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# On the isomorphism classes of weighted spaces of harmonic and holomorphic functions

by

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**Abstract.** Let $\Omega$ be either the complex plane or the open unit disc. We completely determine the isomorphism classes of

$$Hv = \{f : \Omega \to \mathbb{C} \text{ holomorphic} : \sup_{z\in\Omega} |f(z)|v(z) < \infty\}$$

and investigate some isomorphism classes of

$$hv = \{f : \Omega \to \mathbb{C} \text{ harmonic} : \sup_{z\in\Omega} |f(z)|v(z) < \infty\}$$

where $v$ is a given radial weight function. Our main results show that, without any further condition on $v$, there are only two possibilities for $Hv$, namely either $Hv \sim l_\infty$ or $Hv \sim H_\infty$, and at least two possibilities for $hv$, again $hv \sim l_\infty$ and $hv \sim H_\infty$. We also discuss many new examples of weights.

**1. Introduction.** Fix $a > 0$ or $a = \infty$ and put $aD = \{z \in \mathbb{C} : |z| < a\}$ (i.e. $aD = \mathbb{C}$ if $a = \infty$). For $0 < r < a$ and $f : aD \to \mathbb{C}$ put $M_\infty(f, r) = \sup_{|z|=r} |f(z)|$. Recall that $M_\infty(f, r)$ is increasing with respect to $r$ if $f$ is a harmonic function ([5]).

We want to investigate spaces of harmonic and holomorphic functions $f$ where $M_\infty(f, r)$ is unbounded in general but grows in a controlled way. To this end we introduce a *weight function*, i.e. an upper semicontinuous, non-increasing function $v : [0, a[ \to ]0, \infty[$ with $\lim_{r\to a} r^m v(r) = 0$ for all $m \geq 0$. (If $a < \infty$ this is equivalent to $\lim_{r\to a} v(r) = 0$.) We study the growth conditions

$$M_\infty(f, r) = O\left(\frac{1}{v(r)}\right) \text{ and } M_\infty(f, r) = o\left(\frac{1}{v(r)}\right) \quad \text{as } r \to a$$

by defining $\|f\|_v = \sup_{z\in aD} |f(z)|v(|z|)$ and

$$hv = \{f : aD \to \mathbb{C} \text{ harmonic} : \|f\|_v < \infty\},$$

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$$(hv)_0 = \{f \in hv : \lim_{r\to a} M_\infty(f,r)v(r) = 0\},$$
$$Hv = \{f \in hv : f \text{ holomorphic}\},$$
$$(Hv)_0 = (Hv) \cap (hv)_0.$$

These are Banach spaces (with respect to $\|\cdot\|_v$). The condition on $v$ ensures that these spaces contain all polynomials (or trigonometric polynomials, resp.). For example, if $f : aD \to \mathbb{C}$ is harmonic, then clearly

$$M_\infty(f,r) = O\left(\frac{1}{v(r)}\right) \text{ as } r \to a \quad \text{if and only if} \quad f \in hv$$

and

$$M_\infty(f,r) = o\left(\frac{1}{v(r)}\right) \text{ as } r \to a \quad \text{if and only if} \quad f \in (hv)_0.$$

By a simple substitution argument we see that it suffices to consider the two cases $a = 1$ and $a = \infty$. We want to discuss the Banach space nature of $hv$, $(hv)_0$, $Hv$ and $(Hv)_0$. In this respect a lot has already been done for holomorphic and harmonic functions on the unit disc where $v$ is a moderately decreasing weight ([10, 14, 16, 19–21]; see also [2, 3, 6, 7, 17]). But only few results are known for fast decreasing weights and for functions on the complex plane ([8, 9]).

In this article we determine all possible isomorphism classes for $Hv$ and $(Hv)_0$ and some isomorphism classes for $hv$ and $(hv)_0$ without any further condition on $v$.

Let $v : [0, a[ \to \mathbb{R}_+$ be a weight function. For $m > 0$ fix a global maximum point $r_m$ of the function $r \mapsto r^m v(r)$, $r \in [0, a[$, which exists in view of the upper semicontinuity. It is easily seen that $r_m \uparrow a$ as $m \to \infty$, and $m \mapsto r_m^m v(r_m)$, $m > 0$, is a continuous function. We want to compare quotients of the form $(r_m/r_n)^m v(r_m)/v(r_n)$ for different $m$ and $n$. First we introduce the following boundedness condition on $v$:

(B) $\forall b_1 > 1 \exists b_2 > 1\ \exists c > 0\ \forall m,n > 0$:
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \quad \text{and} \quad m,n,|m-n| \ge c \;\Rightarrow\; \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2.$$

Examples of $v$ enjoying (B) include $(1-r)^\alpha$ for $\alpha > 0$, $\exp(-(1-r)^{-1})$, $\exp(-\exp((1-r)^{-1})), \ldots$, if $r \in [0,1[$, and $\exp(-r^\varrho)$ for $\varrho > 0$, $\exp(-\log^\gamma r)$ for $\gamma \ge 2$, $\exp(-\exp(r))$, $\exp(-\exp(\exp(r))), \ldots$ if $r \in \mathbb{R}_+$ (see the next section for details).

Observe that the negation of (B) reads as follows:

$\neg$(B) $\exists b_1 > 1\ \forall b_2 > 1\ \forall c > 0\ \exists m,n > 0$ :
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \text{ and } m,n,|m-n| \ge c \text{ and } \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \ge b_2.$$

For two Banach spaces $X$ and $Y$ we write $X \sim Y$ if they are isomorphic to each other. Let $d(X,Y)$ be the Banach-Mazur distance of $X$ and $Y$, i.e.

$$d(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T : X \to Y \text{ is an (onto) isomorphism}\}.$$

Let $H_n = \operatorname{span}\{1, z^1, z^2, \ldots, z^n\}$ be the space of functions on $\partial D$ with the norm $M_\infty(\cdot, 1)$. It is well known that the Hardy space

$$H_\infty = \{f : D \to \mathbb{C} : f \text{ holomorphic}, \sup_{0<r<1} M_\infty(f,r) < \infty\}$$

is isomorphic to $(\sum_n \oplus H_n)_\infty$ ([22]).

1.1. Theorem.

(a) *Let $v$ satisfy* (B). *Then $Hv \sim l_\infty$ and $(Hv)_0 \sim c_0$.*
(b) *Let $v$ satisfy* $\neg$(B). *Then $Hv \sim H_\infty$ and $(Hv)_0 \sim (\sum_n \oplus H_n)_0$.*

Sections 3–6 are dedicated to the proofs of Theorem 1.1 and the following results.

For the isomorphic classification of $hv$ we need another boundedness condition:

(C) $\exists c_1 > 0\ \exists b_1 > 1\ \forall b_2 > 1\ \forall c_2 > 0\ \exists m, n > 0 :$
$$\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_1, \qquad \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \ge b_2,$$
$$m, n, |n-m| \ge c_2 \quad \text{and} \quad c_1|n-m| < \min(m,n).$$

Observe that (C) $\Rightarrow \neg$(B).

1.2. Theorem.

(a) *If $v$ satisfies* (B) *then $hv \sim l_\infty$ and $(hv)_0 \sim c_0$.*
(b) *If $v$ satisfies* (C) *then $hv \sim H_\infty$ and $(hv)_0 \sim (\sum_n \oplus H_n)_0$.*

If $v$ satisfies (C) then we have the combination $hv \sim Hv \sim H_\infty$ while (B) implies $hv \sim Hv \sim l_\infty$. If $Hv \sim l_\infty$ then it is easily seen that $hv \sim H_v \oplus H_v$ and hence also $hv \sim l_\infty$. However, we can also have the combination $Hv \sim H_\infty$ and $hv \sim l_\infty$ (see the following example). It is likely that these three are the only possibilities.

Example. Let $v(r) = (1 - \log(1-r))^{-1}$, $r \in [0,1[$. It is known that here $Hv \sim H_\infty$ and $hv \sim l_\infty$ ([10, 16]). Hence $v$ satisfies $\neg$(B) and $\neg$(C).

We also investigate under which (sufficient) condition $hv$ is selfadjoint, i.e. we have $f \in hv$ if and only if $\widetilde{f} \in hv$ where $\widetilde{f}$ is the trigonometric conjugate of $f$. ($\widetilde{f}$ is such that $\widetilde{f}(0) = 0$ and $\operatorname{Re} f + i \operatorname{Re} \widetilde{f}$, $\operatorname{Im} f + i \operatorname{Im} \widetilde{f}$ are holomorphic.) This is equivalent to the fact that the Riesz projection $R : hv \to Hv$ with

$$R(r^{|k|} \exp(ik\varphi)) = \begin{cases} r^k \exp(ik\varphi), & k \ge 0, \\ 0, & \text{else}, \end{cases} \qquad k \in \mathbb{Z},$$

is bounded. (We frequently denote the $k$th monomials on $\mathbb{C}$ by $z^k$, $\bar{z}^k$ or $r^k \exp(ik\varphi)$, $r^{|k|}\exp(-ik\varphi)$.) We have $\widetilde{f} = -iRf + i(\mathrm{id} - R)f + if(0)$.

1.3. Theorem. *Let $v$ satisfy* (B). *Then $hv$ is selfadjoint.*

Hence, in particular, a harmonic function $f$ satisfies

$$M_\infty(f,r) = O\left(\frac{1}{v(r)}\right) \text{ as } r \to a \quad \text{if and only if} \quad M_\infty(\widetilde{f},r) = O\left(\frac{1}{v(r)}\right).$$

(B) is a condition about a certain "inner regularity" of $v$ rather than its decay. To give a geometrical interpretation of (B) put $\varphi(t) = -\log(v(e^t))$, where $t \in ]-\infty, 0[$ if $a = 1$ and $t \in \mathbb{R}$ if $a = \infty$. Then $v(r) = \exp(-\varphi(\log r))$. The conditions on $v$ imply that $\varphi$ is increasing and that $\varphi(t) \to \infty$ as $t \to 0$ for $a = 1$, and $\varphi(t)/t \to \infty$ as $t \to \infty$ for $a = \infty$. Due to Hadamard's three circles theorem we may change $v$ on bounded annuli without changing the isomorphic character of $Hv$, $(Hv)_0$, $hv$ or $(hv)_0$. Therefore we may assume without loss of generality that $\varphi$ is twice differentiable. The function $r \mapsto r^m v(r)$ has a maximum only if $\varphi'(\log r) = m$. Put $s = \log r_m$ and $t = \log r_n$. Then we have

$$\log\left(\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)}\right) = \varphi(t) - \varphi(s) - \varphi'(s)(t-s) =: \varrho(t,s);$$

$\varrho(t,s)$ is the distance between the graph of $\varphi$ and its tangent.

