Baire category theorem, the set of all operators in $\overline{F(Z,X)}$ with dense range is residual.

By Theorem 5, the operators $L_{T_n} : \overline{F(Z,X)} \to \overline{F(Z,Y)}$ satisfy condition (C), and so there is a residual set of vectors hypercyclic for (L_{T_n}). Thus there exists an operator $V \in \overline{F(Z,X)}$ with dense range such that V is hypercyclic for (L_{T_n}).

It is easy to see that each nonzero vector in the range VZ is hypercyclic for the sequence (T_n). This completes the proof. ∎

Next we study the existence of a closed infinite-dimensional subspace consisting of hypercyclic vectors for a sequence $(T_n) \subset B(X,Y)$. Such a subspace is known to exist (under a natural additional assumption) if (T_n) is hereditarily hypercyclic. We prove it now for sequences satisfying the more practical condition (C). Moreover, the proof is essentially simplified.

Note that a particularly simple argument is available in the case of a sequence (T^n) satisfying the hypercyclicity criterion (HC) (see [ChT]).

We say for short that a subspace $X_1 \subset X$ is a *hypercyclic subspace for a sequence* $(T_n) \subset B(X,Y)$ if each nonzero vector in X_1 is hypercyclic for (T_n).

THEOREM 19 (cf. [Mo]). *Let $(T_n) \subset B(X,Y)$ be a sequence of operators. Suppose that (n_k) is an increasing sequence of positive integers such that*

(i) *there exists a dense subset $X_0 \subset X$ such that* $\lim_{k\to\infty} T_{n_k} x = 0$ $(x \in X_0)$;
(ii) $\overline{\bigcup_{k\in\mathbb{N}} T_{n_k} B_X} = Y$;

(iii) *there exists a closed infinite-dimensional subspace* $X_1 \subset X$ *with the property that* $\lim_{k\to\infty} T_{n_k}x = 0 \quad (x \in X_1)$.

Then there exists a closed infinite-dimensional hypercyclic subspace for (T_n).

Proof. Without loss of generality we can assume that $\lim_{n\to\infty} T_n x = 0$ for all $x \in X_0 \cup X_1$. Let $\{e_1, e_2, \dots\}$ be a normalized basic sequence in X_1. Let K be the corresponding basic constant and let $\varepsilon < 1/2K$. Let (y_k) be a dense sequence in Y. Let $\prec$ be an order on $\mathbb{N} \times (\mathbb{N} \cup \{0\})$ defined by $(i,j) \prec (i',j')$ if either $i+j < i'+j'$, or $i+j = i'+j'$ and $i < i'$.

Set $z_{i,0} = e_i$ $(i = 1, 2, \dots)$. By induction with respect to the order $\prec$ we construct vectors $z_{i,j} \in X_0$ $(i,j \in \mathbb{N})$ and an increasing sequence $n_{i,j} \subset \mathbb{N}$.

Let $(i,j) \in \mathbb{N} \times \mathbb{N}$ and suppose that $z_{i',j'} \in X_0$ and $n_{i',j'} \in \mathbb{N}$ have already been constructed for all $(i',j') \prec (i,j)$. By definition, there exist $n_{i,j} > \max\{n_{i',j'} : (i',j') \prec (i,j)\}$ and $z_{i,j} \in X_0$ such that

$$\|T_{n_{i,j}} z_{i',j'}\| < \frac{\varepsilon}{2^{i'+j'+i}} \quad ((i',j') \prec (i,j)), \qquad \|T_{n_{i,j}} z_{i,j} - y_i\| < \frac{\varepsilon}{2^{2i+j}},$$
$$\|z_{i,j}\| < \frac{\varepsilon}{2^{i+j} \max\{1, 2^{i'} \|T_{n_{i',j'}}\| : (i',j') \prec (i,j)\}}.$$

In this way, the vectors $z_{i,j} \in X_0$ and numbers $n_{i,j}$ are inductively constructed.

