Hypercyclic sequences of operators

by

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Abstract. A sequence \((T_n)\) of bounded linear operators between Banach spaces \(X, Y\) is said to be hypercyclic if there exists a vector \(x \in X\) such that the orbit \(\{T_n x\}\) is dense in \(Y\). The paper gives a survey of various conditions that imply the hypercyclicity of \((T_n)\) and studies relations among them. The particular case of \(X = Y\) and mutually commuting operators \(T_n\) is analyzed. This includes the most interesting cases \((T^n)\) and \((\lambda_n T^n)\) where \(T\) is a fixed operator and \(\lambda_n\) are complex numbers. We also study when a sequence of operators has a large (either dense or closed infinite-dimensional) manifold consisting of hypercyclic vectors.

I. Introduction. Let \(X\) and \(Y\) be separable Banach spaces. Denote by \(B(X, Y)\) the set of all bounded linear operators from \(X\) to \(Y\). Let \((T_n) \subset B(X, Y)\) be a sequence of operators. A vector \(x \in X\) is called hypercyclic for \((T_n)\) if the set \(\{T_n x\}\) is dense in \(Y\). The sequence \((T_n)\) is called hypercyclic if there is at least one vector hypercyclic for \((T_n)\). We say that an operator \(T : X \to X\) is hypercyclic if the sequence \((T^n)\) of its iterates is hypercyclic.

Similarly, an operator \(T\) is said to be supercyclic if there exists a vector \(x \in X\) such that the set \(\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}\) is dense; the vector \(x\) with this property is called supercyclic for \(T\).

Usually it is not easy to verify whether a sequence \((T_n)\) is hypercyclic or not. There are many criteria that have been studied by a number of authors that imply the hypercyclicity of \((T_n)\) (see e.g. [K], [GS], [BG], [BP]). In the second section we give a survey of various conditions implying the hypercyclicity and study relations among them. A number of illustrative examples are given.

The third section concentrates on the situation when \(Y = X\) and the operators \(T_n : X \to X\) are mutually commuting. The relations among vari-
ous conditions are much simpler in this case. The following section studies
the case when \( T_n = S_1 \cdots S_n \) where \( S_j : X \to X \) are mutually commuting.
This includes the most interesting cases \((T^n)\) and \((\lambda^n T^n)\) where \( T \) is a fixed
operator and \( \lambda_n \) complex numbers.

Sequences of operators with “many hypercyclic vectors” are very impor-
tant in hypercyclicity theory. The interest in them (especially in the cases
of \((T^n)\) and \((\lambda^n T^n)\)) arises from the invariant subspace/subset problem.

There are two research lines in the literature. The first one, which was
initiated by B. Beauzamy [Bea] and continued in [G], [GS], [He], [Bo], [B],
[BC] and recently [Gri], studies the existence of dense manifolds consisting
of hypercyclic vectors. The second, more recent, line studies the existence
of closed infinite-dimensional subspaces all of whose nonzero elements are
hypercyclic; see [Mo], [LMo], [GLM] and recently [BMP]. The questions of
this type are studied in Section V below.

II. Hypercyclicity of sequences of operators. Let \( X,Y \) be separ-
able Banach spaces and let \((T_n) \subset B(X,Y)\) be a sequence of operators. It
is well known that the set of all hypercyclic vectors for \((T_n)\) is a \( G_\delta \) set.
Indeed, \( x \in X \) is hypercyclic for \((T_n)\) if and only if \( x \in \bigcap U \bigcup_{n \in \mathbb{N}} T_n^{-1} U \),
where \( U \) runs over a countable base of open subsets of \( Y \); it is clear that
\( \bigcup_{n \in \mathbb{N}} T_n^{-1} U \) is open for each \( U \).

**Lemma 1 ([GS]).** Let \((T_n) \subset B(X,Y)\) be a sequence of operators. The
following conditions are equivalent:

(i) \((T_n)\) has a dense subset of hypercyclic vectors;
(ii) the set of all hypercyclic vectors for \((T_n)\) is residual (i.e., its com-
plement is of the first category);
(iii) for all nonempty open subsets \( U \subset X \), \( V \subset Y \) there exists \( n \in \mathbb{N} \)
such that \( T_n U \cap V \neq \emptyset \);
(iv) for all \( x \in X \), \( y \in Y \) and \( \varepsilon > 0 \) there exist \( n \in \mathbb{N} \) and \( u \in X \) such
that \( \|u - x\| < \varepsilon \) and \( \|T_n u - y\| < \varepsilon \).

Denote by \( B_X \) the closed unit ball in a Banach space \( X \). The most
practical criteria of hypercyclicity are the following two:

**Definition 2.** We say that a sequence \((T_n) \subset B(X,Y)\) satisfies condi-
tion \((C)\) if there exist an increasing sequence \((n_k)\) of positive integers and a
dense subset \( X_0 \subset X \) such that

(i) \( \lim_{k \to \infty} T_{n_k} x = 0 \) (\( x \in X_0 \));
(ii) \( \bigcup_k T_{n_k} B_X \) is dense in \( Y \).

The second condition is similar:
DEFINITION 3. We say that a sequence \((T_n) \subset B(X,Y)\) satisfies condition \((C_{\text{fin}})\) if there exist an increasing sequence \((n_k)\) of positive integers and a dense subset \(X_0 \subset X\) such that

(i) \(\lim_{k \to \infty} T_{n_k} x = 0 \ (x \in X_0);\)
(ii) \(\bigcup_k (T_{n_k} B_X \oplus \cdots \oplus T_{n_k} B_X)\) is dense in \(Y \oplus \cdots \oplus Y\) for all \(j \in \mathbb{N}\).

Clearly, condition \((C_{\text{fin}})\) implies \((C)\). Condition \((C)\) is the weakest known property which can be practically used to show the hypercyclicity of a sequence \((T_n)\). Moreover, it implies the existence of a dense (and hence residual) set of hypercyclic vectors. Furthermore, under a reasonable additional condition it implies that there is a closed infinite-dimensional subspace of hypercyclic vectors.

Condition \((C_{\text{fin}})\) has a number of equivalent formulations and it implies that there is a dense linear subspace consisting of hypercyclic vectors. The existence of subspaces consisting of hypercyclic vectors will be studied in Section V.

THEOREM 4. Let \((T_n) \subset B(X,Y)\) be a sequence of operators. The following conditions are equivalent:

(i) \((T_n)\) satisfies condition \((C)\);
(ii) for all \(j \in \mathbb{N}\) and nonempty open subsets \(U_0, U_1, \ldots, U_j \subset X\) and \(V_0, V \subset Y\) such that \(U_0\) and \(V_0\) contain the origins of \(X\) and \(Y\), respectively, there exists \(n \in \mathbb{N}\) such that \(T_n U_i \cap V_0 \neq \emptyset\) \((i = 1, \ldots, j)\) and \(T_n U_0 \cap V \neq \emptyset\).

In particular, if \((T_n)\) satisfies \((C)\) then there is a dense set of hypercyclic vectors for \((T_n)\).

Proof. (i)\(\Rightarrow\) (ii). Clear.