Now, (B) is equivalent to the following

$$\forall b_1 > 0\ \exists b_2 > 0\ \exists c > 0\ \forall s,t:$$
$$\varrho(t,s) \le b_1,\ |\varphi'(t)|, |\varphi'(s)|, |\varphi'(t) - \varphi'(s)| \ge c \ \Rightarrow\ \varrho(s,t) \le b_2.$$

This means that the graph of $\varphi$ has no big corners. (See also the remark following Example 2.4.)

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## 2. More examples.

Here we give several examples where (B) holds.

2.1. Example. $v(r) = \exp(-\exp(r))$, $r \in [0, \infty[$. Then $r_{n \log n} = \log n$ for any $n > 0$. Fix $m, n > 0$. For $m' = m \log m$ and $n' = n \log n$ we obtain

$$\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} = \exp(m \log m(\log\log m - \log\log n) + n - m)$$
$$= \exp\left(\frac{(n-m)^2(m\log m)(1 + \log \overline{m})}{2\overline{m}^2 \log^2 \overline{m}}\right)$$

for some $\overline{m}$ between $m$ and $n$. (We have used

$$\log\log n - \log\log m = \frac{n-m}{m\log m} - \frac{1+\log\overline{m}}{2(\overline{m}\log\overline{m})^2}\,(n-m)^2$$

for appropriate $\overline{m}$.) Moreover the function

$$n \mapsto \left(\frac{r_{m'}}{r_{n\log n}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n\log n})}, \qquad n>0 \text{ (for fixed } m),$$

is increasing if $n>m$ and decreasing if $n<m$.

Fix $b_1>1$ and put $\beta = 4\sqrt{\log b_1}$, $c=\max(64\log b_1, 2)$. Hence, if $m\ge c$ then $\beta/\sqrt{m}\le 1/2$. If $|n-m| = \beta\sqrt{m}$, $n,m\ge c$, then we obtain

$$\begin{aligned}
\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} &\ge \exp\left(\frac{\beta^2 m^2(\log m)(1+\log\overline{m})}{2\overline{m}^2\log^2\overline{m}}\right)\\
&\ge \exp\left(\beta^2\,\frac{1}{2}\left(\frac{m}{\overline{m}}\right)^2\frac{\log m}{\log\overline{m}}\right)\\
&\ge \exp\left(\beta^2\,\frac{1}{2}\left(\frac{1}{1+\beta/\sqrt{m}}\right)^2\frac{\log m}{\log m+\log(1+\beta/\sqrt{m})}\right)\\
&\ge \exp\left(\frac{\beta^2}{16}\right) = b_1.
\end{aligned}$$

This implies that $|n-m|\le\beta\sqrt{m}$ whenever

$$\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} \le b_1.$$

In this case we have

$$\begin{aligned}
\left(\frac{r_{n'}}{r_{m'}}\right)^{n'} \frac{v(r_{n'})}{v(r_{m'})} &= \exp\left(\frac{(n-m)^2(n\log n)(1+\log\bar{n})}{2\bar{n}^2\log^2\bar{n}}\right)\\
&\le \exp\left(\frac{(n-m)^2 n\log n}{\bar{n}^2\log\bar{n}}\right)\\
&\le \exp\left(\beta^2 m\,\frac{(m+\beta\sqrt{m})\log(m+\beta\sqrt{m})}{(m-\beta\sqrt{m})^2\log(m-\beta\sqrt{m})}\right)\\
&\le \exp\left(\beta^2\,\frac{(1+\beta/\sqrt{m})(\log m+\log(1+\beta/\sqrt{m}))}{(1-\beta/\sqrt{m})^2(\log m+\log(1-\beta/\sqrt{m}))}\right) \le b_2
\end{aligned}$$

for suitable $b_2$ independent of $m$. (Here $\overline{n}$ is an appropriate number between $m$ and $n$.) Thus $v$ satisfies (B). Similarly one can deal with $\exp(-r^{\varrho})$ for $\varrho>0$, $\exp(-\exp(\exp(r)))$, ....

2.2. Example. $v(r) = \exp(-\log^{\varrho} r)$, $r\in[1,\infty[$, for fixed $\varrho\ge 2$, and $v(r)=1$, $r\in[0,1[$. Here we obtain $r_n = \exp((n/\varrho)^{1/(\varrho-1)})$ (for sufficiently large $n$). We have

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)}$$
$$= \exp\left((\varrho-1)\left(\left(\frac{m}{\varrho}\right)^{\frac{\varrho}{\varrho-1}} - \left(\frac{n}{\varrho}\right)^{\frac{\varrho}{\varrho-1}}\right) + (n-m)\left(\frac{n}{\varrho}\right)^{\frac{1}{\varrho-1}}\right)$$
$$= \exp\left(\frac{(n-m)^2}{2(\varrho-1)\varrho^{\frac{1}{\varrho-1}}\overline{m}^{\frac{\varrho-2}{\varrho-1}}}\right)$$

for suitable $\overline{m}$ between $m$ and $n$. (We used

$$x^\beta - x_0^\beta = \beta x_0^{\beta-1}(x-x_0) + \tfrac{1}{2}\beta(\beta-1)\overline{x}^{\beta-2}(x-x_0)^2$$

for $x = m/\varrho$, $x_0 = n/\varrho$, $\beta = \varrho/(\varrho-1)$ and appropriate $\overline{x}$.) The map

$$n \mapsto \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)}$$

is increasing if $n > m$ and decreasing if $n < m$ (for fixed $m$). Fix $b_1 > 1$ and put

$$\gamma = \frac{\varrho-2}{\varrho-1}, \quad \beta = \sqrt{2^{\gamma+1}(\varrho-1)\varrho^{1/(\varrho-1)}\log b_1}, \quad c = (2\beta)^{2(\varrho-1)/\varrho}.$$

Then $\beta m^{\gamma/2-1} \le 1/2$ provided that $m \ge c$. If $|n-m| = \beta m^{\gamma/2}$ and $n, m \ge c$ we obtain

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \ge \exp\left(2^\gamma(\log b_1)\left(\frac{m}{m+\beta m^{\gamma/2}}\right)^\gamma\right) \ge b_1.$$

Hence, if

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1$$

then $|n-m| \le \beta m^{\gamma/2}$ and

$$\left(\frac{r_n}{r_m}\right)\frac{v(r_m)}{v(r_n)} = \exp\left(\frac{(n-m)^2}{2(\varrho-1)\varrho^{\frac{1}{\varrho-1}}\overline{n}^{\frac{\varrho-2}{\varrho-1}}}\right)$$
$$\le \exp\left(2^\gamma\left(\frac{m}{m-\beta m^{\gamma/2}}\right)^\gamma \log b_1\right) \le b_1^{4^\gamma} =: b_2$$

(for suitable $\overline{n}$ between $m$ and $n$).

2.3. Example. $v(r) = \exp(-1/(1-r))$, $r \in [0,1[$. Here $r_{m^2-m} = 1 - 1/m$. Fix $m, n > 0$. For $m' = m^2 - m$ and $n' = n^2 - n$ we obtain

$$\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} = \left(\frac{1-\frac{1}{m}}{1-\frac{1}{n}}\right)^{m^2-m} \exp(n-m).$$

Hence

$$n \mapsto \left(\frac{r_{m'}}{r_{n^2-n}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n^2-n})}$$

is decreasing if $n < m$ and increasing if $n > m$. Fix $\beta > 0$ and put

$$a_m = \left(\frac{1-\frac{1}{m}}{1-\frac{1}{m\pm\beta\sqrt{m}}}\right)^{m^2-m} \exp(\pm\beta\sqrt{m}).$$

We obtain $\lim_{m\to\infty} a_m = \exp(\beta^2)$. Define $\beta = \sqrt{2\log b_1}$ and take $c$ so large that $a_m \geq \exp(\log b_1) = b_1$ whenever $m \geq c$. Thus, if $|n-m| = \beta\sqrt{m}$ we have

$$\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} \geq b_1.$$

So, if

$$\left(\frac{r_{m'}}{r_{n'}}\right)^{m'} \frac{v(r_{m'})}{v(r_{n'})} \leq b_1$$

we must have $|n-m| \leq \beta\sqrt{m}$. In this case we obtain

$$\begin{aligned}\left(\frac{r_{n'}}{r_{m'}}\right)^{n'} \frac{v(r_{n'})}{v(r_{m'})} &= \left(\frac{1-\frac{1}{n}}{1-\frac{1}{m}}\right)^{n^2-n} \exp(m-n) \\ &= \left(1+\frac{n-m}{m-1}\cdot\frac{1}{n}\right)^{n^2-n} \exp(m-n) \\ &\leq \exp\left(\frac{(n-m)^2}{m-1}\right) \leq \exp(2\beta^2) =: b_2.\end{aligned}$$

Similarly one can show that $\exp(-\exp(1/(1-r)))$, $\exp(-\exp(\exp(1/(1-r))))$, $\ldots$ satisfy (B).

2.4. Example. $v(r) = (1-r)^\alpha$, $r \in [0,1[$, for some fixed $\alpha > 0$. Here $r_n = n/(n+\alpha)$ and, as in the preceding example, we can verify that $v$ satisfies (B).

The weight of Example 2.4 is of moderate decay, it satisfies

$$(\star) \qquad \sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty.$$

Such weights have been studied extensively. Here it is possible to fix $m_1 < m_2 < \cdots$ and $\gamma > 1$ such that

$$\gamma \leq \frac{v(1-2^{-m_n})}{v(1-2^{-m_{n+1}})} \leq \gamma^2 \quad \text{for all } n.$$

This implies the existence of an index $j$ with

$$1-\frac{1}{2^{m_{n-j}}} \leq r_M \leq 1-\frac{1}{2^{m_{n+j}}} \quad \text{whenever} \quad 2^{m_n} \leq M < 2^{m_{n+1}}, \quad n = 1,2,\ldots.$$

Using this one can show that condition (B) is equivalent to

$$(\star\star)\qquad \inf_k \limsup_n \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1$$

provided that ($\star$) holds. Hence Theorem 1.1 includes one of the main results of [16]. (We omit the details.) Weights satisfying ($\star$) and ($\star\star$) are called *normal* (see [4], [13], [19]–[21]).

The following proposition allows us to construct examples for all the cases discussed in Section 1.