Set $z_i = \sum_{j=0}^{\infty} z_{i,j}$ $(i \in \mathbb{N})$. Then

$$\|z_i - e_i\| \le \sum_{j=1}^{\infty} \|z_{i,j}\| < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{i+j}} = \frac{\varepsilon}{2^i}.$$

Hence $\sum_{i=1}^{\infty} \|z_i - e_i\| < \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$, and so (z_i) is a basic sequence.

Let $M = \bigvee\{z_i : i = 1, 2, \dots\}$. Let $z \in M$ be any nonzero vector. Then $z = \sum_{i=1}^{\infty} \alpha_i z_i$ for some complex coefficients α_i. We show that z is hypercyclic for (T_n).

Fix $k \in \mathbb{N}$ with $\alpha_k \neq 0$. Since every nonzero scalar multiple of a hypercyclic vector is also hypercyclic, we can assume that $\alpha_k = 1$. Then

$$\begin{aligned}
\|T_{n_{k,r}} z - y_r\| &\le \sum_{i\neq k} |\alpha_i| \cdot \|T_{n_{k,r}} z_i\| + \|T_{n_{k,r}} z_k - y_r\| \\
&\le \sum_{i\neq k} \sum_{j=0}^{\infty} |\alpha_i| \cdot \|T_{n_{k,r}} z_{i,j}\| + \sum_{j\neq r} \|T_{n_{k,r}} z_{k,j}\| + \|T_{n_{k,r}} z_{k,r} - y_r\| \\
&\le \sum_{(i,j)\prec(k,r)} \max\{|\alpha_i| : i \in \mathbb{N}\} \cdot \|T_{n_{k,r}} z_{i,j}\|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{(k,r)\prec(i,j)} \max\{|\alpha_i| : i\in\mathbb{N}\}\cdot\|T_{n_{k,r}}z_{i,j}\| + \|T_{n_{k,r}}z_{k,r}-y_r\|\\
<&\sum_{(i,j)\prec(k,r)} \max\{|\alpha_i| : i\in\mathbb{N}\}\cdot\frac{\varepsilon}{2^{i+j+k}}\\
&+\sum_{(k,r)\prec(i,j)} \max\{|\alpha_i| : i\in\mathbb{N}\}\cdot\frac{\varepsilon}{2^{i+j+k}}+\frac{\varepsilon}{2^{2k+r}}\\
\le&\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\max\{|\alpha_i| : i\in\mathbb{N}\}\cdot\frac{\varepsilon}{2^{i+j+k}}\le\frac{K\varepsilon}{2^{k-1}}.
\end{aligned}
$$

Hence z is hypercyclic for (T_n). ∎

THEOREM 20. *Let $(T_n)\subset B(X,Y)$ be a sequence of operators satisfying condition* (C) *for a subsequence (n_k). Suppose that there are infinite-dimensional subspaces $M_1, M_2, \ldots$ such that $X\supset M_1\supset M_2\supset\cdots$ and* $\sup_k\|T_{n_k}|M_k\|<\infty$. *Then there exists a closed infinite-dimensional hypercyclic subspace for (T_n).*

Proof. Without loss of generality we can assume that $T_nx\to 0$ for all x in a dense subset $X_0\subset X$. It is sufficient to construct a closed infinite-dimensional subspace $X_1\subset X$ such that $T_nx\to 0$ $(x\in X_1)$.

We can find a basic sequence (x_n) such that $x_i\in M_i$ for all i. Let K be the basic constant of this sequence. Let $\varepsilon<1/2K$ be a positive number. For each n find $e_n\in X_0$ such that

$$
\|x_n-e_n\|<\frac{\varepsilon}{2^n\max\{1,\|T\|,\ldots,\|T^n\|\}}.
$$

Clearly (e_n) is a basic sequence with basic constant $\le 2K$. Let (y_n) be a dense sequence in Y. Choose a subsequence (e_{n_k}) such that $\|T_{n_k}e_{n_i}\|<2^{-(k+i)}$ $(i<k)$ and $\operatorname{dist}\{y_k, T_{n_k}B_X\}<2^{-k}$. Set $X_1=\bigvee\{e_{n_k} : k\in\mathbb{N}\}$. Let $e\in X_1$ be an arbitrary vector. We can write $e=\sum_{i=1}^{\infty}\alpha_ie_{n_i}$ for some complex coefficients α_i. We have