(ii)\(\Rightarrow\) (i). Let \((x_n) \subset X\) and \((y_n) \subset Y\) be dense sequences. Set \(u_{i,i} = x_i\) \((i \in \mathbb{N})\). By induction on \(k\) we construct an increasing sequence \((n_k)\) and vectors \(u_{i,k} \in X\) \((i = 1, \ldots, k - 1)\) and \(v_k \in B_X\) such that

\[
\|T_{n_k} v_k - y_k\| < 2^{-k}, \quad \|T_{n_k} u_{i,k}\| < 2^{-k},
\]

\[
\|u_{i,k} - u_{i,k-1}\| < \frac{1}{2k \max\{1, \|T_{n_1}\|, \ldots, \|T_{n_{k-1}}\|\}},
\]

for all \(i, k\) with \(1 \leq i < k\). For each \(i\) the sequence \((u_{i,k})_k\) is Cauchy. Let \(u_i\) be its limit. Then

\[
\|x_i - u_i\| \leq \sum_{k=i+1}^{\infty} \|u_{i,k} - u_{i,k-1}\| \leq \sum_{k=i+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^i}.
\]

Therefore \((u_i)\) is dense in \(X\).
Clearly the sequence \((T_{n_k} v_k)\) is dense, and so \(\bigcup_k T_{n_k} B_X = Y\). Further,

\[
\lim_{k \to \infty} \|T_{n_k} u_i\| \leq \lim_{k \to \infty} \left( \|T_{n_k} u_{i,k}\| + \sum_{j=k}^{\infty} \|T_{n_k}\| \cdot \|u_{i,j+1} - u_{i,j}\| \right)
\]

\[
\leq \lim_{k \to \infty} \left( \frac{1}{2^k} + \sum_{j=k}^{\infty} \frac{1}{2^j} \right) = \lim_{k \to \infty} \frac{1}{2^{k-1}} = 0
\]

for each \(i\). Thus \((T_n)\) satisfies (C).

To show that condition (C) implies the existence of a dense subset of hypercyclic vectors we use Lemma 1. Let \(x \in X, y \in Y\) and \(\varepsilon > 0\). By (ii), there are \(x_0, x_1 \in X\) and \(n \in \mathbb{N}\) such that \(\|x_0\| < \varepsilon/2\), \(\|T_n x_0 - y\| < \varepsilon/2\), \(\|x - x_1\| < \varepsilon/2\) and \(\|T_n x_1\| < \varepsilon/2\). Then \(\|(x_0 + x_1) - x\| < \varepsilon\) and \(\|T_n(x_0 + x_1) - y\| < \varepsilon\). By Lemma 1, \((T_n)\) has a dense subset of hypercyclic vectors.

**Theorem 5.** Let \((T_n) \subset B(X,Y)\) be a sequence of operators. The following conditions are equivalent:

(i) \((T_n)\) satisfies condition \((C_{\text{fin}})\);

(ii) \(\{T_n \oplus \cdots \oplus T_n\}_{j}\) satisfies condition (C) for all \(j \in \mathbb{N}\);

(iii) \(\{T_n \oplus \cdots \oplus T_n\}_{j}\) has a dense subset of hypercyclic vectors for all \(j \in \mathbb{N}\);

(iv) for all \(j \in \mathbb{N}\) and all nonempty open subsets \(U_1, \ldots, U_j \subset X\) and \(V_1, \ldots, V_j \subset Y\) there is an \(n \in \mathbb{N}\) such that \(T_n U_i \cap V_i \neq \emptyset\) \((i = 1, \ldots, j)\);

(v) there is a subsequence \((T_{n_k})\) such that each of its subsubsequences has a dense set of hypercyclic vectors;

(vi) there are dense subsets \(X_0 \subset X\) and \(Y_0 \subset Y\), an increasing sequence \((n_k) \subset \mathbb{N}\) and mappings \(S_i : Y_0 \to X\) \((i \in \mathbb{N})\) such that

\[
T_{n_k} x \to 0 \quad (x \in X_0),
\]

\[
S_k y \to 0 \quad (y \in Y_0),
\]

\[
T_{n_k} S_k y \to y \quad (y \in Y_0);
\]

(vii) for each Banach space \(Z\) the sequence of operators \(L_{T_n} : \overline{F(Z,X)} \to \overline{F(Z,Y)}\) defined by \(L_{T_n} S = T_n S\) \((S \in \overline{F(Z,X)})\) has a dense set of hypercyclic vectors; here \(F(Z,X)\) denotes the set of all finite rank operators from \(Z\) to \(X\);

(viii) for each Banach space \(Z\) the sequence \((L_{T_n})\) satisfies condition (C).

**Proof.** The equivalences \((vi) \Leftrightarrow (v) \Leftrightarrow (iii)\) were proved in [BG]. The implications \((i) \Rightarrow (ii)\) and \((vi) \Rightarrow (i)\) are obvious. The equivalence \((iii) \Leftrightarrow (iv)\) follows
from Lemma 1 and the implications (viii)⇒(vii) and (ii)⇒(iii) follow from
Theorem 4.

(i)⇒(viii). Let \( X_0 \) be a dense subset of \( X \) and let \((n_k)\) satisfy \( T_{n_k}x \to 0 \)
\((x \in X_0)\) and \( \bigcup (T_{n_k}B_X \oplus \cdots \oplus T_{n_k}B_X) = Y \oplus \cdots \oplus Y \).

Let \( \mathcal{M} \subset B(Z,X) \) be the set of all finite rank operators with range
included in the linear space generated by \( X_0 \). Clearly \( \mathcal{M} \) is dense in \( F(Z,X) \).
For \( G \in \mathcal{M} \) we have \( \lim_k L_{T_{n_k}} G = \lim_k T_{n_k} G = 0 \).

Let \( F \in F(Z,Y) \) and \( \varepsilon > 0 \). We can express \( F = \sum_{i=1}^{j} z_i^* \otimes y_i \) for some
\( y_i \in Y \) and \( z_i^* \in Z^* \). Since \((T_n)\) satisfies condition \( (C_{\text{fin}}) \), there are vectors
\( u_i \in X \ (i = 1, \ldots, j) \) and \( k \in \mathbb{N} \) such that
\[
\|T_{n_k}u_i - y_i\| < \frac{\varepsilon}{j \max\{\|z_1^*\|, \ldots, \|z_j^*\|\}}; \quad \|u_i\| \leq \frac{1}{j \max\{\|z_1^*\|, \ldots, \|z_j^*\|\}}.
\]
Set \( F_0 = \sum_{i=1}^{j} z_i^* \otimes u_i \in F(Z,X) \). Then \( \|F_0\| \leq \sum_{i=1}^{j} \|z_i^*\| \cdot \|u_i\| \leq 1 \) and
\[
\|L_{T_{n_k}} F_0 - F\| = \|T_{n_k} F_0 - F\| = \left\| \sum_{i=1}^{j} z_i^* \otimes T_{n_k} u_i - \sum_{i=1}^{j} z_i^* \otimes y_i \right\|
\]
\[
= \left\| \sum_{i=1}^{j} z_i^* \otimes (T_{n_k} u_i - y_i) \right\| \leq \sum_{i=1}^{j} \|z_i^*\| \cdot \|T_{n_k} u_i - y_i\| < \varepsilon.
\]

(vii)⇒(iii). Let \( j \in \mathbb{N} \) and let \( Z \) be a \( j \)-dimensional Banach space. Then
\( F(Z,X) \) is isomorphic to \( X \oplus \cdots \oplus X \) and \( F(Z,Y) \) to \( Y \oplus \cdots \oplus Y \). In the
same way \( L_{T_n} \) can be identified with \( T_n \oplus \cdots \oplus T_n \). \( \blacksquare \)

For completeness we also mention other conditions that have been studied
in the literature, for instance, in [BG, Theorem 2.2] (see also [BP, Remark
2.6(3)] in the case of commuting operators with dense range). In the diagram
below we show the relations among them. The abbreviations there mean:

(HC) (Hypercyclicity criterion) There exist dense subsets \( X_0 \subset X \)
and \( Y_0 \subset Y \), an increasing sequence \((n_k) \subset \mathbb{N} \) and mappings
\( S_k : Y_0 \to X \) such that
\[
T_{n_k}x \to 0 \quad (x \in X_0),
\]
\[
S_ky \to 0 \quad (y \in Y_0),
\]
\[
T_{n_k}S_ky = y \quad (y \in Y_0, k \in \mathbb{N}).
\]

(hc) \( (T_n) \) is hypercyclic.

dense hc \( (T_n) \) has a dense set of hypercyclic vectors.