2.5. Proposition. *Fix numbers* $1 \le n_1 < n_2 < \cdots$, $0 < s_1 < s_2 < \cdots$ *and* $v_1 > v_2 > \cdots > 0$ *such that* $\sup_k n_k < \infty$, $\lim_{k\to\infty} s_k = a$ *and*

$$(2.1)\qquad s_m^{n_m} v_m = \sup_k s_k^{n_m} v_k,$$

$$(2.2)\qquad \lim_{k\to\infty} s_k^{n_m} v_k = 0 \quad \text{for each } m.$$

*Put* $v(s) = v_m$ *if* $s_{m-1} < s \le s_m$. *Then* $v$ *is a weight on* $[0, a[$ *with* $r_{n_m} = s_m$ *for all* $m$. *Moreover, if* $n_{m-1} < j < n_m$ *then*

$$r_j = \begin{cases} s_{m-1} & \text{if } s_{m-1}^j v_{m-1} \ge s_m^j v_m, \\ s_m & \text{else.} \end{cases}$$

*Proof.* $v$ is upper semicontinuous, non-increasing and $\lim_{r\to a} r^m v(r) = 0$ for all $m \ge 0$. Fix $m$. If $s_{k-1} < s \le s_k$ then $s^{n_m} v(s) = s^{n_m} v_k \le s_k^{n_m} v_k \le s_m^{n_m} v_m$. Hence $r_{n_m} = s_m$.

Now, let $n_{m-1} < j < n_m$. If $k \le m-1$ and $s_{k-1} < s \le s_k$ then

$$s^j v(s) \le s_k^j v_k \le s_k^{j-n_{m-1}} s_{m-1}^{n_{m-1}} v_{m-1} \le s_{m-1}^j v_{m-1}.$$

If $k \ge m$ and $s_k < s \le s_{k+1}$ then

$$s^j v(s) \le s^{j-n_m} s_m^{n_m} v_m \le s_m^j v_m.$$

Finally, if $s_{m-1} < s \le s_m$ then $s^j v(s) = s^j v_m \le s_m^j v_m$. Hence $r_j = s_{m-1}$ if $s_{m-1}^j v_{m-1} \ge s_m^j v_m$, and $r_j = s_m$ otherwise. ∎

2.6. Example. Using Proposition 2.5 we construct a weight $v$ on $[0, \infty[$ which satisfies (C). To this end put

$$s_m = m!, \qquad n_m = \sum_{j=1}^m j, \qquad v_m = \prod_{j=1}^m \frac{1}{j^{n_j}}.$$

Then $s_m^{n_m} v_m = \prod_{j=1}^m j^{n_m - n_j}$. Moreover

$$s_k^{n_m} v_k = \begin{cases} \prod_{j=1}^{k} j^{n_m - n_j} & \text{if } k \le m, \\ \Big(\prod_{j=1}^{m} j^{n_m - n_j}\Big)\Big(\prod_{j=m+1}^{k} \frac{1}{j^{n_j - n_m}}\Big) & \text{if } k > m. \end{cases}$$

This implies (2.1) and (2.2). Hence Proposition 2.5 yields a weight $v$ with $r_{n_m} = s_m$. We obtain $|n_{m+1} - n_m| = m + 1 \le \min(n_m, n_{m+1})$ and

$$\left(\frac{s_m}{s_{m+1}}\right)^{n_m} \frac{v_m}{v_{m+1}} = (m+1)^{m+1} \quad \text{and} \quad \left(\frac{s_{m+1}}{s_m}\right)^{n_{m+1}} \frac{v_{m+1}}{v_m} = 1.$$

This shows that $v$ satisfies (C). Hence $Hv \sim hv \sim H_\infty$.

**3. Trigonometric polynomials.** In the following let $[x]$ be the largest integer $\le x$ for a given number $x \in \mathbb{R}$. We need

3.1. Lemma. *Let $0 < r < s$ and $m, n > 0$.*

(a) *Then, for any trigonometric polynomial $f$ of degree $\le n$, we have*

$$M_\infty(f, s) \le \left(\frac{s}{r}\right)^n M_\infty(f, r).$$

(b) *Let $g \in \operatorname{span}\{t^{|k|} \exp(ik\varphi) : |k| > m\}$. Then*

$$M_\infty(g, r) \le \frac{2(r/s)^m}{(r/s)^{2m} + 1} M_\infty(g, s) \le 2\left(\frac{r}{s}\right)^m M_\infty(g, s).$$

*Proof.* (a) See [15, Lemma 3.1(i)].

(b) Put $p = [m] + 1$. Let

$$h(\exp(i\varphi)) = \frac{1}{2}\,(\exp(ip\varphi) + \exp(-ip\varphi)) \sum_{k\in\mathbb{Z}} \left(\frac{r}{s}\right)^{|k|} \exp(ik\varphi).$$

Then $h$ is a Poisson kernel up to the factor $2^{-1}(\exp(ip\varphi) + \exp(-ip\varphi))$. Hence $(2\pi)^{-1} \int_0^{2\pi} |h(\exp(i\varphi))|\, d\varphi \le 1$ and

$$h(\exp(i\varphi)) = \frac{1}{2} \sum_{j\ge p} \left( \left(\frac{r}{s}\right)^{j-p} + \left(\frac{r}{s}\right)^{j+p} \right) \exp(ij\varphi)$$
$$+ \frac{1}{2} \sum_{j\le -p} \left( \left(\frac{r}{s}\right)^{p-j} + \left(\frac{r}{s}\right)^{-j-p} \right) \exp(ij\varphi) + \sum_{|j|<p} \alpha_j \exp(ij\varphi)$$

for some $\alpha_j$. If $g = \sum_{|k|>m} \beta_k t^{|k|} \exp(ik\varphi)$ for some $\beta_k$ we obtain

$$\frac{1}{2}\left(\left(\frac{r}{s}\right)^p + \left(\frac{s}{r}\right)^p\right) g(r \exp(i\varphi)) = \frac{1}{2\pi} \int_0^{2\pi} h(\exp(i(\varphi - \psi)))g(s \exp(i\psi))\, d\psi.$$

This implies, since $0 < (r/s)^p < (r/s)^m < 1$,

$$|g(r\exp(i\varphi))| = 2\left(\left(\frac{r}{s}\right)^p + \left(\frac{s}{r}\right)^p\right)^{-1} \times (2\pi)^{-1}\left|\int_0^{2\pi} h(\exp(i(\varphi-\psi)))g(s\exp(i\psi))\,d\psi\right|$$
$$\le \frac{2(r/s)^p}{(r/s)^{2p}+1}\,M_\infty(g,s)(2\pi)^{-1}\int_0^{2\pi}|h(\exp(i(\varphi-\psi)))|\,d\psi$$
$$\le \frac{2(r/s)^m}{(r/s)^{2m}+1}\,M_\infty(g,s).$$

Hence

$$M_\infty(g,r) \le \frac{2(r/s)^m}{(r/s)^{2m}+1}\,M_\infty(g,s). \ \blacksquare$$

Now, fix a weight $v : [0,a[ \to \mathbb{R}_+$. As before, let $r_m$ be a maximum point of the function $r \mapsto r^m v(r)$, $r > 0$.

3.2. Corollary.

(a) *Fix* $m > 0$ *and consider* $f \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z},\ |k| \le m\}$, $g \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z}, |k| > m\}$. *Then*

$$\|f\|_v \le \sup_{r\le r_m} M_\infty(f,r)v(r) \quad \textit{and} \quad \|g\|_v \le 2\sup_{r\ge r_m} M_\infty(g,r)v(r).$$

(b) *Fix* $0 < m < n$ *and put*

$$\alpha = \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)}, \qquad \beta = \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)}.$$

*Then any* $h \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z},\ m < |k| \le n\}$ *satisfies*

$$\|h\|_v \le 2\alpha M_\infty(h,r_n)v(r_n) \quad \textit{and} \quad \|h\|_v \le 2\beta M_\infty(h,r_m)v(r_m).$$

*Proof.* (a) If $r > r_m$ then we obtain, by Lemma 3.1,

$$M_\infty(f,r)v(r) \le \left(\frac{r}{r_m}\right)^m \frac{v(r)}{v(r_m)}\,M_\infty(f,r_m)v(r_m) \le M_\infty(f,r_m)v(r_m).$$

If $0 < r < r_m$ Lemma 3.1 implies

$$M_\infty(g,r)v(r) \le 2\left(\frac{r}{r_m}\right)^m \frac{v(r)}{v(r_m)}\,M_\infty(g,r_m)v(r_m) \le 2M_\infty(g,r_m)v(r_m).$$

This yields (a).

(b) According to (a) we have

$$\|h\|_v \le \sup_{r\le r_n} M_\infty(h,r)v(r) \le 2\sup_{r\le r_n}\left(\frac{r}{r_n}\right)^m \frac{v(r)}{v(r_n)}\,M_\infty(h,r_n)v(r_n)$$
$$\le 2\alpha M_\infty(h,r_n)v(r_n)$$

and

$$\|h\|_v \le 2 \sup_{r\ge r_m} M_\infty(h,r)v(r) \le 2 \sup_{r\ge r_m} \left(\frac{r}{r_m}\right)^n \frac{v(r)}{v(r_m)} M_\infty(h,r_m)v(r_m)$$
$$\le 2\beta M_\infty(h,r_m)v(r_m). \ \blacksquare$$

We want to study special operators on $hv$. Note that any linear operator $T : hv \to hv$ is bounded provided that $T$, restricted to the trigonometric polynomials, is bounded with respect to $M_\infty(\cdot,1)$. Let $\|T\|_v$ be the operator norm with respect to $\|\cdot\|_v$ and $\|T\|_\infty$ the operator norm with respect to $M_\infty(\cdot,1)$. We always have $\|T\|_v \le \|T\|_\infty$. Indeed, put $z = r\exp(i\varphi)$ and $f = \sum_k \alpha_k r^{|k|}\exp(ik\varphi)$. Then

$$|(Tf)(z)|v(|z|) = \Big|T\Big(\sum_k \alpha_k r^{|k|}\exp(ik\varphi)\Big)\Big|v(r)$$
$$\le \|T\|_\infty \sup_\varphi \Big|\sum_k \alpha_k r^{|k|}\exp(ik\varphi)\Big| v(r) \le \|T\|_\infty \|f\|_v.$$

Hence $\|Tf\|_v \le \|T\|_\infty \|f\|_v$. ∎

Sometimes $T$ is bounded with respect to $\|\cdot\|_v$ but unbounded with respect to $M_\infty(\cdot,1)$ (see below).

