STUDIA MATHEMATICA 175 (1) (2006)

On joint spectral radii in locally convex algebras

by

Andrzej Sołtysiak (Poznań)

Abstract. We present several notions of joint spectral radius of mutually commuting elements of a locally convex algebra and prove that all of them yield the same value in case the algebra is pseudo-complete. This generalizes a result proved by the author in 1993 for elements of a Banach algebra.

1. Notation and definitions. The term *locally convex algebra* will always mean a Hausdorff locally convex space over the complex field $\mathbb{C}$ which is a linear associative algebra with jointly continuous multiplication. We assume that a locally convex algebra A has a unit e. The topology of A can be given by a family $\mathcal{P}$ of seminorms which satisfy the following conditions (see [7]):

(i) for every $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ such that for all $a, b \in A$,

$$p(ab) \le q(a)q(b);$$

(ii) for every $p_1, \ldots, p_n \in \mathcal{P}$ there exists $q \in \mathcal{P}$ such that for all $a \in A$,

$$\max\{p_1(a), \ldots, p_n(a)\} \le q(a);$$

(iii) $p(e) = 1$ for all $p \in \mathcal{P}$.

For convenience of the reader we recall some basic facts from the spectral theory of locally convex algebras as introduced by G. R. Allan in [1].

Let A be a locally convex algebra. An element $a \in A$ is *bounded* if there exists a non-zero complex number λ such that the set $\{(\lambda a)^n : n = 1, 2, \ldots\}$ is a bounded subset of A or, what is the same, $(\lambda a)^n \to 0$ as $n \to \infty$. We denote by A_{b} the set of all bounded elements of the algebra A. An important role in this theory is played by the *radius of boundedness* $\beta(a)$ of an element a defined by the formula

$$\beta(a) = \inf\{\lambda > 0 : (\lambda^{-1}a)^n \to 0 \text{ as } n \to \infty\},$$

2000 *Mathematics Subject Classification*: Primary 46H05, 46J05; Secondary 46H30.
Key words and phrases: joint spectral radius, locally convex algebra, bounded element.

where as usual $\inf \emptyset = +\infty$. Clearly $a \in A_{\mathrm{b}}$ if and only if $\beta(a) < \infty$. Moreover for each $a \in A$ and every non-negative integer n,

$$\beta(a^n) = \beta(a)^n \tag{1}$$

(see [10]). Hence an element a is bounded if and only if a^n is bounded for all n. For bounded and commuting elements a and b we have

$$\beta(ab) \le \beta(a)\beta(b). \tag{2}$$

Let $\mathcal{B}$ denote the family of all subsets B of A such that

(i) B is absolutely convex and $B^2 \subset B$,
(ii) B is bounded and closed,
(iii) $e \in B$.

For each $B \in \mathcal{B}$ let $A(B)$ denote the subalgebra of A generated by B, i.e.

$$A(B) = \{\lambda a \colon \lambda \in \mathbb{C},\ a \in B\}.$$

$A(B)$ is a normed algebra with the norm defined by

$$\|a\|_B = \inf\{\lambda > 0 \colon a \in \lambda B\}.$$

The locally convex algebra A is *pseudo-complete* if each of the normed algebras $A(B)$ $(B \in \mathcal{B})$ with the norm $\|\cdot\|_B$ is a Banach algebra. Notice that pseudo-completeness is weaker than completeness or even sequential completeness (cf. [1, Prop. 2.6]). It is known (see [1, Cor. 2.11]) that if the algebra A is commutative and pseudo-complete then A_{b} is a subalgebra of A.

Let A be a locally convex algebra and let $a \in A$. The *Allan spectrum* of a in A, denoted by $\sigma_{\mathrm{b}}^A(a)$, or simply by $\sigma_{\mathrm{b}}(a)$, is the subset of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ defined in the following way:

(i) if $\lambda \neq \infty$, then $\lambda \in \sigma_{\mathrm{b}}(a)$ if $\lambda e - a$ has no inverse in A_{b};
(ii) $\infty \in \sigma_{\mathrm{b}}(a)$ if and only if $a \notin A_{\mathrm{b}}$.