$$
\begin{aligned}
\|T_{n_k}e\|&\le\sum_{i=1}^{k-1}\|T_{n_k}\alpha_ie_{n_i}\|+\Big\|\sum_{i=k}^{\infty}T_{n_k}\alpha_ix_{n_i}\Big\|+\sum_{i=k}^{\infty}\|T_{n_k}\alpha_i(e_{n_i}-x_{n_i})\|\\
&\le 2K\sum_{i=1}^{k-1}\frac{1}{2^{i+k}}+\sup_n\|T_n|M_n\|\cdot\Big\|\sum_{i=k}^{\infty}\alpha_ix_{n_i}\Big\|+2K\sum_{i=k}^{\infty}\frac{\varepsilon}{2^i}\\
&\le\frac{K}{2^{k-1}}+\sup_n\|T_n|M_n\|\cdot\Big\|\sum_{i=k}^{\infty}\alpha_ix_{n_i}\Big\|+\frac{K\varepsilon}{2^{k-2}}\to 0
\end{aligned}
$$

as $k\to\infty$. Further $\overline{\bigcup_j T_{n_j}B_X}=Y$, and so there is a closed infinite-dimensional subspace consisting of hypercyclic vectors for (T_n). ∎

We now give a negative result—a condition implying that there is no closed infinite-dimensional subspace consisting of hypercyclic vectors.

Recall the quantity $j_\mu(T) = \sup\{j(T|M) : M \subset X,\ \operatorname{codim} M < \infty\}$, where j denotes the minimum modulus, $j(S) = \inf\{\|Sx\| : \|x\| \le 1\}$. The number $j_\mu(T)$ can be called the *essential minimum modulus* of T.

LEMMA 21. *Let $T_1, \ldots, T_k \in B(X, Y)$ and let $X_1 \subset X$ be a closed infinite-dimensional subspace. Let $\varepsilon > 0$. Then there exists $x \in X_1$ of norm one such that $\|T_i x\| > j_\mu(T_i) - \varepsilon$ $(i = 1, \ldots, k)$.*

Proof. For $i = 1, \ldots, k$ there is a subspace $M_i \subset X$ of finite codimension such that $j(T_i|M_i) > j_\mu(T_i) - \varepsilon$. Let x be any vector of norm one in $X_1 \cap \bigcap_{i=1}^k M_i$. Then

$$\|T_i x\| \ge j(T_i|M_i) > j_\mu(T_i) - \varepsilon$$

for all $i = 1, \ldots, k$. ∎

THEOREM 22. *Let X, Y be Banach spaces, let $(T_n) \subset B(X, Y)$ $(n = 1, 2, \ldots)$, let (a_n) be a sequence of positive numbers such that $\lim_{i\to\infty} a_i = 0$ and let $X_1 \subset X$ be a closed infinite-dimensional subspace. Let $\delta > 0$. Then there exists a vector $x \in X_1$ with $\|x\| \le \sup_i a_i + \delta$ and $\|T_n x\| \ge a_n j_\mu(T_n)$ for all $n \in \mathbb{N}$.*

Moreover, there is a subset X_2 dense in X_1 with the property that for each $x \in X_2$ there exists n_0 such that $\|T_n x\| \ge a_n j_\mu(T_n)$ $(n \ge n_0)$.

Proof. Without loss of generality we can assume that $a_1 \ge a_2 \ge \cdots$. Let $\varepsilon > 0$ satisfy $(1-\varepsilon)^2(a_1 + \delta/2) > a_1$. Find numbers $r_0 < r_1 < \cdots$ such that $a_{r_k} < (1-\varepsilon)^3 \delta/2^{k+3}$ for all k. Find $x_0 \in X_1$ such that $\|x_0\| = a_1 + \delta/2$ and $\|T_n x_0\| > (1-\varepsilon)(a_1 + \delta/2) j_\mu(T_n)$ $(n \le r_0)$.