Now fix $0 < m < n$ (not necessarily integers) and consider the trigonometric polynomial $f = \sum_{k\in\mathbb{Z}} \alpha_k r^{|k|}\exp(ik\varphi)$. We define the operator $V_{n,m}$ by

$$V_{n,m}f = \sum_{|k|\le m} \alpha_k r^{|k|}\exp(ik\varphi) + \sum_{m<|k|\le n} \frac{[n]-|k|}{[n]-[m]}\alpha_k r^{|k|}\exp(ik\varphi). \tag{3.1}$$

Moreover, we consider the Riesz projection

$$Rf = \sum_{k\ge 0} \alpha_k r^{|k|}\exp(ik\varphi). \tag{3.2}$$

3.3. Lemma. *We have*

(a) $\|V_{n,m}\|_\infty \le \dfrac{[n]+[m]}{[n]-[m]}$,

(b) $M_\infty(Rh,r) \le \left(1+\dfrac{[n]-[m]}{[m]}\right) M_\infty(h,r)$
*for any* $r>0$ *and* $h \in \mathrm{span}\{r^{|k|}\exp(ik\varphi) : k\in\mathbb{Z},\ m<|k|\le n\}$,

(c) $\|V_{n_4,n_3} - V_{n_2,n_1}\|_\infty \le 4\dfrac{[n_4]-[n_1]}{[n_2]-[n_1]}\left(3+4\dfrac{[n_4]-[n_1]}{[n_4]-[n_3]}\right)$
*if* $0<n_1<n_2<n_3<n_4$,

(d) $\|V_{n_4,n_3} - V_{n_2,n_1}\|_\infty \le 2([n_4]-[n_1])$,
$\|R(V_{n_4,n_3} - V_{n_2,n_1})\|_\infty \le [n_4]-[n_1]$ *if* $0<n_1<n_2<n_3<n_4$.

*Proof.* (a) By definition we have $V_{n,m} = V_{[n],[m]}$. Fix $p \in \mathbb{Z}_+$. Then

$$V_{p,0} f = \sum_{|k| \le p} \frac{p - |k|}{p} \, \alpha_k r^{|k|} \exp(ik\varphi).$$

It is well known ([11]) that $\|V_{p,0}\|_\infty = 1$. Since

$$V_{n,m} = \frac{[n]V_{[n],0} - [m]V_{[m],0}}{[n] - [m]}$$

we obtain (a).

(b) Let $m$ and $n$ be integers. Fix $k \in \mathbb{Z}$ and put, for the trigonometric polynomial $f$, $(S_k f)(r\exp(i\varphi)) = \exp(ik\varphi) f(r\exp(i\varphi))$. If $h$ is as indicated in (b) we obtain $Rh = S_n V_{n+m,n-m} S_{-n} h$ (compare the Fourier coefficients on both sides). We conclude that $M_\infty(Rh, r) \le 2n(2m)^{-1} M_\infty(h, r)$. From this the result follows.

(c) Retain the notation $S_k$ of (b). Let $0 \le n_1 < n_2 < n_3 < n_4$ be integers. Put $(Uf)(z) = f(\overline{z})$ for any trigonometric polynomial $f$. Set $T = V_{n_4+n_2-2n_1, n_3+n_2-2n_1} - V_{2(n_2-n_1), n_2-n_1}$. Then

$$V_{n_4,n_3} - V_{n_2,n_1} = US_{2n_1-n_2} RTS_{-(2n_1-n_2)} U + S_{2n_1-n_2} RTS_{-(2n_1-n_2)}.$$

Hence (a) and (b) imply

$$\begin{aligned}\|V_{n_4,n_3} - V_{n_2,n_1}\|_\infty &\le 2\,\frac{n_4 + n_2 - 2n_1}{n_2 - n_1}\left(3 + \frac{n_4 + n_3 + 2n_2 - 4n_1}{n_4 - n_3}\right)\\ &\le 4\,\frac{n_4 - n_1}{n_2 - n_1}\left(3 + 4\,\frac{n_4 - n_1}{n_4 - n_3}\right).\end{aligned}$$

(d) Put $f = \sum_k \alpha_k \exp(ik\varphi)$. Then, by definition, there are $\varrho_k \in [0,1]$ with

$$(V_{n_4,n_3} - V_{n_2,n_1}) f = \sum_{n_1 < |k| \le n_4} \alpha_k \varrho_k \exp(ik\varphi),$$

$$R(V_{n_4,n_3} - V_{n_2,n_1}) f = \sum_{n_1 < k \le n_4} \alpha_k \varrho_k \exp(ik\varphi).$$

Since $|\alpha_k| \le \|f\|_\infty$ for all $k$, (d) follows. ∎

3.4. Proposition. *Suppose that, for some* $n, m > 0$,

$$\alpha := \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} > 2.$$

(a) *Then there is* $\beta(\alpha) > 0$ *such that* $\|f\|_v \le \beta(\alpha)\|f + g\|_v$ *whenever* $f \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z},\ |k| \le \min(m,n)\}$ *and* $g \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : k \in \mathbb{Z},\ |k| > \max(m,n)\}$; *moreover,* $\limsup_{\alpha\to\infty} \beta(\alpha) < \infty$.

(b) *There is a constant* $\gamma(\alpha) > 0$ *such that* $V := V_{\max(m,n),\min(m,n)} : hv \to hv$ *satisfies* $\|V\|_v \le \gamma(\alpha)$; *moreover*, $\limsup_{\alpha\to\infty} \gamma(\alpha) < \infty$.

*Proof.* (a) First consider the case $m < n$. By Lemma 3.1 and Corollary 3.2, we have

$$\begin{aligned}\|f+g\|_v &\ge \sup_{r\le r_m} M_\infty(f+g,r)v(r)\\ &\ge \sup_{r\le r_m}\Big(M_\infty(f,r)v(r) - M_\infty(g,r)v(r)\Big)\\ &\ge \|f\|_v - 2\left(\frac{r_m}{r_n}\right)^n \frac{v(r_m)}{v(r_n)}\left(\sup_{r\le r_m}\left(\frac{r}{r_m}\right)^n \frac{v(r)}{v(r_m)}\right) M_\infty(g,r_n)v(r_n)\\ &\ge \|f\|_v - \frac{2}{\alpha}\sup_{r\le r_m}\left(\frac{r}{r_m}\right)^m \frac{v(r)}{v(r_m)}\,\|g\|_v\\ &\ge \|f\|_v - \frac{2}{\alpha}\,\|g\|_v \ge \|f\|_v - \frac{2}{\alpha}\,\|f+g\|_v - \frac{2}{\alpha}\,\|f\|_v.\end{aligned}$$

Hence $\|f\|_v \le (1-2/\alpha)^{-1}(1+2/\alpha)\|f+g\|_v$.

For $n < m$ we have, by Lemma 3.1,

$$\begin{aligned}\|f+g\|_v &\ge \sup_{r\ge r_m} M_\infty(f+g,r)v(r)\\ &\ge \frac{1}{2}\|g\|_v - \left(\sup_{r\ge r_m}\left(\frac{r}{r_m}\right)^n \frac{v(r)}{v(r_m)}\right) M_\infty(f,r_n)v(r_n)\left(\frac{r_m}{r_n}\right)^n \frac{v(r_m)}{v(r_n)}\\ &\ge \frac{1}{2}\left(\|g\|_v - \frac{2}{\alpha}\sup_{r\ge r_m}\left(\frac{r}{r_m}\right)^m \frac{v(r)}{v(r_m)}\,\|f\|_v\right)\\ &= \frac{1}{2}\left(\|f\|_v - \|f+g\|_v - \frac{2}{\alpha}\,\|f\|_v\right)\end{aligned}$$

We obtain $\|f\|_v \le (3/(1-2/\alpha))\|f+g\|_v$.

(b) Assume without loss of generality that $m < n$. Fix $h \in hv$, say $h = \sum_k \alpha_k r^{|k|}\exp(ik\varphi)$.

First consider the case $[m]=[n]$. Then, by definition, $Vh=\sum_{|k|\le m}\alpha_k r^{|k|}\exp(ik\varphi)$. In view of (a) this means that $V$ is bounded by $\beta(\alpha)$.

Now assume $[n]-[m]\ge 1$. It suffices to assume $[n]\le 2[m]$ (otherwise Proposition 3.4 follows from Lemma 3.3). Put $T = V_{2[n]-[m],[n]} - V_{[m],2[m]-[n]}$. Lemma 3.3(a), (c) implies that $T$ is uniformly bounded. The definition of $T$ yields moreover $T(r^{|k|}\exp(ik\varphi)) = r^{|k|}\exp(ik\varphi)$ whenever $[m]\le|k|\le[n]$. Since $V = V_{[n],[m]}$ we obtain

$$VTh = (V_{[n],[m]} - V_{[m],2[m]-[n]})h.$$

Lemma 3.3(c) implies $\|VTh\|_v \le 88\|h\|_v$. Now put

$$Ph = \sum_{|k|<m} \alpha_k r^{|k|}\exp(ik\varphi), \qquad Qh = \sum_{|k|>n} \alpha_k r^{|k|}\exp(ik\varphi)$$

and $f = P(\mathrm{id}-T)h$, $g = Q(\mathrm{id}-T)h$. We obtain $Th + f + g = h$.

(a) and the definitions of $V$ and $g$ imply

$$\begin{aligned}\|Vh\|_v = \|f + VTh\|_v &\le \|f\|_v + 88\|h\|_v \le \beta(\alpha)\|f+g\|_v + 88\|h\|_v\\ &\le \beta(\alpha)\|f+g+Th\|_v + \beta(\alpha)\|Th\|_v + 88\|h\|_v\\ &\le (\beta(\alpha)(1+\|T\|_v) + 88)\|h\|_v. \ \blacksquare\end{aligned}$$

**4. Conditions** (B) **and** ¬(B)**.** Let $v : [0,a[\ \to \mathbb{R}_+$ be a weight. First we prove

4.1. Proposition. *Let $v$ satisfy* (B) *and let $c > 0$ be the corresponding constant in* (B). *Fix $c < m < n < p$ and $b, d > 1$ such that $b \le \alpha, \beta, \gamma, \delta \le d$ where*

$$\alpha = \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)}, \qquad \beta = \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)},$$
$$\gamma = \left(\frac{r_n}{r_p}\right)^n \frac{v(r_n)}{v(r_p)}, \qquad \delta = \left(\frac{r_p}{r_n}\right)^p \frac{v(r_p)}{v(r_n)}.$$

*Then there are constants $d' > 1$ and $\kappa, \eta > 0$ depending only on $b$ and $d$ but not on $m$, $n$ or $p$ such that either $p - m \le c$ or*

$$\eta \le \frac{p-n}{n-m} \le \kappa \quad \textit{and} \quad \max\left(\left(\frac{r_m}{r_p}\right)^m \frac{v(r_m)}{v(r_p)}, \left(\frac{r_p}{r_m}\right)^p \frac{v(r_p)}{v(r_m)}\right) \le d'.$$

*Proof.* Our assumptions imply

$$\frac{r_m}{r_n} \le \left(\frac{1}{b}\right)^{\frac{2}{n-m}} \quad \text{and} \quad \frac{r_n}{r_p} \le \left(\frac{1}{b}\right)^{\frac{2}{p-n}}.$$

Assume $p - m > c$.