(4 nbhd) (4 neighbourhoods condition) for all nonempty open subsets
\( U, U_0 \subset X \) and \( V, V_0 \subset Y \) such that \( U_0 \) and \( V_0 \) contain the
origins in \( X \) and \( Y \), respectively, there exists \( n \in \mathbb{N} \) such
that $T_n U \cap V_0 \neq \emptyset$ and $T_n U_0 \cap V_n \neq \emptyset$ (when $X = Y$ this condition reduces to “the three open sets condition”, which was introduced in [GS, Section III]).

(her hc) (Hereditarily hypercyclic) There is a subsequence $(T_{n_k})$ such that each of its subsequences is hypercyclic.

The relations among these conditions are given in the following diagram:

\[
\begin{array}{c}
(HC) \rightarrow (C_{\text{fin}}) \rightarrow (\text{her hc}) \rightarrow (hc) \\
\downarrow \\
(C) \rightarrow (4 \text{ nbhd}) \rightarrow (\text{dense hc})
\end{array}
\]

Moreover, there are no other implications among the conditions in question.

The implications $(HC) \rightarrow (C_{\text{fin}})$ and $(C_{\text{fin}}) \rightarrow (\text{her hc})$ were proved in Theorem 5, the implication $(C) \rightarrow (4 \text{ nbhd})$ in Theorem 4. For the implication $(4 \text{ nbhd}) \rightarrow (\text{dense hc})$ see Proposition 6 below.

The remaining implications are trivial.

The negative results follow from the examples below. Note that it is sufficient to show $(C_{\text{fin}}) \not\rightarrow (HC)$, $(C) \not\rightarrow (\text{her hc})$, $(\text{her hc}) \not\rightarrow (\text{dense hc})$ and $(\text{dense hc}) \not\rightarrow (C)$.

**Proposition 6 (cf. [GS]).** If $(T_n) \subset B(X,Y)$ satisfies $(4 \text{ nbhd})$ then there is a dense subset of hypercyclic vectors for $(T_n)$.

**Proof.** The result was essentially proved already in the proof of Theorem 4. Let $x \in X$, $y \in Y$ and $\varepsilon > 0$. Then there are $n \in \mathbb{N}$, $u \in X$ and $v \in Y$ such that $\|u - x\| < \varepsilon/2$, $\|T_n u\| < \varepsilon/2$, $\|v\| < \varepsilon/2$ and $\|T_n u - y\| < \varepsilon/2$. Set $x' = u + v$. Then $\|x - x'\| \leq \|x - u\| + \|v\| < \varepsilon$ and $\|T_n x' - y\| < \|T_n v - y\| + \|T_n u\| < \varepsilon$. By Lemma 1, this implies that $(T_n)$ has a dense subset of hypercyclic vectors.

**Example 7.** Let $X$ be a Hilbert space with an orthonormal basis $\{e_F : F \subset \mathbb{N}, \text{ card } F < \infty\}$. Let $Y = X$ and let the operators $T_n : X \rightarrow X$ be defined by

\[T_n e_F = \begin{cases} ne_F \setminus \{n\} & (n \in F), \\ 0 & (n \notin F).\end{cases}\]

It is easy to verify that the sequence $(T_n)$ satisfies condition $(C_{\text{fin}})$ for the dense subspace $X_0 \subset X$ generated by the vectors $\{e_F : F \subset \mathbb{N}\}$. However, $(T_n)$ does not satisfy $(HC)$ since the operators $T_n$ have nondense ranges, $T_n X = \bigvee\{e_F : n \notin F\}$.

Note that the operators $T_n$ are even commuting.

**Example 8.** Let $X$ and $T_n : X \rightarrow X$ be as in the previous example. Note that $e_{\{n\}} \perp T_n X$ for each $n$. Consider the operators $S_n : X \oplus \mathbb{C} \rightarrow X$
defined by
\[ S_n(x \oplus \lambda) = T_n x + \lambda e_{\{n\}} \quad (x \in X, \lambda \in \mathbb{C}). \]

Since the operators \((T_n)\) satisfy \((C_{\text{fin}})\) and hence are hereditarily hypercyclic, it is easy to see that the sequence \((S_n)\) is also hereditarily hypercyclic. On the other hand, the set of all vectors hypercyclic for \((S_n)\) is not dense. Indeed, let \(x \in X, \lambda \in \mathbb{C}, \lambda \neq 0\). Then
\[ \|S_n(x \oplus \lambda)\| = \|T_n x + \lambda e_{\{n\}}\| \geq |\lambda| > 0, \]
and so \(x \oplus \lambda\) is not hypercyclic. This shows that \((\text{her hc}) \not\implies (\text{dense hc})\).

**Example 9.** Let \(X\) be a separable Hilbert space with an orthonormal basis \(\{e_1, e_2, \ldots\}\). Let \(Y = \mathbb{C}^2\) and \((y_n)\) be a dense sequence of elements of \(Y\). Define \(T_n : X \to Y\) by \(T_n e_n = y_n\) and \(T_n e_i = 0\) \((i \neq n)\). Clearly \(T_n x \to 0\) for each \(x\) as is a finite linear combination of the vectors \(e_i\) \((i \in \mathbb{N})\). Further \(\bigcup_n T_n B_X \supset \{y_n : n \in \mathbb{N}\}^- = Y\). Thus \((T_n)\) satisfies condition \((C)\).

On the other hand, let \((n_k)\) be any increasing sequence of positive integers such that \((T_{n_k})\) is hypercyclic. Let \(U \subset Y\) be a nonempty open set such that \(\overline{U} \neq Y\) and \(\mathbb{C} \cdot U \subset U\). Choose a subsequence of those indices \(n_k\) for which \(y_{n_k} \in U\). For such an \(n_k\) we have \(T_{n_k} X = \mathbb{C} \cdot y_{n_k} \subset U\), and so \((T_n)\) is not hereditarily hypercyclic. Consequently, \((C) \not\implies (\text{her hc})\).

**Example 10.** Let \(\dim X = 1\) \((\text{i.e.}, X = \mathbb{C})\) and let \(Y\) be a separable Hilbert space. Let \((y_n)\) be a dense sequence in \(Y\). Define \(T_n : X \to Y\) by \(T_n(\lambda) = \lambda y_n\). Clearly each nonzero \(\lambda \in X\) is hypercyclic for \((T_n)\). It is easy to see that \((T_n)\) does not satisfy condition \((4 \text{ nbhd})\). Indeed, consider the neighbourhoods \(U = \{z \in \mathbb{C} : |z| > 2\}\), \(V_0 = \{y \in Y : \|y\| < 1\}\), \(U_0 = \{z \in \mathbb{C} : |z| < 1\}\) and \(V = \{y \in Y : \|y\| > 2\}\). Thus \((\text{dense hc}) \not\implies (4 \text{ nbhd})\).