Let $k \ge 0$ and suppose that $x_0, \ldots, x_k$ have already been constructed. Let $E_k = \bigvee\{T_n x_i : 0 \le i \le k,\ 1 \le n \le r_{k+1}\}$. Let M_k be a subspace of X of finite codimension such that

$$\|e + m\| \ge (1-\varepsilon)\max\{\|e\|, \|m\|/2\} \quad (e \in E_k, m \in M_k)$$

(see [M]). Since the space $L_k = \bigcap_{i=1}^k \bigcap_{n=1}^{r_{k+1}} T_n^{-1} M_i < \infty$ is of finite codimension, we can choose $x_{k+1} \in X_1 \cap L_k$ such that $\|x_{k+1}\| = \delta 2^{-(k+2)}$ and

$$\|T_n x_{k+1}\| \ge (1-\varepsilon)\delta 2^{-(k+2)} j_\mu(T_n) \quad (1 \le n \le r_{k+1}).$$

Set $x = \sum_{i=0}^\infty x_i$. Then $x \in X_1$ and

$$\|x\| \le \sum_{i=0}^\infty \|x_i\| \le a_1 + \delta/2 + \sum_{i=1}^\infty \delta 2^{-(i+1)} = a_1 + \delta.$$

For $n = 1, \ldots, r_0$ we have

$$\|T_n x\| = \Big\|T_n x_0 + \sum_{i=1}^\infty T_n x_i\Big\| \ge (1-\varepsilon)\|T_n x_0\| > a_1 j_\mu(T_n) \ge a_n j_\mu(T_n).$$

Let $k \ge 0$ and $r_k < n \le r_{k+1}$. Then

$$\begin{aligned}\|T_n x\| &= \Big\|\sum_{i=0}^{\infty} T_n x_i\Big\| \ge (1-\varepsilon)\Big\|\sum_{i=0}^{k+1} T_n x_i\Big\| \\ &\ge \frac{(1-\varepsilon)^2}{2}\,\|T_n x_{k+1}\| \ge \frac{(1-\varepsilon)^3}{2}\cdot\frac{\delta}{2^{k+2}}\, j_\mu(T_n) \ge a_n j_\mu(T_n).\end{aligned}$$

Thus $\|T_n x\| \ge a_n j_\mu(T_n)$ for all $n \in \mathbb{N}$.

To show the second statement, let $u \in X_1$ and $\varepsilon > 0$. Find n_0 such that $a_n < \varepsilon$ for all $n \ge n_0$. As in the first part, taking $x_0 = u$, construct a vector $x \in X_1$ with $\|x - u\| \le \varepsilon$ and $\|T_n x\| \ge a_n j_\mu(T_n)$ $(n \ge n_0)$. ∎

Corollary 23. *Let X, Y be Banach spaces and let $(T_n) \subset B(X,Y)$ be a sequence of operators satisfying $\lim_{n\to\infty} j_\mu(T_n) = \infty$. Then there is no closed infinite-dimensional hypercyclic subspace for (T_n).*

Proof. Let M be a closed infinite-dimensional subspace of X. By the previous result for the numbers $\alpha_n = (j_\mu(T_n))^{-1/2}$, there exists $x \in M$ such that $\|T_n x\| \to \infty$. Therefore x is not hypercyclic for (T_n). ∎

We apply the previous results to the sequences of the form $(\lambda_n T^n)$ where $T \in B(X)$ and λ_n are complex numbers. Denote by $\sigma_{\mathrm{e}}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not Fredholm$\}$ the essential spectrum of T.

Corollary 24. *Let $T \in B(X)$ be an operator and let (λ_n) be a sequence of complex numbers. Suppose that $(\lambda_n T^n)$ satisfies condition* (C) *and $\sup_n |\lambda_n| d^n < \infty$ where $d = \operatorname{dist}\{0, \sigma_{\mathrm{e}}(T)\}$. Then there exists a closed infinite-dimensional hypercyclic subspace for $(\lambda_n T^n)$.*

Proof. Since $(\lambda_n T^n)$ satisfies condition (C), the range of T is dense. Without loss of generality we can assume that the numbers λ_n are nonzero.