If $n - m \le p - n$ we have

$$\left(\frac{r_m}{r_p}\right)^m \frac{v(r_m)}{v(r_p)} = \alpha\gamma\left(\frac{r_p}{r_n}\right)^{n-m} \le \alpha\gamma\left(\frac{r_p}{r_n}\right)^{p-n} \le \alpha\gamma^2\delta \le d^4.$$

(B) provides us with a constant $b' = b'(d^4) > 1$ such that $(r_p/r_m)^p v(r_p)/v(r_m) \le b'$. In this case we have

$$\left(\frac{1}{b'd^4}\right)^{\frac{1}{p-m}} \le \frac{r_m}{r_p} \le \left(\frac{1}{b}\right)^{\frac{2}{n-m}+\frac{2}{p-n}},$$

which implies

$$2(\log b)\left(\frac{1}{n-m} + \frac{1}{p-n}\right) \le \frac{\log(b'd^4)}{p-m}.$$

Since $p - m = (p - n) + (n - m)$ we deduce

$$1 \le \max\left(\frac{p-n}{n-m}, \frac{n-m}{p-n}\right) \le \frac{\log(b'd^4)}{2\log b}.$$

If $p - n < n - m$ we have

$$\left(\frac{r_p}{r_m}\right)^p \frac{v(r_p)}{v(r_m)} = \delta\beta\left(\frac{r_n}{r_m}\right)^{p-n} \le \delta\beta\left(\frac{r_n}{r_m}\right)^{n-m} \le \delta\beta^2\alpha \le d^4$$

and we proceed exactly as before. Put $d' = \max(d^4, b')$. ∎

In order to discuss some consequences of 4.1 we need two technical lemmas.

4.2. Lemma. *Let $b_1, b_2 > 1$ and $m, n > 0$ be such that*

$$\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \ge b_2 \quad \text{and} \quad \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1.$$

*Then for any $N \in \mathbb{Z}_+$ and $p = n2^{-N} + (1 - 2^{-N})m$, we have*

$$\left(\frac{r_p}{r_m}\right)^p \frac{v(r_p)}{v(r_m)} \ge b_2^{1/2^N} b_1^{-1+1/2^N}, \qquad \left(\frac{r_m}{r_p}\right)^m \frac{v(r_m)}{v(r_p)} \le b_1$$

*and $|p - m|2^N = |n - m|$.*

*Proof.* First, for $n_1 = (m + n)/2$ we easily obtain

$$\left(\frac{r_n}{r_m}\right)^{n_1} \frac{v(r_n)}{v(r_m)} \ge \sqrt{\frac{b_2}{b_1}}.$$

Hence

$$\left(\frac{r_{n_1}}{r_m}\right)^{n_1} \frac{v(r_{n_1})}{v(r_m)} = \left(\frac{r_n}{r_m}\right)^{n_1} \frac{v(r_n)}{v(r_m)} \left(\frac{r_{n_1}}{r_n}\right)^{n_1} \frac{v(r_{n_1})}{v(r_n)} \ge \sqrt{\frac{b_2}{b_1}}.$$

Since $(r_n/r_{n_1})^m \le (r_n/r_{n_1})^{n_1}$ for $m \le n_1 \le n$ as well as for $n \le n_1 \le m$ we also obtain

$$\left(\frac{r_m}{r_{n_1}}\right)^m \frac{v(r_m)}{v(r_{n_1})} = \left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \left(\frac{r_n}{r_{n_1}}\right)^m \frac{v(r_n)}{v(r_{n_1})} \le b_1 \left(\frac{r_n}{r_{n_1}}\right)^{n_1} \frac{v(r_n)}{v(r_{n_1})} \le b_1.$$

In the next step we repeat the procedure with $n_1$ instead of $n$ and $\sqrt{b_2/b_1}$ instead of $b_2$. This yields $n_2 = (n_1 + m)/2$ and

$$\left(\frac{r_{n_2}}{r_m}\right)^{n_2} \frac{v(r_{n_2})}{v(r_m)} \ge b_2^{1/4} b_1^{-1/2-1/4}, \qquad \left(\frac{r_m}{r_{n_2}}\right)^m \frac{v(r_m)}{v(r_{n_2})} \le b_1.$$

Continuation proves Lemma 4.2. ∎

4.3. Lemma. *Fix $M, q \in \mathbb{Z}_+$ and put*

$$P_{q,M}(f) = \sum_j \alpha_{q+jM} r^{|q+jM|} \exp(i(q + jM)\varphi)$$

*for any trigonometric polynomial $f = \sum_k \alpha_k r^{|k|} \exp(ik\varphi)$. Then $\|P_{q,M}\|_\infty = 1$.*

*Proof.* We obtain

$$\frac{1}{M}\sum_{l=0}^{M-1}\exp\Big(-i\,\frac{2\pi}{M}\,lq\Big)f\Big(\exp\Big(i\,\frac{2\pi}{M}\,l\Big)\cdot r\exp(i\varphi)\Big)$$
$$=\frac{1}{M}\sum_{k}\alpha_k\Big(\sum_{l=0}^{M-1}\exp\Big(i\,\frac{2\pi}{M}\,l(k-q)\Big)\Big)r^{|k|}\exp(ik\varphi)$$
$$=\sum_{j}\alpha_{q+jM}r^{|q+jM|}\exp(i(q+jM)\varphi).$$

This implies that $P_{q,M}$ has norm one. ∎

Again let $H_n=\operatorname{span}\{1,z,\ldots,z^n\}$ be endowed with $M_\infty(\cdot,1)$. Now we are ready to prove

4.4. Proposition. *Assume* ¬(B). *Fix* $M,N\in\mathbb{Z}_+$. *Then there is a subspace* $A\subset\operatorname{span}\{z^k : k\geq M\}\subset(Hv)_0$ *and a projection* $Q:Hv\to A$ *such that* $\|Q\|_v$ *and the Banach–Mazur distance* $d(A,H_N)$ *do not depend on* $M$ *or* $N$. *If, in addition,* $v$ *satisfies* (C) *then* $Q$ *is defined and uniformly bounded on all of* $hv$.

*Proof.* ¬(B) yields the existence of $b>1$ and $m,n\geq\max(N,M)$, with

$$\Big(\frac{r_m}{r_n}\Big)^m\frac{v(r_m)}{v(r_n)}\leq b,\qquad\Big(\frac{r_n}{r_m}\Big)^n\frac{v(r_n)}{v(r_m)}\geq b^{2^{N+1}}$$

and $|m-n|\geq N2^N$. We may even assume that

$$b>2. \tag{4.1}$$

According to Lemma 4.2 we find $p$ between $m$ and $n$ with

$$|n-m|=2^N|p-m|, \tag{4.2}$$

$$\Big(\frac{r_p}{r_m}\Big)^p\frac{v(r_p)}{v(r_m)}\geq b\quad\text{and}\quad\Big(\frac{r_m}{r_p}\Big)^m\frac{v(r_m)}{v(r_p)}\leq b \tag{4.3}$$

In particular we have $|n-p|\geq(2^N-1)|p-m|$. Corollary 3.2 implies

$$\|f\|_v\leq 2bM_\infty(f,r_n)v(r_n) \tag{4.4}$$

whenever $f\in\operatorname{span}\{r^{|k|}\exp(ik\varphi) : |k|\text{ between }n\text{ and }m\}$.

Case $m<p<n$. Then, in view of Proposition 3.4(b), $\|V_{p,m}\|_v$ does not depend on $m$ or $p$ (see (4.1) and (4.3)). We may assume without loss of generality from now on that $m$ and $p$ are integers. Otherwise we take $[m]$ and $[p]$ instead.

Put $Q_1=P_{m,p-m}(\mathrm{id}-V_{p,m})$ ($P_{m,p-m}$ as in Lemma 4.3). Then, for $k\geq0$,

$$Q_1(z^k)=\begin{cases}z^k & \text{if } k=p+j(p-m)\text{ for some integer } j\geq0,\\ 0 & \text{else.}\end{cases} \tag{4.5}$$

Define $T_1 : H_N \to (Hv)_0$ by

$$T_1 z^j = \frac{z^{p+j(p-m)}}{r_n^{p+j(p-m)} v(r_n)}, \qquad j = 0, 1, \dots, N. \tag{4.6}$$

Since $p+N(p-m) = m+(N+1)(p-m) \le n$ (see (4.2)) we obtain $\|T_1\| \le 2b$ (see (4.4)).

Define $\widetilde{S}_1 : Hv \to L_\infty(\partial D)$ by

$$(\widetilde{S}_1 f)(z) = (Q_1 f)(r_n z^{1/(p-m)}) \cdot \overline{z}^{p/(p-m)} v(r_n), \qquad f \in Hv,$$

which implies

$$\widetilde{S}_1 z^k = \begin{cases} r_n^k z^j v(r_n) & \text{if } k = p + j(p-m) \text{ for some integer } j \ge 0, \\ 0 & \text{else} \end{cases} \tag{4.7}$$

(see (4.5)). Finally, put

$$S_1 = V_{N,0} \widetilde{S}_1. \tag{4.8}$$

Then (4.3), Proposition 3.4 and the definition of $Q_1$ imply that $\|S_1\| \le \gamma(b)$ for some $\gamma(b) > 0$ which does not depend on $m$, $n$ or $p$. (Recall that $N \le n$.) Moreover, (4.6) and (4.7) show that $S_1 T_1 = V_{N,0}|_{H_N}$.

Case $n < p < m$. Here $\|V_{m,p}\|_v$ does not depend on $m$ or $p$. As before, we may assume from now on that $m$ and $p$ are integers.

Put $Q_1 = P_{m,m-p} V_{m,p}$. Then

$$Q_1 z^k = \begin{cases} z^k & \text{if } k = p - j(m-p) \text{ for some integer } j \ge 0, \\ 0 & \text{else.} \end{cases}$$

Define $\widetilde{S}_1 : Hv \to L_\infty(\partial D)$ by

$$(\widetilde{S}_1 f)(z) = (Q_1 f)(r_n \overline{z}^{1/(m-p)}) \cdot z^{p/(m-p)} v(r_n), \qquad f \in Hv,$$

so that

$$\widetilde{S}_1 z^k = \begin{cases} r_n^k z^j v(r_n) & \text{if } k = p - j(m-p) \text{ for some integer } j \ge 0, \\ 0 & \text{else.} \end{cases}$$

Then put $S_1 = V_{N,0} \widetilde{S}_1$. Finally, define $T_1 : H_N \to (Hv)_0$ by

$$T_1 z^j = \frac{z^{p-j(m-p)}}{r_n^{p-j(m-p)} v(r_n)}, \qquad j = 0, 1, \dots, N.$$

As before we obtain $S_1 T_1 = V_{N,0}|_{H_N}$ and $\|S_1\| \le \gamma(b)$, $\|T_1\| \le 2b$.