**Example 11.** Let \(X = \mathbb{C}^2\) and \(\dim Y = \infty\). Let \((y_n)\) be a dense sequence in \(Y\) and \((x_n)\) dense in \(X\). For each \(n\) find \(u_n \in X\) linearly independent of \(x_n\) such that \(\|u_n\| = 1/n\). For \(m, n \in \mathbb{N}\) define \(T_{m,n} : X \to Y\) by \(T_{m,n} x_n = 0\) and \(T_{m,n} u_n = y_m\). Then \((T_{m,n})\) is a countable set of operators satisfying condition \((4 \text{ nbhd})\).

Let \((T_{m_k,n_k})\) be any subsequence such that \(T_{m_k,n_k} x \to 0\) for all \(x\) in a dense subset of \(X\). Then \(T_{m_k,n_k} \to 0\) in the strong operator topology, and therefore this subsequence is bounded by the Banach–Steinhaus theorem. Thus \((T_{m,n})\) does not satisfy \((C)\). Hence \((4 \text{ nbhd}) \not\implies (C)\).

**III. Sequences of commuting operators.** In this section we assume that \(Y = X\) and \((T_n) \subset B(X)\) is a sequence of mutually commuting operators. The situation is much simpler in this case.

**Theorem 12.** Let \((T_n) \subset B(X)\) be a sequence of mutually commuting operators. The following conditions are equivalent:
(i) \((T_n)\) satisfies condition (C);
(ii) \((T_n)\) satisfies condition \((C_{\text{fin}})\);
(iii) \((T_n)\) is hereditarily hypercyclic;
(iv) \((T_n)\) satisfies (4 nbhd); in fact in this case the 4 neighbourhoods condition reduces to the “3 neighbourhoods condition”: for all nonempty open subsets \(U, V, W \subset X\) with \(0 \in W\) there exists \(n\) such that 
\[T_n U \cap W \neq \emptyset \text{ and } T_n W \cap V \neq \emptyset.\]

**Proof.** (i)⇒(ii). Let \(X_0 \subset X\) be a dense subset and \((n_k) \subset \mathbb{N}\) an increasing sequence such that 
\[T_{n_k} x \to 0 \quad (x \in X_0) \text{ and } \bigcup_k T_{n_k} B_x = X.\]

By Theorem 4, \((T_{n_k})\) is hypercyclic. Let \(x \in X\) be a hypercyclic vector for \((T_{n_k})\).

Let \(y_1, \ldots, y_r \in X\) and \(\varepsilon > 0\). Since \(x\) is hypercyclic, there are \(k_1, \ldots, k_r\) such that 
\[\|T_{n_{k_i}} x - y_i\| < \varepsilon/2 \quad (i = 1, \ldots, r).\]

Further, there are \(u \in X\), 
\[\|u\| \leq \max\{\|T_{n_{k_i}}\| : i = 1, \ldots, r\}^{-1} \text{ and } s \in \mathbb{N}\] such that 
\[\|T_{n_{k_i}} u - x\| < \frac{\varepsilon}{2\max\{\|T_{n_{k_1}}\|, \ldots, \|T_{n_{k_r}}\|\}}.\]

Set \(x_i = T_{n_{k_i}} u\) \((i = 1, \ldots, r)\). Then \(x_i \in B_X\) and 
\[\|T_{n_{k_i}} x_i - y_i\| = \|T_{n_{k_i}} T_{n_{k_i}} u - y_i\| \leq \|T_{n_{k_i}} (T_{n_{k_i}} u - x)\| + \|T_{n_{k_i}} x - y_i\| < \varepsilon\]
for all \(i = 1, \ldots, r\).

(ii)⇒(iii). Clear.

(iii)⇒(iv). Let \(U, V, W \subset X\) be nonempty open sets, \(0 \in W\). Let \((n_k)\) be a sequence of positive integers such that each subsequence of \((T_{n_k})\) is hypercyclic. Let \(x\) be a hypercyclic vector for \((T_{n_k})\). Since each nonzero multiple of \(x\) is also hypercyclic, we can assume that \(x \in W\). Consider the subsequence \((T_{n_k})_{k \in F}\) where \(F = \{k : T_{n_k} x \in V\}\). Consequently, each \(k \in F\) satisfies \(T_{n_k} W \cap V \neq \emptyset\).

Let \(y\) be a vector hypercyclic for this subsequence. Thus there exists \(k_0 \in F\) such that \(T_{n_{k_0}} y \in U\). Moreover, we can choose increasing numbers \(k_i \in F\) such that \(T_{n_{k_i}} y \to 0 \quad (i \to \infty)\). Thus 
\[\lim_{i \to \infty} T_{n_{k_i}} T_{n_{k_0}} y = \lim_{i \to \infty} T_{n_{k_0}} T_{n_{k_i}} y = 0\]
and there is an \(i\) with \(T_{n_{k_i}} T_{n_{k_0}} y \in W\). Hence \(T_{n_{k_i}} U \cap W \neq \emptyset\).

(iv)⇒(i). By Proposition 6, the sequence \((T_n)\) has a dense subset of hypercyclic vectors.

Let \(U_1, \ldots, U_r, V, W \subset X\) be nonempty open subsets, \(0 \in W\). Let \(x\) be a hypercyclic vector for the sequence \((T_n)\). Find \(n_1, \ldots, n_r \in \mathbb{N}\) such that 
\[T_{n_i} x \in U_i \quad (i = 1, \ldots, r).\]
Let \(\varepsilon > 0\) satisfy \(\{y : \|y - T_{n_i} x\| < \varepsilon\} \subset U_i\) \((i = 1, \ldots, r)\) and \(\{y : \|y\| < \varepsilon\} \subset W\). By assumption, there are \(x' \in X\) and \(n_0 \in \mathbb{N}\) such that 
\[\|x' - x\| < \varepsilon \max\{\|T_{n_1}\|, \ldots, \|T_{n_r}\|\}^{-1}, \|T_{n_0} x'\| < \varepsilon \max\{\|T_{n_1}\|, \ldots, \|T_{n_r}\|\}^{-1}\]
and \(T_{n_0} W \cap V \neq \emptyset\). Then 
\[\|T_{n_i} x' - T_{n_i} x\| \leq \varepsilon \max\{\|T_{n_1}\|, \ldots, \|T_{n_r}\|\}^{-1}.\]
\[ \|T_n\| \cdot \|x' - x\| < \varepsilon, \text{ and so } T_n x' \in U_i \quad (i = 1, \ldots, r). \]

Further \[ \|T_{n_0} T_n x'\| = \|T_{n_0} T_n x'\| \leq \|T_{n_0}\| \cdot \|T_n x'\| < \varepsilon, \text{ and so } T_{n_0} T_n x' \in W \].

Thus for commuting operators \( T_n : X \rightarrow X \) we have the following situation:

\[
\begin{array}{c}
(\text{HC}) \quad \leftrightarrow \quad (\text{C}) \quad \leftrightarrow \quad (\text{dense hc}) \quad \leftrightarrow \quad (\text{hc})
\end{array}
\]

A sequence \((T_n)\) of commuting operators satisfying condition (C) but not (HC) was given in Example 7.

An example of commuting operators with a dense set of hypercyclic vectors but not satisfying condition (C) is the space \( X = \mathbb{C} \) and \( T_n(\lambda) = r_n \lambda \) \((\lambda \in \mathbb{C})\) where \((r_n)\) is a dense sequence in \( \mathbb{C} \).

The existence of a hypercyclic sequence of commuting operators with a nondense set of hypercyclic vectors is an open problem:

**Problem 13.** Let \((T_n)\) be a hypercyclic sequence of mutually commuting operators acting on a Banach space \( X \). Is the set of all vectors hypercyclic for \((T_n)\) dense in \( X \)?