Choose $\lambda \in \sigma_{\mathrm{e}}(T)$ with $|\lambda| = d$. Thus $T - \lambda$ is not Fredholm. We show that $T - \lambda$ is not upper semi-Fredholm. This is clear if $d = 0$ since the range of T is dense. If $d > 0$ then $\lambda \in \partial\sigma_{\mathrm{e}}(T)$ and $T - \lambda$ is not upper semi-Fredholm by [HW].

By [LS], there is a compact operator $K \in B(X)$ with $\dim\ker(T - \lambda - K) = \infty$. Set $M_0 = \ker(T - \lambda - K)$. For each n we have $T^n = (T-K)^n + K_n$ for some compact operator K_n. Find subspaces $M_n' \subset X$ of finite codimension such that $\|K_n | M_n'\| \le |\lambda_n|^{-1}$. Set $M_n = M_0 \cap \bigcap_{i \le n} M_i'$. Then $M_1 \supset M_2 \supset \cdots$ and $\dim M_n < \infty$. For $z \in M_n$ with $\|z\| = 1$ we have

$$\|\lambda_n T^n z\| \le \|\lambda_n (T-K)^n z\| + \|\lambda_n K_n\| \le |\lambda_n \lambda^n| + 1 = |\lambda_n| d^n + 1,$$

and so $\sup_n \|\lambda_n T^n | M_n\| < \infty$. The statement now follows from Theorem 20. ∎

COROLLARY 25. *Let $T : X \to X$ and suppose that $(\lambda_n T^n)$ satisfies* (C) *and T is not Fredholm. Then there is an infinite-dimensional closed hypercyclic subspace for $(\lambda_n T^n)$.*

Proof. We have $d = \operatorname{dist}\{0, \sigma_{\mathrm{e}}(T)\} = 0$, and so the statement follows from the previous corollary. ∎

COROLLARY 26. *Let $T \in B(X)$ and suppose that (T^n) satisfies* (C). *The following conditions are equivalent*:

(i) *there exists a closed infinite-dimensional hypercyclic subspace for (T^n)*;
(ii) *the essential spectrum of T intersects the closed unit ball.*

Proof. Write $d = \operatorname{dist}\{0, \sigma_{\mathrm{e}}(T)\}$.

(ii)⇒(i). If $d \le 1$ then Corollary 24 implies (i).

(i)⇒(ii). Let $d > 1$. Then T is Fredholm. Recall the following standard construction from operator theory (see [S], [BHW]): let $\ell^\infty(X)$ be the space of all bounded sequences of elements of X; with the naturally defined algebraic operations and sup-norm it is a Banach space. Let $\widetilde{X} = \ell^\infty(X)/m(X)$ where $m(X)$ is the subspace of all precompact sequences. Let $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ be the operator induced by T. It is well known that $\widetilde{T}$ is invertible and $\sigma(\widetilde{T}) = \sigma_{\mathrm{e}}(T)$. By the spectral radius formula we have $d = \operatorname{dist}\{0, \sigma(\widetilde{T})\} = r(\widetilde{T}^{-1})^{-1} = \lim_{n\to\infty} \|\widetilde{T}^{-n}\|^{-1/n} = \lim_{n\to\infty} j(\widetilde{T}^n)^{1/n}$ where r denotes the spectral radius. By [F] we find that $j_\mu(T^n) \le 2j(\widetilde{T}^n) \le 4j_\mu(T^n)$ for all n. Thus $1 < d = \lim_{n\to\infty} j_\mu(T^n)^{1/n}$ and $\lim_{n\to\infty} j_\mu(T^n) = \infty$.

Let M be a closed infinite-dimensional subspace of X. By Theorem 22 for the numbers $\alpha_n = (j_\mu(T^n))^{-1/2}$, there exists $x \in M$ such that $\lim_{n\to\infty} \|T^n x\| = \infty$. Hence x is not hypercyclic for (T^n). ∎

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