In both cases we have $S_1 z^k = 0$ if $k$ is not between $n$ and $m$ (see (4.2), (4.7), (4.8) and take into account that $\min(m,n) + N|m-p| \le \max(m,n)$). Now, fix $M_1 > \max(M, m, n)$. Repeat the same procedure with $M_1$ instead of $M$ to find $m' \ge M_1$, $n' \ge M_1$ and linear operators $T_2 : H_N \to (Hv)_0$ and $S_2 : Hv \to H_N$ such that $\|S_2\| \le \gamma(b)$, $\|T_2\| \le 2b$, $S_2 T_2 = V_{N,0}|_{H_N}$, and

$S_2 z^k = 0$ if $k$ is not between $m'$ and $n'$. In particular

$$S_2 T_1 = 0 \quad \text{and} \quad S_1 T_2 = 0. \tag{4.9}$$

For a complex function $f$ put $(Wf)(z) = f(\overline{z})$. Finally, define $V : (H_N \oplus H_N)_\infty \to Hv$ by $V(f,g) = T_1 f + T_2 g$ and $U : Hv \to (H_N \oplus H_N)_\infty$ by

$$Uf = (S_1 f + z^N W S_2 f, S_2 f + z^N W S_1 f).$$

Then $\|U\| \le 2\gamma(b)$ and $\|V\| \le 4b$. It is easily seen that $UHv = \operatorname{span}\{(z^j, z^{N-j}) : j = 0, 1, \ldots, N\}$, which is isometrically isomorphic to $H_N$. Moreover, by (4.9),

$$\begin{aligned} UV(z^j, z^{N-j}) &= U(T_1 z^j + T_2 z^{N-j}) \\ &= (V_{N,0} z^j + z^N V_{N,0} \overline{z}^{N-j}, V_{N,0} z^{N-j} + z^N V_{N,0} \overline{z}^j) \\ &= (z^j, z^{N-j}). \end{aligned}$$

This implies that $Q = VU : Hv \to Hv$ is a projection and $d(QHv, H_N)$ and $\|Q\|_v$ depend only on $b$. The construction of $Q$ and $U$ furthermore shows that $Qz^k = 0$ if $k$ is neither between $m$ and $n$ nor between $m'$ and $n'$.

Now assume that, moreover, (C) holds. Then we can choose $m$, $m'$ and $n$, $n'$ such that, in addition,

$$\begin{aligned} &\min(m', n') \ge 3\max(m,n), \qquad \min(m,n) \ge d|n-m|, \\ &\min(m', n') \ge d|n' - m'| \end{aligned} \tag{4.10}$$

for some $d > 0$, say $m < n < m' < n'$. Again we may assume that $m$, $m'$, $n$, $n'$ are integers (otherwise take $[m]$, $[m']$, $[n]$, $[n']$ instead). Using (C) we can assume that

$$\frac{d}{2}(n-m) > 1 \quad \text{and} \quad \frac{d}{2}(n'-m') > 1. \tag{4.11}$$

Define $W : hv \to Hv$ by

$$W = R(V_{n+\frac{d}{2}(n-m),n} - V_{m,m-\frac{d}{2}(n-m)}) + R(V_{n'+\frac{d}{2}(n'-m'),n'} - V_{m',m'-\frac{d}{2}(n'-m')})$$

where $R$ is the Riesz projection. From (4.10) we infer that $n + 2^{-1}d(n-m) < m' - 2^{-1}d(n'-m')$. Lemma 3.3(b), (c) provides us with a constant $\alpha > 0$ such that

$$\begin{aligned} \|W\|_v &= \alpha\left(1 + \frac{(1+d)(n-m)}{m - \frac{d}{2}(n-m)} + 1 + \frac{(1+d)(n'-m')}{m' - \frac{d}{2}(n'-m')}\right) \\ &\le \alpha\left(2 + 4\,\frac{1+d}{d}\right). \end{aligned}$$

The construction yields $Wz^j = z^j$ if $m \le j \le n$ or $m' \le j \le n'$. Finally, define $\widehat{Q} : hv \to QHv$ by $\widehat{Q} = QW$. ■

We deduce

4.5. Corollary. *Under the assumptions of Proposition* 4.4 *the spaces* $Hv$ *and* $hv$ *each contain a complemented subspace isomorphic to* $H_\infty$ *while* $(Hv)_0$ *and* $(hv)_0$ *each contain a complemented subspace isomorphic to* $(\sum_n \oplus H_n)_0$.

*Proof.* Let $c$ be a constant such that $d(A, H_N) \le c$ and $\|Q\| \le c$ for $A$, $H_N$, $Q$ of Proposition 4.4. Observe that for every $\varepsilon, M > 0$ there is $K > 0$ such that if $f \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : |k| \le M\}$ and $g \in \operatorname{span}\{r^{|k|}\exp(i\varphi) : |k| \ge N\}$ with $N - M \ge K$, then

$$(1-\varepsilon)\max(\|f\|_v, \|g\|_v) \le \|f+g\|_v \le (1+\varepsilon)\max(\|f\|_v, \|g\|_v).$$

This follows since $\lim_{r\to a} v(r) = 0$.

Using Proposition 4.4, by induction, we find integers $0 < M_1 < M_2 < \cdots$ (sufficiently far apart), subspaces $A_k \subset (Hv)_0$ and projections $Q_k : Hv \to A_k$ (or $Q_k : hv \to A_k$) such that $d(A_k, H_k) \le c$, $\|Q_k\| \le c$ and, for $T_k = V_{M_{4k+3},M_{4k+2}} - V_{M_{4k+1},M_{4k}}$,

$$\frac{1}{2}\sup_k \|T_k f\|_v \le \Big\|\sum_k T_k f\Big\|_v \le 2\sup_k \|T_k f\|_v \tag{4.12}$$

for all $f \in hv$ and

$$T_k h = h \quad \text{ for all } h \in A_k,\ k = 1, 2, \ldots. \tag{4.13}$$

Put $Q = \sum_k Q_k T_k$. Then, in view of (4.12) and (4.13), $Q$ is a bounded projection from $(Hv)_0$ (or $(hv)_0$) onto the closure of $\operatorname{span}(\bigcup_{k=1}^\infty A_k)$ in $(Hv)_0$.

Moreover, if the $f_k \in A_k$ are such that $\sup_k \|f_k\|_v < \infty$ then, in view of (4.12) and Montel's theorem, $\sum_k f_k$ converges (uniformly on compact subsets) to a holomorphic function (called $\sum_k f_k$ again) with $\|\sum_k f_k\|_v < \infty$. Hence $\sum_k f_k \in Hv$. We conclude that $\{\sum_k f_k : f_k \in A_k,\ k = 1, 2, \ldots, \sup_k \|f_k\|_v < \infty\}$ is complemented in $Hv$ (or $hv$). Finally, this space is isomorphic to $(\sum_n \oplus H_n)_{(\infty)} \sim H_\infty$. ∎

**5. Norms equivalent to $\|\cdot\|_v$.** First we prove, for a given weight $v : [0, a[ \to \mathbb{R}_+$,

5.1. Lemma. *Fix* $b > 1$. *Then there are numbers* $0 < m_1 < m_2 < \cdots$ *such that*

$$\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \ge b \quad \text{and} \quad \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} \ge b,$$

*and, for each* $n$, *one of these inequalities is an equality*; *moreover*, $\lim_{n\to\infty} m_n = \infty$.

*Proof.* Start with $m_1 = 1$. Then assume that we already have $m_n$ for some $n$. Use $\lim_{M\to\infty} r_M^{m_n} v(r_M) = 0$ (by assumption on $v$) to find $M_0 > m_n$

with

$$\left(\frac{r_{m_n}}{r_M}\right)^{m_n}\frac{v(r_{m_n})}{v(r_M)} \geq b \quad \text{for any } M \geq M_0.$$

Fix $M \geq M_0$ with $r_M > r_{m_n}$ and use

$$\lim_{N\to\infty}\left(\frac{r_M}{r_{m_n}}\right)^{N}\frac{v(r_M)}{v(r_{m_n})} = \infty$$

to find $N > M$ with

$$\left(\frac{r_M}{r_{m_n}}\right)^{N}\frac{v(r_M)}{v(r_{m_n})} \geq b.$$

Since $r_N^N v(r_N) \geq r_M^N v(r_M)$ by definition of $r_N$, this implies

$$\left(\frac{r_N}{r_{m_n}}\right)^{N}\frac{v(r_N)}{v(r_{m_n})} \geq b \quad \text{and} \quad \left(\frac{r_{m_n}}{r_N}\right)^{m_n}\frac{v(r_{m_n})}{v(r_N)} \geq b.$$

Now let $N$ be the smallest number $> m_n$ which satisfies the last two inequalities and put $m_{n+1} = N$ (which exists since $m \mapsto r_m^m v(r_m)$ is continuous). Then, in particular, one of the above inequalities is an equality.

Finally, if $\sup_n m_n < \infty$ we would obtain

$$\begin{aligned} b &\leq \lim_{n\to\infty}\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}}\frac{v(r_{m_{n+1}})}{v(r_{m_n})} \\ &= \lim_{n\to\infty} r_{m_n}^{m_n - m_{n+1}}\frac{r_{m_{n+1}}^{m_{n+1}} v(r_{m_{n+1}})}{r_{m_n}^{m_n} v(r_{m_n})} = 1 \end{aligned}$$

by continuity, a contradiction. ∎

In the following let $b$, $m_n$ be the numbers of Lemma 5.1.