**IV. Commuting chains of operators.** The most important case of a sequence of operators is the sequence \((T^n)\) of powers of a fixed operator \( T \in B(X) \). Of importance are also sequences of the form \((\lambda_n T^n)\) where \( T \in B(X) \) and \( \lambda_n \) are nonzero complex numbers. Hypercyclicity of these sequences is closely connected with the supercyclicity of the operator \( T \). Indeed, an operator \( T \in B(X) \) is supercyclic (i.e., there exists \( x \in X \) such that the set \( \{\lambda T^n x : \lambda \in \mathbb{C}, n \geq 0\} \) is dense) if and only if there are complex numbers \( \lambda_n \) such that the sequence \((\lambda_n T^n)\) is hypercyclic. In this way problems concerning the supercyclicity of operators reduce to problems concerning hypercyclicity of sequences of operators.

It turns out that the most important property of sequences \((T^n)\) or \((\lambda_n T^n)\) is that they form a chain of commuting operators. We call a sequence \((T_n) \subset B(X)\) a chain of commuting operators if there are mutually commuting operators \( S_j \in B(X) \) such that \( T_n = S_1 \cdots S_n \) for all \( n \).

For chains of commuting operators the situation is even simpler. A hypercyclic chain always has a dense subset of hypercyclic vectors and condition (C) is equivalent to (HC).

**Proposition 14.** Let \( S_j \in B(X) \quad (j \in \mathbb{N}) \) be mutually commuting operators and \( T_n = S_1 \cdots S_n \). Suppose that the sequence \((T_n)\) is hypercyclic. Then there exists a dense subset of vectors hypercyclic for \((T_n)\).

**Proof.** Let \( x \in X \) be a vector hypercyclic for \((T_n)\). Clearly \( T_1 X \supset T_2 X \supset \cdots \), and so \( T_n \) has dense range for all \( n \). We show that \( T_j x \) is hypercyclic for all \( j \). We have \( \{T_n T_j x : n \in \mathbb{N}\} \supset \supset T_j \{T_n x : n \in \mathbb{N}\} = T_j X \), which is
a dense subset of $X$. Hence $T_j x$ is hypercyclic and the sequence $(T_n)$ has a dense subset of hypercyclic vectors. ■

**Theorem 15.** Let $S_j : X \rightarrow X \ (j \in \mathbb{N})$ be mutually commuting operators and let $T_n = S_1 \cdots S_n$. Suppose that the sequence $(T_n)$ satisfies condition $C_{\text{fin}}$. Then it satisfies $(HC)$.

**Proof.** Since any subsequence of $(T_n)$ is again a chain of commuting operators, without loss of generality we can assume that $(T_n)$ satisfies condition $C_{\text{fin}}$ for the whole sequence $(T_n)$, i.e., $T^n x \rightarrow 0$ for all $x$ in a dense subset of $X$.

Note first that for all $k, j \in \mathbb{N}$ we have

$$\bigcup_{n > k} (S_{k+1} \cdots S_n B_X)^j = Y^j.$$  

Indeed, we have $T_k B_X \subset \|T_k\| B_X$, and so

$$\bigcup_{n > k} (S_{k+1} \cdots S_n B_X)^j \supset \|T_k\|^{-1} \bigcup_{n > k} (S_1 \cdots S_n B_X)^j = \|T_k\|^{-1} \bigcup_{n > k} (T_n B_X)^j,$$

which is dense in $Y^j$.

Let $(x_k)$ be a sequence dense in $X$. By induction on $j$ we construct an increasing sequence $n_j$ and vectors $u_{k,j} \in X \ (k, j \in \mathbb{N}, j \geq k)$. Set formally $n_0 = 0$ and $u_{k,k} = x_k$.

Let $j \geq 2$ and suppose that $n_{j-1}$ and $u_{k,j-1} \in X \ (k \in \mathbb{N})$ have already been constructed. By (1), we can find $n_j > n_{j-1}$ and vectors $u_{k,j} \in X \ (k = 1, \ldots, j - 1)$ such that

$$\|S_{n_{j-1}+1} \cdots S_{n_j} u_{k,j} - u_{k,j-1}\| < \frac{1}{2^{k+j} \prod_{i \leq n_{j-1}} \max\{1, \|S_i\|\}}$$

and $\|u_{k,j}\| < 1/2^{k+j}$, which completes the construction.

Write for short $R_j = S_{n_{j-1}+1} \cdots S_{n_j}$. Then

$$\|R_j u_{k,j} - u_{k,j-1}\| < \frac{1}{2^{k+j} \prod_{i \leq j-1} \max\{1, \|R_i\|\}}$$

for all $k, j$, and $\|u_{k,j}\| < 2^{-(k+j)} \ (k < j)$.

For fixed $k, j \in \mathbb{N}$ consider the sequence $(R_{j+1} \cdots R_{m} u_{k,m})_{m=j}^{\infty}$. Since

$$\|R_{j+1} \cdots R_{m+1} u_{k,m+1} - R_{j+1} \cdots R_{m} u_{k,m}\| \leq \|R_{j+1} \cdots R_{m}\| \cdot \|R_{m+1} u_{k,m+1} - u_{k,m}\| \leq 1/2^{k+m+1},$$

the sequence $(R_{j+1} \cdots R_{m} u_{k,m})_{m=j}^{\infty}$ is Cauchy. Denote by $v_{k,j}$ its limit. For all $k, j$ we have

$$R_{j+1} v_{k,j+1} = \lim_{m \rightarrow \infty} R_{j+1} R_{j+2} \cdots R_{m} u_{k,m} = v_{k,j}.$$
In particular, \( T_n v_{k,j} = R_1 \cdots R_j v_{k,j} = v_{k,0} \) for all \( k, j \). Furthermore,

\[
\|v_{k,0} - x_k\| = \lim_{m \to \infty} \|R_1 \cdots R_m u_{k,m} - u_{k,k}\| \\
\leq \sum_{m=k}^{\infty} \|R_1 \cdots R_{m+1} u_{k,m+1} - R_1 \cdots R_m u_{k,m}\| \\
\leq \sum_{m=0}^{\infty} \frac{1}{2k+m+1} = \frac{1}{2k},
\]

and so the sequence \((v_{k,0})\) is dense in \( X \).

Finally, for \( j > k \) we have

\[
\|v_{k,j}\| = \lim_{m \to \infty} \|R_{j+1} \cdots R_m u_{k,m}\| \\
\leq \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_{j+1} \cdots R_{m+1} u_{k,m+1} - R_{j+1} \cdots R_m u_{k,m}\| \\
\leq \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_1 \cdots R_m\| \cdot \|R_{m+1} u_{k,m+1} - u_{k,m}\| \\
\leq \frac{1}{2k+j} + \sum_{m=j}^{\infty} \frac{1}{2k+m+1} = \frac{1}{2k+j-1},
\]

and so \( \lim_{j \to \infty} \|v_{k,j}\| = 0 \). Hence the sequence \((T_n)\) satisfies condition (HC) for the sequence \((n_j)\) and the dense set \( \{v_{k,0} : k \in \mathbb{N}\} \).

**Corollary 16.** Let \( T \in B(X) \) and let \((\lambda_n)\) be a sequence of complex numbers. Then all the conditions \((C), (C_{\text{fin}}), (HC), (\text{her } hc)\) and \((4 \text{ nbhd})\) are equivalent for the sequence \((\lambda_n T^n)\).

If \((\lambda_n T^n)\) is hypercyclic then there is a dense subset of hypercyclic vectors.

**Problem 17.** Is there a chain of commuting operators (and in particular a sequence of the form \((T^n)\)) which is hypercyclic but does not satisfy the hypercyclicity criterion (or any equivalent conditions)?