The *spectral radius* of a, $r^A(a)$ or simply $r(a)$, is defined by

$$r(a) = \sup\{|\lambda| \colon \lambda \in \sigma_{\mathrm{b}}(a)\},$$

where $|\infty| = \infty$.

It is well known (see [1, Cor. 3.9 and Thm. 3.12]) that for an element a of a locally convex algebra A we have $\sigma_{\mathrm{b}}(a) \neq \emptyset$ and $\beta(a) \le r(a)$. Moreover, if A is pseudo-complete then $\sigma_{\mathrm{b}}(a)$ is closed (in $\overline{\mathbb{C}}$) and

$$\beta(a) = r(a). \tag{3}$$

From (3) it follows that the spectral radius of an element a of a pseudo-complete locally convex algebra A does not depend upon the algebra, i.e. it stays the same in any (closed and unital) subalgebra of A containing a.

Notice that (1) and (3) also imply that $r(a^n) = r(a)^n$ for every element a of such an algebra and each non-negative integer n. From (2) and (3) we get $r(ab) \le r(a)r(b)$ for bounded and commuting elements a and b in A.

We also need the notion of joint spectrum of elements in A_{b}. Let A be a commutative locally convex algebra and let $a=(a_1,\dots,a_n)$ be an n-tuple of elements of A_{b}. The *joint spectrum* of this tuple is defined as follows (cf. [2]):

$$\sigma_{A_{\mathrm{b}}}(a)=\sigma_{\mathrm{b}}(a)=\sigma_{\mathrm{b}}(a_1,\dots,a_n)=\Big\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n\colon e\notin\sum_{j=1}^{n}(\lambda_j e-a_j)A_{\mathrm{b}}\Big\}.$$

Notice that for a single element $a\in A_{\mathrm{b}}$ this notion coincides with the Allan spectrum of a.

It follows from [1, Lemma 6.6] (cf. also [2]) that for any n-tuple $a=(a_1,\dots,a_n)$ of bounded elements in a commutative pseudo-complete locally convex algebra A,

$$\sigma_{\mathrm{b}}(a)=\{(\varphi(a_1),\dots\varphi(a_n))\colon \varphi\in\mathfrak{M}(A_{\mathrm{b}})\},\tag{4}$$

where $\mathfrak{M}(A_{\mathrm{b}})$ denotes the set of all non-zero multiplicative linear functionals on the algebra A_{b}.

From (4) it follows that the joint spectrum $\sigma_{\mathrm{b}}(a_1,\dots,a_n)$ has *the spectral mapping property* on A_{b} with respect to polynomials, i.e.

$$\sigma_{\mathrm{b}}(w(a_1,\dots,a_n))=w(\sigma_{\mathrm{b}}(a_1,\dots,a_n)),$$

where $w=(w_1,\dots,w_m)$ is an arbitrary m-tuple of polynomials over $\mathbb{C}$ in n indeterminates.

If $a=(a_1,\dots,a_n)$ is an n-tuple of elements of a locally convex algebra A, then $\mathcal{A}(a_1,\dots,a_n)$, or briefly $\mathcal{A}(a)$, is the unital algebra generated by these elements, i.e. $\mathcal{A}(a)$ is the smallest closed subalgebra of A containing $a_1,\dots,a_n$ and the unit. If A is pseudo-complete then so is $\mathcal{A}(a)$. Notice that $\mathcal{A}(a)_{\mathrm{b}}=A_{\mathrm{b}}\cap\mathcal{A}(a)$. The algebra $\mathcal{A}(a)$ is commutative if the elements a_j are pairwise commuting. Let $a=(a_1,\dots,a_n)$ be an n-tuple of mutually commuting bounded elements of a locally convex algebra A. Let $\widehat{\sigma}(a)$ denote the joint spectrum of $a=(a_1,\dots,a_n)$ in the algebra $\mathcal{A}(a)$, i.e. $\widehat{\sigma}(a)=\sigma_{\mathrm{b}}^{\mathcal{A}(a)}(a)$. If A is pseudo-complete we have

$$\widehat{\sigma}(a)=\{(\varphi(a_1),\dots,\varphi(a_n))\colon \varphi\in\mathfrak{M}(\mathcal{A}(a)_{\mathrm{b}})\}.\tag{5}$$

2. Spectral radius and joint spectral radius. Let us recall several notions of spectral radius (see [3], [16]) in locally convex algebras.