5.2. Proposition. *Assume that $b > 2$. Then there are constants $c_1, c_2 > 0$ such that, for any $f \in hv$ and $f_n = (V_{m_{n+1},m_n} - V_{m_n,m_{n-1}})f$, we have*

$$c_1 \sup_n \sup_{r_{m_{n-1}} \leq r \leq r_{m_{n+1}}} M_\infty(f_n, r)v(r) \leq \|f\|_v \leq c_2 \sup_n \sup_{r_{m_{n-1}} \leq r \leq r_{m_{n+1}}} M_\infty(f_n, r)v(r).$$

*Proof.* The left-hand inequality is clear since, according to Proposition 3.4, the operators $V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$ are uniformly bounded with respect to $\|\cdot\|_v$. It suffices to assume that $f$ is a trigonometric polynomial. We have $f = \sum_k f_k$ and $f_k \in \operatorname{span}\{r^{|j|}\exp(ij\varphi) : [m_{k-1}] + 1 \leq |j| \leq [m_{k+1}]\}$. Fix $n$ and $r$ such that $r_{m_{n-1}} \leq r \leq r_{m_n}$. Then we obtain, using Lemma 3.1,

$$M_\infty(f, r)v(r) \leq \sum_k M_\infty(f_k, r)v(r)$$

$$\le \sum_{k\le n-2}\left(\frac{r}{r_{m_{k+1}}}\right)^{m_{k+1}}\frac{v(r)}{v(r_{m_{k+1}})}\,M_\infty(f_k,r_{m_{k+1}})v(r_{m_{k+1}})$$
$$+\sum_{j=-1}^{1}M_\infty(f_{n+j},r)v(r)$$
$$+2\sum_{k\ge n+2}\left(\frac{r}{r_{m_{k-1}}}\right)^{m_{k-1}}\frac{v(r)}{v(r_{m_{k-1}})}\,M_\infty(f_k,r_{m_{k-1}})v(r_{m_{k-1}}).$$

We have

$$\left(\frac{r}{r_{m_{k+1}}}\right)^{m_{k+1}}\frac{v(r)}{v(r_{m_{k+1}})}\le\left(\frac{r_{m_{k+2}}}{r_{m_{k+1}}}\right)^{m_{k+1}}\frac{v(r_{m_{k+2}})}{v(r_{m_{k+1}})}\left(\frac{r_{m_{k+3}}}{r_{m_{k+2}}}\right)^{m_{k+2}}\frac{v(r_{m_{k+3}})}{v(r_{m_{k+2}})}$$
$$\cdots\left(\frac{r_{m_{n-1}}}{r_{m_{n-2}}}\right)^{m_{n-2}}\frac{v(r_{m_{n-1}})}{v(r_{m_{n-2}})}\left(\frac{r}{r_{m_{n-1}}}\right)^{m_{n-1}}\frac{v(r)}{v(r_{m_{n-1}})}\le\left(\frac{1}{b}\right)^{n-k-2}$$

if $k\le n-2$ and, similarly, if $k\ge n+2$,

$$\left(\frac{r}{r_{m_{k-1}}}\right)^{m_{k-1}}\frac{v(r)}{v(r_{m_{k-1}})}\le\left(\frac{r}{r_{m_{n+1}}}\right)^{m_{n+1}}\frac{v(r)}{v(r_{m_{n+1}})}\left(\frac{r_{m_{n+1}}}{r_{m_{n+2}}}\right)^{m_{n+2}}\frac{v(r_{m_{n+1}})}{v(r_{m_{n+2}})}$$
$$\cdots\left(\frac{r_{m_{k-2}}}{r_{m_{k-1}}}\right)^{m_{k-1}}\frac{v(r_{m_{k-2}})}{v(r_{m_{k-1}})}\le\left(\frac{1}{b}\right)^{k-1-n}.$$

Since $b>1$ we obtain

$$M_\infty(f,r)v(r)\le c_2\sup_n\sup_{r_{m_{n-1}}\le r\le r_{m_{n+1}}}M_\infty(f_n,r)v(r)$$

for some constant $c_2$ which depends only on $b$. ∎

Using Proposition 5.2 it might be possible to exactly describe all the weights $\widetilde{v}$ such that the differentiation operator $\mathrm{Diff}: Hv\to H\widetilde{v}$, where $\mathrm{Diff}(f)=f'$, is bounded.

We want to strengthen Proposition 5.2. To this end fix $n$ and find $p_n$, $q_n$ with $m_{n-1}<p_n<m_n<q_n<m_{n+1}$ such that

$$\left(\frac{r_{p_n}}{r_{m_n}}\right)^{p_n}\frac{v(r_{p_n})}{v(r_{m_n})}=\sqrt{b}\quad\text{and}\quad\left(\frac{r_{q_n}}{r_{m_n}}\right)^{q_n}\frac{v(r_{q_n})}{v(r_{m_n})}=\sqrt{b}.$$

(Again, use the continuity of $p\mapsto r_p^p v(r_p)$.)

5.3. Lemma. *Assume that $b>4$. Then there are universal constants $d_1,d_2>0$ such that, for every $n$, there is $s_n\in\{r_{m_n},r_{m_{n+1}}\}$ satisfying the following.*

*For every $f\in\mathrm{span}\{r^{|k|}\exp(ik\varphi): m_{n-1}\le|k|\le m_{n+1}\}$ and $u_n=V_{m_n,p_n}f$, $v_n=(V_{q_n,m_n}-V_{m_n,p_n})f$, $w_n=(\mathrm{id}-V_{q_n,m_n})f$, we have*

$$\|u_n\|_v\le d_2M_\infty(u_n,s_{n-1})v(s_{n-1}),$$
$$\|v_n\|_v\le d_2M_\infty(v_n,r_{m_n})v(r_{m_n}),$$
$$\|w_n\|_v\le d_2M_\infty(w_n,s_n)v(s_n).$$

*In particular*,

$$d_1 \max(M_\infty(u_n, s_{n-1})v(s_{n-1}), M_\infty(v_n, r_{m_n})v(r_{m_n}), M_\infty(w_n, s_n)v(s_n)) \le \|f\|_v$$
$$\le d_2 \max(M_\infty(u_n, s_{n-1})v(s_{n-1}), M_\infty(v_n, r_{m_n})v(r_{m_n}), M_\infty(w_n, s_n)v(s_n)).$$

*Proof.* According to the choice of $p_n$ and $q_n$, in view of Proposition 3.4, the norms of the operators $V_{m_n,p_n}$ and $V_{q_n,m_n}$ depend only on $b$. We have

$$u_n \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : m_{n-1} \le |k| \le m_n\},$$
$$v_n \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : p_n \le |k| \le q_n\},$$
$$w_n \in \operatorname{span}\{r^{|k|}\exp(ik\varphi) : m_n \le |k| \le m_{n+1}\}.$$

Fix $j$. If

$$\left(\frac{r_{m_{j+1}}}{r_{m_j}}\right)^{m_{j+1}} \frac{v(r_{m_{j+1}})}{v(r_{m_j})} = b$$

put $s_j = r_{m_j}$. If this is not the case then, in view of Lemma 5.1, we have

$$\left(\frac{r_{m_j}}{r_{m_{j+1}}}\right)^{m_j} \frac{v(r_{m_j})}{v(r_{m_{j+1}})} = b.$$

Here put $s_j = r_{m_{j+1}}$. Using Corollary 3.2 we deduce

$$\|u_n\|_v \le 2bM_\infty(u_n, s_{n-1})v(s_{n-1}),$$
$$\|v_n\|_v \le 2\max\Big(\sup_{r_{p_n}\le r\le r_{m_n}} M_\infty(v_n, r)v(r), \sup_{r_{m_n}\le r\le r_{q_n}} M_\infty(v_n, r)v(r)\Big)$$
$$\le 2\sqrt{b}M_\infty(v_n, r_{m_n})v(r_{m_n}),$$
$$\|w_n\|_v \le 2bM_\infty(w_n, s_n)v(s_n).$$

Since $f = u_n + v_n + w_n$ the result follows. ■

Combining Lemma 5.3 and Proposition 5.2 we obtain

5.4. Corollary. *Assume that $b > 4$. Then there are constants $c_1, c_2 > 0$, indices $0 \le k_1 \le k_2 \le \ldots$, radii $0 < t_1 \le t_2 \le \cdots$ and uniformly bounded linear operators*

$$T_n : hv \to \operatorname{span}\{r^{|j|}\exp(ij\varphi) : k_{n-2} < |j| \le k_{n+1}\}$$

*satisfying the following.*

*For every trigonometric polynomial $f$ we have $f = \sum_n T_n f$,*

$$c_1 \sup_n M_\infty(T_n f, t_n)v(t_n) \le \|f\|_v \le c_2 \sup_n M_\infty(T_n f, t_n)v(t_n)$$

*and $T_m T_n f = 0$ if $|n - m| > 4$.*

*Finally*,

$$\|h\|_v \le c_2 M_\infty(h, t_n)v(t_n) \quad \text{whenever } h \in T_n hv,\ n = 1, 2, \ldots.$$

**6. The Banach space geometry of $hv$ and $Hv$.** First we show

6.1. Lemma.

(a) *Let $m,n,p\in\mathbb{Z}_+$ with $m\le n\le p$. Then $H_m$ is isometrically isomorphic to a 2-complemented subspace of $(H_n\oplus H_p)_\infty$.*

(b) *Consider integers $0<m<n$ and let $B_{n,m}=\operatorname{span}\{r^{|j|}\exp(ij\varphi): j\in\mathbb{Z}$ and $m\le|j|\le n\}$ be endowed with the norm $M_\infty(\cdot,1)$. Then there is an integer $N>0$ such that $B_{n,m}$ is isometrically isomorphic to a 16-complemented subspace of $(H_N\oplus H_N)_\infty$.*

*Proof.* (a) For a complex function $f$ put $(Wf)(z)=f(\bar z)$. Identify $z^j\in H_m$ with $(z^j,z^{m-j})\in(H_n\oplus H_p)_\infty$. Put

$$P(f,g)=(V_{m,0}f+z^mWV_{m,0}g,V_{m,0}g+z^mWV_{m,0}f).$$

Then $P$ is a projection from $(H_n\oplus H_p)_\infty$ onto $\{(z^j,z^{m-j}):j=0,1,\ldots,m\}$, which is isometrically isomorphic to $H_m$. We have $\|P\|\le 2$.

(b) If $n\le 2m$, then, according to Lemma 3.3, the Riesz projection $R: B_{n,m}\to z^mH_{n-m}$ satisfies $\|R|_{B_{n,m}}\|_\infty\le 2$. Hence it follows that $d(B_{n,m},(H_{n-m}\oplus H_{n-m})_\infty)\le 4$, which yields (b) with $N=n-m$.