**V. Subspaces of hypercyclic vectors.** In this section we study the existence of a dense (closed infinite-dimensional, respectively) subspace consisting of hypercyclic vectors.

In the case of a hypercyclic sequence \((T^n)\) where \( T \in B(X) \) is a fixed operator it is known that there is always a dense subspace consisting of hypercyclic vectors. The proof, however, uses special properties of the sequence \((T^n)\).

Our first result gives the existence of a dense subspace consisting of hypercyclic vectors for any sequence \((T_n) \subset B(X,Y)\) satisfying \( C_{\text{fin}} \).
Theorem 18. Let \((T_n) \subset B(X, Y)\) be a sequence of operators satisfying condition \((C_{\text{fin}})\). Then there exists a dense subspace \(X_1 \subset X\) such that each nonzero vector in \(X_1\) is hypercyclic for \((T_n)\).

Proof. Let \(Z\) be any separable infinite-dimensional Banach space. Let \(x \in X, x \neq 0\) and \(\varepsilon > 0\). Set \(\mathcal{M} = \{V \in F(Z, X) : \text{dist}\{x, VZ\} < \varepsilon\}\). Clearly \(\mathcal{M}\) is open. We show that it is dense in \(F(Z, X)\).

Let \(W \in F(Z, X)\) and \(\delta > 0\). Then there exists a finite rank operator \(W_1 : Z \rightarrow X\) such that \(\|W - W_1\| < \delta/2\). Let \(z \in \ker W_1\) and \(z^* \in Z^*\) satisfy \(\langle z, z^* \rangle = 1\). Set
\[
W_2 = W_1 + \frac{\delta \cdot (z^* \otimes x)}{2\|x\| \cdot \|z^*\|}.
\]

Then
\[
\|W - W_2\| \leq \|W - W_1\| + \|W_1 - W_2\| < \delta \quad \text{and} \quad W_2z = \frac{\delta x}{2\|x\| \cdot \|z^*\|}.
\]

Thus \(W_2 \in \mathcal{M}\) and \(\mathcal{M}\) is dense in \(F(Z, X)\).

Let \((x_k) \subset X\) be a dense sequence of nonzero vectors. Clearly \(V \in F(Z, X)\) has dense range if and only if \(\text{dist}\{x_k, VZ\} < 1/k\) for all \(k\). By the Baire category theorem, the set of all operators in \(F(Z, X)\) with dense range is residual.

By Theorem 5, the operators \(L_{T_n} : F(Z, X) \rightarrow F(Z, Y)\) satisfy condition \((C)\), and so there is a residual set of vectors hypercyclic for \((L_{T_n})\). Thus there exists an operator \(V \in F(Z, X)\) with dense range such that \(V\) is hypercyclic for \((L_{T_n})\).

It is easy to see that each nonzero vector in the range \(VZ\) is hypercyclic for the sequence \((T_n)\). This completes the proof. \(\blacksquare\)

Next we study the existence of a closed infinite-dimensional subspace consisting of hypercyclic vectors for a sequence \((T_n) \subset B(X, Y)\). Such a subspace is known to exist (under a natural additional assumption) if \((T_n)\) is hereditarily hypercyclic. We prove it now for sequences satisfying the more practical condition \((C)\). Moreover, the proof is essentially simplified.

Note that a particularly simple argument is available in the case of a sequence \((T^n)\) satisfying the hypercyclicity criterion \((HC)\) (see [ChT]).

We say for short that a subspace \(X_1 \subset X\) is a hypercyclic subspace for a sequence \((T_n) \subset B(X, Y)\) if each nonzero vector in \(X_1\) is hypercyclic for \((T_n)\).

Theorem 19 (cf. [Mo]). Let \((T_n) \subset B(X, Y)\) be a sequence of operators. Suppose that \((n_k)\) is an increasing sequence of positive integers such that
\begin{itemize}
  \item[(i)] there exists a dense subset \(X_0 \subset X\) such that \(\lim_{k \rightarrow \infty} T_{n_k}x = 0\) \((x \in X_0)\);
  \item[(ii)] \(\bigcup_{k \in \mathbb{N}} T_{n_k}B_X = Y\);
\end{itemize}
(iii) there exists a closed infinite-dimensional subspace $X_1 \subset X$ with the property that $\lim_{k \to \infty} T_{n_k} x = 0 \ (x \in X_1)$.

Then there exists a closed infinite-dimensional hypercyclic subspace for $(T_n)$. 

Proof. Without loss of generality we can assume that $\lim_{n \to \infty} T_n x = 0$ for all $x \in X_0 \cup X_1$. Let $\{e_1,e_2,\ldots\}$ be a normalized basic sequence in $X_1$. Let $K$ be the corresponding basic constant and let $\varepsilon < 1/2K$. Let $(y_k)$ be a dense sequence in $Y$. Let $\prec$ be an order on $\mathbb{N} \times (\mathbb{N} \cup \{0\})$ defined by $(i,j) \prec (i',j')$ if either $i + j < i' + j'$, or $i + j = i' + j'$ and $i < i'$.

Set $z_{i,0} = e_i \ (i = 1,2,\ldots)$. By induction with respect to the order $\prec$ we construct vectors $z_{i,j} \in X_0 \ (i,j \in \mathbb{N})$ and an increasing sequence $n_{i,j} \subset \mathbb{N}$.

Let $(i,j) \in \mathbb{N} \times \mathbb{N}$ and suppose that $z_{i',j'} \in X_0$ and $n_{i',j'} \in \mathbb{N}$ have already been constructed for all $(i',j') \prec (i,j)$. By definition, there exist $n_{i,j} > \max\{n_{i',j'} : (i',j') \prec (i,j)\}$ and $z_{i,j} \in X_0$ such that

$$\|T_{n_{i,j}} z_{i',j'}\| < \frac{\varepsilon}{2^{i'+j'+i}} \ ( (i',j') \prec (i,j)), \quad \|T_{n_{i,j}} z_{i,j} - y_i\| < \frac{\varepsilon}{2^{i+j}},$$

$$\|z_{i,j}\| < \frac{\varepsilon}{2^{i+j} \max\{1,2^{\nu}\|T_{n_{i',j'}}\| : (i',j') \prec (i,j)\}}.$$

In this way, the vectors $z_{i,j} \in X_0$ and numbers $n_{i,j}$ are inductively constructed.

Set $z_i = \sum_{j=0}^{\infty} z_{i,j} \ (i \in \mathbb{N})$. Then

$$\|z_i - e_i\| \leq \sum_{j=1}^{\infty} \|z_{i,j}\| < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{i+j}} = \frac{\varepsilon}{2^i}.$$

Hence $\sum_{i=1}^{\infty} \|z_i - e_i\| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$, and so $(z_i)$ is a basic sequence.

Let $M = \bigvee\{z_i : i = 1,2,\ldots\}$. Let $z \in M$ be any nonzero vector. Then $z = \sum_{i=1}^{\infty} \alpha_i z_i$ for some complex coefficients $\alpha_i$. We show that $z$ is hypercyclic for $(T_n)$.