Let A be a locally convex algebra and let $a\in A$. Consider the following formulas:

$$\widehat{r}(a)=\sup\{\limsup_{n\to\infty}p(a^n)^{1/n}\colon p\in\mathcal{P}\},$$
$$r'(a)=\sup\{\limsup_{n\to\infty}|f(a^n)|^{1/n}\colon f\in A'\},$$

where A' denotes the dual space of A.

It is well known that $\widehat{r}(a) = r'(a) = \beta(a)$ (see [1, Prop. 2.18]) and moreover, if the algebra A is pseudo-complete, then in view of (3),

$$\widehat{r}(a) = r'(a) = \beta(a) = r(a). \tag{6}$$

REMARK 1. W. Żelazko in [16] proposed another definition of spectrum of an element of a locally convex convex algebra. Namely, let

$$\sigma(a) = \{\lambda \in \mathbb{C} \colon \lambda e - a \text{ has no inverse in } A\},$$
$$\sigma_{\mathrm{d}}(a) = \{\lambda_0 \in \mathbb{C} \colon (\lambda e - a)^{-1} \text{ is discontinuous at } \lambda = \lambda_0\},$$
$$\sigma_\infty(a) = \begin{cases} \{\infty\} & \text{if and only if } (e - \lambda a)^{-1} \text{ is discontinuous at } \lambda = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The *extended spectrum* of a is the set

$$\Sigma(a) = \sigma(a) \cup \sigma_{\mathrm{d}}(a) \cup \sigma_\infty(a)$$

and the *extended spectral radius* of a is

$$R(a) = \sup\{|\lambda| \colon \lambda \in \Sigma(a)\}.$$

The Allan spectrum $\sigma_{\mathrm{b}}(a)$ need not coincide with the extended spectrum $\Sigma(a)$. However, Żelazko proved that $\widehat{r}(a) = r'(a) = R(a)$ in a complete locally convex algebra (cf. [16]; in fact it is enough to assume that the algebra is sequentially complete). Therefore if a is an element of a sequentially complete locally convex algebra A, then

$$\widehat{r}(a) = r'(a) = \beta(a) = r(a) = R(a).$$

The notion of spectral radius can be generalized to the case of a finite or bounded set of elements (see [4]–[6], [8], [11]–[15]). We concentrate on the case of a finite set of commuting elements in order to keep links between joint spectral radius and joint spectrum. Let us recall the definitions of joint spectral radii used in the theory of Banach algebras. Let A be a Banach algebra and let $a = (a_1, \ldots, a_n)$ be an n-tuple of pairwise commuting elements of A. The *geometric joint spectral radius* of a, $r(a)$, is defined as follows (see [5]):

$$r(a) = \sup\{|\lambda|_\infty \colon (\lambda_1, \ldots, \lambda_n) \in \sigma_{\mathrm{H}}(a)\},$$

where $|\lambda|_\infty = \max\{|\lambda_j| \colon j = 1, \ldots, n\}$ and $\sigma_{\mathrm{H}}(a)$ denotes the Harte spectrum of the elements $a_1, \ldots, a_n$. In fact, the Harte spectrum can be replaced by any other "reasonable" joint spectrum (cf. [5], [9, Prop. 2, p. 288], and [14]). The *Rota–Strang joint spectral radius* of a is defined as (see [12])

$$\widehat{r}(a) = \lim_{s\to\infty} \max_{|\alpha|=s} \|a_1^{\alpha_1} \cdots a_n^{\alpha_n}\|^{1/s},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\|\cdot\|$ stands for the norm of the algebra A. The *Berger–Wang joint spectral radius* is defined as (see [4])

$$r_*(a) = \lim_{s\to\infty} \max_{|\alpha|=s} r(a_1^{\alpha_1} \cdots a_n^{\alpha_n})^{1/s}.$$