If $2m<n$, then Lemma 3.3 implies $\|V_{2n,n+m}\|_\infty\le(n-m)^{-1}(3n+m)$ $\le 7$. Let $W$ be as in (a). Consider the space

$$A=\operatorname{span}\{z^j: j\in\mathbb{Z}_+,\ 0\le j\le n-m \text{ or } n+m\le j\le 2n\},$$

endowed with the norm $M_\infty(\cdot,1)$, which is isometrically isomorphic to $B_{n,m}$. Define $P:(H_{2n}\oplus H_{2n})_\infty\to(H_{2n}\oplus H_{2n})_\infty$ by

$$\begin{aligned}P(f,g)&\\ &=(V_{n-m,0}f+(\mathrm{id}-V_{2n,n+m})f+z^{2n}WV_{n-m,0}g+z^{2n}W(\mathrm{id}-V_{2n,n+m})g,\\ &\quad V_{n-m,0}g+(\mathrm{id}-V_{2n,n+m})g+z^{2n}WV_{n-m,0}f+z^{2n}W(\mathrm{id}-V_{2n,n+m})f)\end{aligned}$$

We easily check that $P$ is a projection onto

$$\operatorname{span}\{(z^j,z^{n+m-j}): j\in\mathbb{Z}_+,\ 0\le j\le n-m \text{ or } n+m\le j\le 2n\},$$

which is isometrically isomorphic to $A$. (Observe that $0\le j\le n-m$ if and only if $n+m\le 2n-j\le 2n$.) We obtain $\|P\|\le 16$, which proves (b) with $N=2n$. ∎

6.2. Corollary. *Consider integers $0<m_k\le n_k$ with $\lim_{k\to\infty}(n_k-m_k)=\infty$ and let $B_k=\operatorname{span}\{r^{|j|}\exp(ij\varphi): j\in\mathbb{Z}_+$ and $m_k\le|j|\le n_k\}$ be endowed with $M_\infty(\cdot,1)$. Then*

$$\Big(\sum_k\oplus H_{n_k}\Big)_\infty\sim\Big(\sum_k\oplus B_k\Big)_\infty\sim H_\infty.$$

*Proof.* Put $X=(\sum_m\oplus H_m)_\infty$. Then $X$ is isomorphic to $H_\infty$ ([22]). Moreover, put $Y=(\sum_k\oplus H_{n_k})_\infty$. We conclude that $Y$ is complemented in $X$. Using Lemma 6.1(a) we see that $X$ is complemented in $Y$. Since $H_\infty \sim (H_\infty \oplus H_\infty \oplus \ldots)_\infty$ ([22]) this shows that $Y \sim H_\infty$. Using Lemma 6.1(a) we also see that every $H_m$ is 2-complemented in $(B_k \oplus B_{k'})_\infty$ for suitable $k$ and $k'$. Hence $(\sum_k \oplus B_k)_\infty$ contains a complemented subspace isomorphic to $H_\infty$. Finally, Lemma 6.1(b) implies that $(\sum_k \oplus B_k)_\infty$ is complemented in $H_\infty$. Hence $(\sum_k \oplus B_k)_\infty \sim H_\infty$. ∎

6.3. Proposition. *For any weight $v$ the spaces $hv$ and $Hv$ are isomorphic to complemented subspaces of $H_\infty$, while $(hv)_0$ and $(Hv)_0$ are isomorphic to complemented subspaces of $(\sum_n \oplus H_n)_0$.*

*Proof.* Let $c_1$, $c_2$, $k_m$, $t_m$ and $T_n : hv \to \operatorname{span}\{r^{|j|}\exp(ij\varphi) : k_{n-2} < |j| \le k_{n+1}\} =: B_n$ be as in Corollary 5.4, where $B_n$ is endowed with $\|\cdot\|_v$. Put $X = (\sum_n \oplus (B_n, \|\cdot\|_v))_\infty$. Define $U : X \to hv$ by $U(h_n) = \sum_n h_n$. Then, according to Corollary 5.4, $U$ is bounded. Indeed, we have $T_m h_n = 0$ if $|n-m| > 4$ and

$$\|U(h_n)\|_v \le c_2 \sup_m M_\infty\Big(T_m \sum_n h_n, t_m\Big) v(t_m) \le 6c_2^2 \sup_n \|h_n\|_v.$$

Conversely, define $V : hv \to X$ by

$$Vf = (T_n f)_{n=1}^\infty.$$

We have $\|V\| \le c_1^{-1}$ and $UV = \mathrm{id}_{hv}$, which implies that $hv$ is isomorphic to a complemented subspace of $X$.

If $\sup_n (k_{n+1} - k_{n-2}) < \infty$, then $\sup_n \dim B_n < \infty$ and hence $(\sum_n \oplus B_n)_\infty \sim l_\infty$. Since $l_\infty$ is complemented in $H_\infty$ the assertion of Proposition 6.3 follows.

If $\sup_n (k_{n+1} - k_{n-2}) = \infty$, then in view of Corollary 5.4 we have

$$\sup_n d((B_n, \|\cdot\|_v), (B_n, M_\infty(\cdot, 1))) < \infty$$

(since $(B_n, M_\infty(\cdot, t_n)v(t_n))$ is isometrically isomorphic to $(B_n, M_\infty(\cdot, 1))$). We conclude, by Corollary 6.2, that $X = (\sum_n \oplus B_n)_\infty$ is isomorphic to $H_\infty$. Again, the assertion follows in this case.

The proof for $Hv$ instead of $hv$ is identical. Here, instead of $B_n$, we consider $\operatorname{span}\{r^j \exp(ij\varphi) : k_{n-2} < j \le k_{n+1}\}$, which is isometrically isomorphic to $H_{k_{n+1} - k_{n-2} - 1}$.

Also the proof for $(Hv)_0$ and $(hv)_0$ instead of $Hv$ and $hv$ is identical. ∎

Corollary 4.5 and Proposition 6.3 together with the decomposition method ([12]) prove Theorems 1.1(b) and 1.2(b). Theorems 1.1(a), 1.2(a) and 1.3 follow from

6.4. Proposition. *Let $v$ satisfy* (B). *Then $Hv$ and $hv$ are isomorphic to $l_\infty$, while $(Hv)_0$ and $(hv)_0$ are isomorphic to $c_0$. Moreover, the Riesz projection $R : hv \to Hv$ is bounded.*

*Proof.* Let $m_n$ be the numbers of Lemma 5.1 with respect to some $b > 2$. Then, using (B) and Proposition 4.1, we obtain universal constants $\eta$, $\kappa$ and $c$, $d$ such that $m_{n+1} - m_{n-1} \le c$ or

$$\eta \le \frac{[m_{n+1}] - [m_n]}{[m_n] - [m_{n-1}]} \le \kappa \tag{6.1}$$

and

$$\max\left(\left(\frac{r_{m_{n+1}}}{r_{m_{n-1}}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_{n-1}})}, \left(\frac{r_{m_{n-1}}}{r_{m_{n+1}}}\right)^{m_{n-1}} \frac{v(r_{m_{n-1}})}{v(r_{m_{n+1}})}\right) \le d$$

for all $n$ with $m_{n-1} \ge c$. To prove the proposition it suffices to consider only those $n$ with $m_{n-1} \ge c$.

Put $T_n = V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$. By (6.1), Lemma 3.3(c), (d) the operators $T_n$ are uniformly bounded with respect to $M_\infty(\cdot, 1)$ and hence with respect to $\|\cdot\|_v$ and to the norms $M_\infty(\cdot, r_{m_n})v(r_{m_n})$. From Corollary 3.2 we deduce

$$\begin{aligned} \|T_n h\|_v &\le 2dM_\infty(T_n h, r_{m_{n+1}})v(r_{m_{n+1}}) \\ &\le 2d(\sup_n \|T_n\|_\infty) M_\infty(h, r_{m_{n+1}})v(r_{m_{n+1}}) \end{aligned} \tag{6.2}$$

whenever $h \in hv$.

Let $Y_n$ be the space of all harmonic functions on $r_{m_{n+1}}D$ whose radial limits are $L_\infty$-functions on $\{z \in \mathbb{C} : |z| = r_{m_{n+1}}\}$. On $Y_n$ we consider the norm $M_\infty(\cdot, r_{m_{n+1}})v(r_{m_{n+1}})$ which is equivalent to $M_\infty(\cdot, r_{m_{n+1}})$. Hence $Y_n$ is isometrically isomorphic to $L_\infty$. Note that the operators $V_{m,\widetilde{m}}$ make sense on $Y_n$ and $V_{m,\widetilde{m}}h$ is a trigonometric polynomial for every $h \in Y_n$.

If $m_{n+1} - m_{n-1} > c$ find finite-dimensional subspaces $X_n \subset Y_n$ with

$$V_{m_{n+2},m_{n+1}} Y_n \subset X_n \tag{6.3}$$

and $\sup_n d(X_n, l_\infty^{\dim X_n}) < \infty$. If $m_{n+1} - m_{n-1} \le c$ take $X_n = T_n hv$. Then $\dim X_n \le c$. Altogether we obtain $(\sum_n \oplus X_n)_0 \sim (\sum_n \oplus l_\infty^{\dim X_n})_0 \sim c_0$.

Define $U : (\sum_n \oplus X_n)_0 \to (hv)_0$ by $U(h_k) = \sum_k T_k h_k$. (The functions $T_k h_k$ are trigonometric polynomials and therefore can be regarded as elements of $hv$.) Since $T_n T_m = 0$ if $|n - m| \ge 2$ we have

$$T_n U(h_k) = T_n T_{n-1} h_{n-1} + T_n^2 h_n + T_n T_{n+1} h_{n+1}.$$

Hence $\|T_n U(h_k)\|_v \le c_1 \sup_{j=n-1,n,n+1} \|T_j h_j\|_v$ for a universal constant $c_1$. Proposition 5.2, (6.2) and the uniform boundedness of the $T_n$ imply that $U$ is bounded.

If $m_{n+1} - m_{n-1} \le c$ define, for $f = \sum_k \alpha_k r^{|k|} \exp(ik\varphi)$,

$$S_n f = \sum_{m_{n-1} < |k| \le m_{n+1}} \alpha_k r^{|k|} \exp(ik\varphi) \in X_n.$$

Otherwise put $S_n = (\mathrm{id} - V_{m_{n-2},m_{n-2}/2}) V_{m_{n+2},m_{n+1}}$. Define $V : (hv)_0 \to (\sum_n \oplus X_n)_0$ by $Vf = (S_n f)$, which makes sense in view of (6.3). Recall that, since $b > 2$ in view of Proposition 3.4, we have $\sup_n \|V_{m_{n+2},m_{n+1}}\|_v < \infty$. Therefore, $V$ is bounded. Moreover, $UVf = \sum_n T_n f = f$. This implies that $(hv)_0$ is isomorphic to a complemented subspace of $(\sum_n \oplus X_n)_0 \sim c_0$ and hence $(hv)_0 \sim c_0$ ([12]). In view of Proposition 5.2, (6.1) and Lemma 3.3 the Riesz projection $R : (hv)_0 \to (Hv)_0$ is bounded. As a consequence we also have $(Hv)_0 \sim c_0$.

To prove the result for $hv$ instead of $(hv)_0$ we proceed exactly as before. Define $U : (\sum_n \oplus X_n)_\infty \to hv$ by $U(h_k) = \sum_k T_k h_k$. From Proposition 5.2 and (6.2), looking at the Fourier series, we see that the series $\sum_k T_k h_k$ converges pointwise to a harmonic function (called $\sum_k T_k h_k$ again) with $\|\sum_k T_k h_k\|_v < \infty$. Hence $\sum_k T_k h_k \in hv$. The definition of $V$ can be repeated literally for the operator $hv \to (\sum_n \oplus X_n)_\infty$ with $UV = \mathrm{id}_{hv}$. Hence we obtain $hv \sim l_\infty$ and the Riesz projection $R : hv \to Hv$ is bounded. Therefore we also have $Hv \sim l_\infty$. (Alternatively, we could have used Proposition 5.2 or [1, 18] to see that $hv \sim (hv)_0^{**} \sim l_\infty$ and $Hv \sim (Hv)_0^{**} \sim l_\infty$.) ∎

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