Fix $k \in \mathbb{N}$ with $\alpha_k \neq 0$. Since every nonzero scalar multiple of a hypercyclic vector is also hypercyclic, we can assume that $\alpha_k = 1$. Then

$$\|T_{n_{k,r}} z - y_r\| \leq \sum_{i \neq k} |\alpha_i| \cdot \|T_{n_{k,r}} z_i\| + \|T_{n_{k,r}} z_k - y_r\|$$

$$\leq \sum_{i \neq k} \sum_{j=0}^{\infty} |\alpha_i| \cdot \|T_{n_{k,r}} z_{i,j}\| + \sum_{j \neq r} \|T_{n_{k,r}} z_{k,j}\| + \|T_{n_{k,r}} z_k - y_r\|$$

$$\leq \sum_{(i,j) \prec (k,r)} \max\{|\alpha_i| : i \in \mathbb{N}\} \cdot \|T_{n_{k,r}} z_{i,j}\|$$
+ \sum_{(k,r) \prec (i,j)} \max\{ |\alpha_i| : i \in \mathbb{N} \} \cdot \|T_{nk,r} z_{i,j}\| + \|T_{nk,r} z_{k,r} - y_r\|

< \sum_{(i,j) \prec (k,r)} \max\{ |\alpha_i| : i \in \mathbb{N} \} \cdot \frac{\varepsilon}{2i+j+k}

+ \sum_{(k,r) \prec (i,j)} \max\{ |\alpha_i| : i \in \mathbb{N} \} \cdot \frac{\varepsilon}{2i+j+k} + \frac{\varepsilon}{22k+r}

\leq \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \max\{ |\alpha_i| : i \in \mathbb{N} \} \cdot \frac{\varepsilon}{2i+j+k} \leq \frac{K\varepsilon}{2^{k-1}}.

Hence \( z \) is hypercyclic for \( (T_n) \). ■

**Theorem 20.** Let \( (T_n) \subset B(X,Y) \) be a sequence of operators satisfying condition (C) for a subsequence \((n_k)\). Suppose that there are infinite-dimensional subspaces \( M_1, M_2, \ldots \) such that \( X \supset M_1 \supset M_2 \supset \cdots \) and \( \sup_k \|T_{nk}|M_k\| < \infty \). Then there exists a closed infinite-dimensional hypercyclic subspace for \( (T_n) \).

**Proof.** Without loss of generality we can assume that \( T_n x \to 0 \) for all \( x \) in a dense subset \( X_0 \subset X \). It is sufficient to construct a closed infinite-dimensional subspace \( X_1 \subset X \) such that \( T_n x \to 0 \) \((x \in X_1)\).

We can find a basic sequence \( (x_n) \) such that \( x_i \in M_i \) for all \( i \). Let \( K \) be the basic constant of this sequence. Let \( \varepsilon < 1/2K \) be a positive number. For each \( n \) find \( e_n \in X_0 \) such that

\[ \|x_n - e_n\| < \frac{\varepsilon}{2^n \max\{1,\|T\|, \ldots, \|T^n\}\}}. \]

Clearly \( (e_n) \) is a basic sequence with basic constant \( \leq 2K \). Let \( (y_n) \) be a dense sequence in \( Y \). Choose a subsequence \( (e_{nk}) \) such that \( \|T_{nk} e_{nk}\| < 2^{-(k+i)} \) \((i < k)\) and \( \text{dist}\{y_k, T_{nk} B_X\} < 2^{-k} \). Set \( X_1 = \bigvee \{e_{nk} : k \in \mathbb{N}\} \). Let \( e \in X_1 \) be an arbitrary vector. We can write \( e = \sum_{i=1}^{\infty} \alpha_i e_{ni} \) for some complex coefficients \( \alpha_i \). We have

\[ \|T_{nk} e\| \leq \sum_{i=1}^{k-1} \|T_{nk} \alpha_i e_{ni}\| + \sum_{i=k}^{\infty} \|T_{nk} \alpha_i e_{ni}\| + \sum_{i=k}^{\infty} \|T_{nk} \alpha_i (e_{ni} - x_{ni})\| \]

\[ \leq 2K \sum_{i=1}^{k-1} \frac{1}{2^{i+k}} + \sup_n \|T_n|M_n\| \cdot \left\| \sum_{i=k}^{\infty} \alpha_i x_{ni} \right\| + 2K \sum_{i=k}^{\infty} \frac{\varepsilon}{2^i} \]

\[ \leq \frac{K}{2^{k-1}} + \sup_n \|T_n|M_n\| \cdot \left\| \sum_{i=k}^{\infty} \alpha_i x_{ni} \right\| + \frac{K\varepsilon}{2^{k-2}} \to 0 \]

as \( k \to \infty \). Further \( \bigcup_j T_{nj} B_X = Y \), and so there is a closed infinite-dimensional subspace consisting of hypercyclic vectors for \( (T_n) \). ■
We now give a negative result—a condition implying that there is no closed infinite-dimensional subspace consisting of hypercyclic vectors.

Recall the quantity \( j_\mu(T) = \sup \{ j(T|M) : M \subset X, \text{ codim } M < \infty \} \), where \( j \) denotes the minimum modulus, \( j(S) = \inf \{ \|Sx\| : \|x\| \leq 1 \} \). The number \( j_\mu(T) \) can be called the essential minimum modulus of \( T \).

**Lemma 21.** Let \( T_1, \ldots, T_k \in B(X, Y) \) and let \( X_1 \subset X \) be a closed infinite-dimensional subspace. Let \( \varepsilon > 0 \). Then there exists \( x \in X_1 \) of norm one such that \( \|T_ix\| > j_\mu(T_i) - \varepsilon \) (\( i = 1, \ldots, k \)).

**Proof.** For \( i = 1, \ldots, k \) there is a subspace \( M_i \subset X \) of finite codimension such that \( j(T_i|M_i) > j_\mu(T_i) - \varepsilon \). Let \( x \) be any vector of norm one in \( X_1 \cap \bigcap_{i=1}^k M_i \). Then

\[
\|T_ix\| \geq j(T_i|M_i) > j_\mu(T_i) - \varepsilon
\]

for all \( i = 1, \ldots, k \). \( \blacksquare \)

**Theorem 22.** Let \( X, Y \) be Banach spaces, let \( (T_n) \subset B(X, Y) \) \((n = 1, 2, \ldots)\), let \( (a_n) \) be a sequence of positive numbers such that \( \lim_{i \to \infty} a_i = 0 \) and let \( X_1 \subset X \) be a closed infinite-dimensional subspace. Let \( \delta > 0 \). Then there exists a vector \( x \in X_1 \) with \( \|x\| \leq \sup_i a_i + \delta \) and \( \|T_nx\| \geq a_n j_\mu(T_n) \) for all \( n \in \mathbb{N} \).

Moreover, there is a subset \( X_2 \) dense in \( X_1 \) with the property that for each \( x \in X_2 \) there exists \( n_0 \) such that \( \|T_nx\| \geq a_n j_\mu(T_n) \) \((n \geq n_0)\).

**Proof.** Without loss of generality we can assume that \( a_1 \geq a_2 \geq \cdots \). Let \( \varepsilon > 0 \) satisfy \((1 - \varepsilon)^2(a_1 + \delta/2) > a_1\). Find numbers \( r_0 < r_1 < \cdots \) such that \( a_{r_k} < (1 - \varepsilon)^3 \delta/2^{k+3} \) for all \( k \). Find \( x_0 \in X_1 \) such that \( \|x_0\| = a_1 + \delta/2 \) and \( \|T_nx_0\| > (1 - \varepsilon)(a_1 + \delta/2) j_\mu(T_n) \) \((n \leq r_0)\).