It was shown in [15] (cf. also [9, Thm. 5, p. 290]) that if $a = (a_1, \ldots, a_n)$ is a commuting n-tuple of elements of a Banach algebra A, then

$$r(a) = r_*(a) = \widehat{r}(a).$$

Moreover, in [6] E. Yu. Emel'yanov and Z. Ercan proved that all these radii are equal to

$$r'(a) = \sup\{\limsup_{s\to\infty} \max_{|\alpha|=s} |f(a_1^{\alpha_1} \cdots a_n^{\alpha_n})|^{1/s} \colon f \in A'\}.$$

We propose the following generalizations of these formulas to the case of a locally convex algebra.

Definition. Let A be a locally convex algebra and let $a = (a_1, \ldots, a_n)$ be an n-tuple of mutually commuting elements of A. Define

$$r(a) = \begin{cases} \sup\{|\lambda|_\infty \colon (\lambda_1, \ldots, \lambda_n) \in \widehat{\sigma}(a)\} & \text{if } a_j \in A_{\mathrm{b}} \text{ for } j = 1, \ldots, n, \\ \infty & \text{otherwise;} \end{cases}$$

$$r_*(a) = \limsup_{s\to\infty} \max_{|\alpha|=s} r(a_1^{\alpha_1} \ldots a_n^{\alpha_n})^{1/s};$$

$$\widehat{r}(a) = \sup\{\limsup_{s\to\infty} \max_{|\alpha|=s} p(a_1^{\alpha_1} \ldots a_n^{\alpha_n})^{1/s} \colon p \in \mathcal{P}\};$$

$$r'(a) = \sup\{\limsup_{s\to\infty} \max_{|\alpha|=s} |f(a_1^{\alpha_1} \ldots a_n^{\alpha_n})|^{1/s} \colon f \in A'\}.$$

Remark 2. If the algebra A is pseudo-complete, then the upper limit in the definition of $r_*(a)$ can be replaced by the limit. To show this let $a = (a_1, \ldots, a_n)$ be an n-tuple of mutually commuting elements of a locally convex pseudo-complete algebra A. If all elements a_j ($j = 1, \ldots, n$) are bounded, then submultiplicativity of the spectral radius for bounded and commuting elements implies

$$\max_{|\alpha|=s+t} r(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) \le \max_{|\mu|=s} r(a_1^{\mu_1} \cdots a_n^{\mu_n}) \max_{|\nu|=t} r(a_1^{\nu_1} \cdots a_n^{\nu_n})$$

and by the standard technique (see e.g. [9, Lemma 21, p. 8]) the limit of $\max_{|\alpha|=s} r(a_1^{\alpha_1} \ldots a_n^{\alpha_n})^{1/s}$ exists.

If the n-tuple $a = (a_1, \ldots, a_n)$ contains at least one unbounded element, say a_j, then by (1) all of its powers a_j^s are unbounded and so $r(a_j^s) = \infty$ for all s. Since

$$\max_{|\alpha|=s} r(a_1^{\alpha_1} \ldots a_n^{\alpha_n}) \ge r(a_j^s)$$

it follows that the limit of $\max_{|\alpha|=s} r(a_1^{\alpha_1} \ldots a_n^{\alpha_n})^{1/s}$ is ∞.

Remark 3. In the definition of $\widehat{r}(a)$ one can replace $\mathcal{P}$ by the set of all continuous seminorms on A. This will not change the value of $\widehat{r}(a)$.