Let \( k \geq 0 \) and suppose that \( x_0, \ldots, x_k \) have already been constructed. Let \( E_k = \bigvee \{ T_n x_i : 0 \leq i \leq k, 1 \leq n \leq r_{k+1} \} \). Let \( M_k \) be a subspace of \( X \) of finite codimension such that

\[
\|e + m\| \geq (1 - \varepsilon) \max \{ \|e\|, \|m\|/2 \} \quad (e \in E_k, m \in M_k)
\]

(see [M]). Since the space \( L_k = \bigcap_{i=1}^k \bigcap_{n=1}^{r_{k+1}} T_n^{-1} M_i \subset \infty \) is of finite codimension, we can choose \( x_{k+1} \in X_1 \cap L_k \) such that \( \|x_{k+1}\| = \delta 2^{-(k+2)} \) and

\[
\|T_nx_{k+1}\| \geq (1 - \varepsilon) \delta 2^{-(k+2)} j_\mu(T_n) \quad (1 \leq n \leq r_{k+1}).
\]

Set \( x = \sum_{i=0}^\infty x_i \). Then \( x \in X_1 \) and

\[
\|x\| \leq \sum_{i=0}^\infty \|x_i\| \leq a_1 + \delta/2 + \sum_{i=1}^\infty \delta 2^{-(i+1)} = a_1 + \delta.
\]

For \( n = 1, \ldots, r_0 \) we have

\[
\|T_nx\| = \|T_n x_0 + \sum_{i=1}^\infty T_n x_i\| \geq (1 - \varepsilon) \|T_n x_0\| > a_1 j_\mu(T_n) \geq a_n j_\mu(T_n).
\]
Let $k \geq 0$ and $r_k < n \leq r_{k+1}$. Then
\[
\|T_n x\| = \left\| \sum_{i=0}^{\infty} T_n x_i \right\| \geq (1 - \varepsilon) \left\| \sum_{i=0}^{k+1} T_n x_i \right\|
\geq \frac{(1 - \varepsilon)^2}{2} \|T_n x_{k+1}\| \geq \frac{(1 - \varepsilon)^3}{2} \cdot \frac{\delta}{2^{k+2}} j_\mu(T_n) \geq a_n j_\mu(T_n).
\]
Thus $\|T_n x\| \geq a_n j_\mu(T_n)$ for all $n \in \mathbb{N}$.

To show the second statement, let $u \in X_1$ and $\varepsilon > 0$. Find $n_0$ such that $a_n < \varepsilon$ for all $n \geq n_0$. As in the first part, taking $x_0 = u$, construct a vector $x \in X_1$ with $\|x - u\| \leq \varepsilon$ and $\|T_n x\| \geq a_n j_\mu(T_n) \ (n \geq n_0)$. ■

**Corollary 23.** Let $X, Y$ be Banach spaces and let $(T_n) \subset B(X, Y)$ be a sequence of operators satisfying $\lim_{n \to \infty} j_\mu(T_n) = \infty$. Then there is no closed infinite-dimensional hypercyclic subspace for $(T_n)$.

**Proof.** Let $M$ be a closed infinite-dimensional subspace of $X$. By the previous result for the numbers $\alpha_n = (j_\mu(T_n))^{-1/2}$, there exists $x \in M$ such that $\|T_n x\| \to \infty$. Therefore $x$ is not hypercyclic for $(T_n)$. ■

We apply the previous results to the sequences of the form $(\lambda_n T^n)$ where $T \in B(X)$ and $\lambda_n$ are complex numbers. Denote by $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not Fredholm} the essential spectrum of $T$.

**Corollary 24.** Let $T \in B(X)$ be an operator and let $(\lambda_n)$ be a sequence of complex numbers. Suppose that $(\lambda_n T^n)$ satisfies condition (C) and $\sup_n |\lambda_n| d^n < \infty$ where $d = \text{dist}\{0, \sigma_e(T)\}$. Then there exists a closed infinite-dimensional hypercyclic subspace for $(\lambda_n T^n)$.

**Proof.** Since $(\lambda_n T^n)$ satisfies condition (C), the range of $T$ is dense. Without loss of generality we can assume that the numbers $\lambda_n$ are nonzero.

Choose $\lambda \in \sigma_e(T)$ with $|\lambda| = d$. Thus $T - \lambda$ is not Fredholm. We show that $T - \lambda$ is not upper semi-Fredholm. This is clear if $d = 0$ since the range of $T$ is dense. If $d > 0$ then $\lambda \in \partial \sigma_e(T)$ and $T - \lambda$ is not upper semi-Fredholm by [HW].

By [LS], there is a compact operator $K \in B(X)$ with $\dim \ker(T - \lambda - K) = \infty$. Set $M_0 = \ker(T - \lambda - K)$. For each $n$ we have $T^n = (T - K)^n + K_n$ for some compact operator $K_n$. Find subspaces $M'_n \subset X$ of finite codimension such that $\|K_n M'_n\| \leq |\lambda_n|^{-1}$. Set $M_n = M_0 \cap \bigcap_{i \leq n} M'_i$. Then $M_1 \supset M_2 \supset \cdots$ and $\dim M_n < \infty$. For $z \in M_n$ with $\|z\| = 1$ we have
\[
\|\lambda_n T^n z\| \leq \|\lambda_n (T - K)^n z\| + \|\lambda_n K_n\| \leq |\lambda_n| n + 1 = |\lambda_n| d^n + 1,
\]
and so $\sup_n \|\lambda_n T^n \| M_n < \infty$. The statement now follows from Theorem 20. ■
Corollary 25. Let $T : X \to X$ and suppose that $(\lambda_n T^n)$ satisfies (C) and $T$ is not Fredholm. Then there is an infinite-dimensional closed hypercyclic subspace for $(\lambda_n T^n)$.

Proof. We have $d = \text{dist}\{0, \sigma_e(T)\} = 0$, and so the statement follows from the previous corollary.

Corollary 26. Let $T \in B(X)$ and suppose that $(T^n)$ satisfies (C). The following conditions are equivalent:

(i) there exists a closed infinite-dimensional hypercyclic subspace for $(T^n)$;
(ii) the essential spectrum of $T$ intersects the closed unit ball.

Proof. Write $d = \text{dist}\{0, \sigma_e(T)\}$.

(ii)⇒(i). If $d \leq 1$ then Corollary 24 implies (i).

(i)⇒(ii). Let $d > 1$. Then $T$ is Fredholm. Recall the following standard construction from operator theory (see [S], [BHW]): let $\ell^\infty(X)$ be the space of all bounded sequences of elements of $X$; with the naturally defined algebraic operations and sup-norm it is a Banach space. Let $\widetilde{X} = \ell^\infty(X)/m(X)$ where $m(X)$ is the subspace of all precompact sequences. Let $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ be the operator induced by $T$. It is well known that $\widetilde{T}$ is invertible and $\sigma(\widetilde{T}) = \sigma_e(T)$. By the spectral radius formula we have $d = \text{dist}\{0, \sigma(\widetilde{T})\} = r(\widetilde{T}^{-1}) = \lim_{n \to \infty} \|\widetilde{T}^{-n}\|^{-1/n} = \lim_{n \to \infty} j(\widetilde{T}^{-n})^{1/n}$ where $r$ denotes the spectral radius. By [F] we find that $j_{\mu}(T^n) \leq 2 j(\widetilde{T}^{-n}) \leq 4 j_{\mu}(T^n)$ for all $n$. Thus $1 < d = \lim_{n \to \infty} j_{\mu}(T^n)^{1/n}$ and $\lim_{n \to \infty} j_{\mu}(T^n) = \infty$.

Let $M$ be a closed infinite-dimensional subspace of $X$. By Theorem 22 for the numbers $\alpha_n = (j_{\mu}(T^n))^{-1/2}$, there exists $x \in M$ such that $\lim_{n \to \infty} \|T^n x\| = \infty$. Hence $x$ is not hypercyclic for $(T^n)$.

References


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