3. Main result

THEOREM. *Let A be a pseudo-complete locally convex algebra and let $a = (a_1, \ldots, a_n)$ be an n-tuple of mutually commuting elements of A. Then*

$$r(a) = \max_{j=1,\ldots,n} r(a_j) = \max_{j=1,\ldots,n} \beta(a_j) = r_*(a) = r'(a) = \widehat{r}(a). \tag{7}$$

Proof. Notice first that if $a_j \notin A_{\mathrm{b}}$ for some j ($j = 1, \ldots, n$), then in view of (1), $a_j^s \notin A_{\mathrm{b}}$ for every positive integer s and by (6) we have

$$r(a_j^s) = \beta(a_j^s) = \widehat{r}(a_j^s) = r'(a_j^s) = \infty.$$

This implies that all expressions in (7) are equal to ∞.

From now on we assume that $a_1, \ldots, a_n \in A_{\mathrm{b}}$. Let $j \in \{1, \ldots, n\}$. The relations

$$r(a_j)^s = r(a_j^s) \le \max_{|\alpha|=s} r(a_1^{\alpha_1} \cdots a_n^{\alpha_n})$$

imply

$$\max_{j=1,\ldots,n} r(a_j) \le r_*(a_1, \ldots, a_n). \tag{8}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $|\alpha| = s$. Since

$$r(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) \le r(a_1)^{\alpha_1} \cdots r(a_n)^{\alpha_n} \le \max_{j=1,\ldots,n} r(a_j)^s,$$

we have

$$r_*(a_1, \ldots, a_n) \le \max_{j=1,\ldots,n} r(a_j). \tag{9}$$

From (3), (8), and (9) we obtain

$$\max_{j=1,\ldots,n} r(a_j) = \max_{j=1,\ldots,n} \beta(a_j) = r_*(a_1, \ldots, a_n). \tag{10}$$

Now we prove that

$$r(a_1, \ldots, a_n) = r_*(a_1, \ldots, a_n). \tag{11}$$

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \widehat{\sigma}(a_1, \ldots, a_n)$, then $\lambda_j \in \sigma_{\mathrm{b}}^{\mathcal{A}(a)}(a_j)$ by (5). Therefore $|\lambda_j| \le r(a_j)$ and

$$|\lambda|_\infty = \max_{j=1,\ldots,n} |\lambda_j| \le \max_{j=1,\ldots,n} r(a_j) = r_*(a_1, \ldots, a_n).$$

Consequently,

$$r(a_1, \ldots, a_n) \le r_*(a_1, \ldots, a_n). \tag{12}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $|\alpha| = s$. The spectral mapping property of $\sigma_b^{\mathcal{A}(a)}$ implies that

$$\sigma_{\mathrm{b}}^{\mathcal{A}(a)}(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) = w(\sigma_{\mathrm{b}}^{\mathcal{A}(a)}(a_1, \ldots, a_n)) = w(\widehat{\sigma}(a_1, \ldots, a_n)),$$

where w is the polynomial $w(z_1, \ldots, z_n) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Take any $\lambda \in \sigma^{\mathcal{A}(a)}(a_1^{\alpha_1} \cdots a_n^{\alpha_n})$. There exists $\mu = (\mu_1, \ldots, \mu_n) \in \widehat{\sigma}(a_1, \ldots, a_n)$ such that

$\lambda = \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n}$. We have

$$|\lambda| = |\mu_1|^{\alpha_1} \cdots |\mu_n|^{\alpha_n} \le |\mu|_\infty^s \le r(a_1, \ldots, a_n)^s.$$

Since λ was an arbitrary element of $\sigma_{\mathrm{b}}^{\mathcal{A}(a)}(a_1^{\alpha_1} \cdots a_n^{\alpha_n})$ we get

$$r(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) \le r(a_1, \ldots, a_n)^s$$

and finally

$$r_*(a_1, \ldots, a_n) = \lim_{s\to\infty} \max_{|\alpha|=s} r(a_1^{\alpha_1} \cdots a_n^{\alpha_n})^{1/s} \le r(a_1, \ldots, a_n). \tag{13}$$

Equality (11) follows from (12) and (13).

The next step in the proof is the inequality

$$\widehat{r}(a_1, \ldots, a_n) \le r_*(a_1, \ldots, a_n). \tag{14}$$

Notice first that it is enough to prove that

$$r_*(a_1, \ldots, a_n) < 1 \;\Rightarrow\; \widehat{r}(a_1, \ldots, a_n) \le 1. \tag{15}$$

Indeed, if we take an $\varepsilon > 0$ and the elements $b_j = (r_*(a_1, \ldots, a_n) + \varepsilon)^{-1} a_j$ $(j = 1, \ldots, n)$, then $r_*(b_1, \ldots, b_n) < 1$ and hence by (15) we get $\widehat{r}(b_1, \ldots, b_n)$ ≤ 1. But

$$\widehat{r}(b_1, \ldots, b_n) = \frac{\widehat{r}(a_1, \ldots, a_n)}{r_*(a_1, \ldots, a_n) + \varepsilon},$$

which gives

$$\widehat{r}(a_1, \ldots, a_n) \le r_*(a_1, \ldots, a_n) + \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, (14) follows.

Now we proceed to the proof of (15). Let $r_*(a_1, \ldots, a_n) < 1$. By (10) we have $\beta(a_j) < 1$ for $j = 1, \ldots, n$. Therefore $a_j^s \to 0$ as $s \to \infty$. This implies that for every seminorm $\widetilde{q} \in \mathcal{P}$ there exists $s_0 = s_0(\widetilde{q})$ such that for every $s \ge s_0$ and each $j = 1, \ldots, n$ we have

$$\widetilde{q}(a_j^s) < 1.$$

Take any $p \in \mathcal{P}$. There exists $q \in \mathcal{P}$ such that for all $x_1, \ldots, x_n \in A$,

$$p(x_1 \cdots x_n) \le q(x_1) \cdots q(x_n).$$

Thus for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have

$$p(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) \le q(a_1^{\alpha_1}) \cdots q(a_n^{\alpha_n}).$$

Take $\widetilde{q} \in \mathcal{P}$ such that

$$q(xy) \le \widetilde{q}(x)\widetilde{q}(y) \quad \text{ for all } x, y \in A.$$

Set $s_0 = s_0(\widetilde{q})$. Let $\alpha_j = k_j s_0 + l_j$ with k_j, $l_j \in \mathbb{Z}_+$ and $0 \le l_j < s_0$, and let

$M_j = \max\{1, \widetilde{q}(a_j), \ldots, \widetilde{q}(a_j^{s_0-1})\}$ $(j = 1, \ldots, n)$. Then

$$\begin{aligned} p(a_1^{\alpha_1} \cdots a_n^{\alpha_n}) &\le q(a_1^{\alpha_1}) \cdots q(a_n^{\alpha_n}) \\ &\le \widetilde{q}(a_1^{k_1 s_0}) \widetilde{q}(a_1^{l_1}) \cdots \widetilde{q}(a_n^{k_n s_0}) \widetilde{q}(a_1^{l_n}) \\ &\le M_1 \cdots M_n = M. \end{aligned}$$

Consequently,

$$\limsup_{s\to\infty} \max_{|\alpha|=s} p(a_1^{\alpha_1} \cdots a_n^{\alpha_n})^{1/s} \le 1,$$

which gives $\widehat{r}(a_1, \ldots, a_n) \le 1$ and completes the proof of (15).

Now we prove that

$$r'(a_1, \ldots, a_n) \le \widehat{r}(a_1, \ldots, a_n). \tag{16}$$

For every $f \in A'$ there exist a constant $M > 0$ and a seminorm $p \in \mathcal{P}$ such that

$$|f(x)| \le Mp(x) \quad \text{for all } x \in A.$$

Thus for each multi-index α with $|\alpha| = s$ we have

$$|f(a_1^{\alpha_1} \cdots a_n^{\alpha_n})|^{1/s} \le M^{1/s} p(a_1^{\alpha_1} \cdots a_n^{\alpha_n})^{1/s}.$$

This implies

$$\begin{aligned} \limsup_{s\to\infty} \max_{|\alpha|=s} |f(a_1^{\alpha_1} \cdots a_n^{\alpha_n})|^{1/s} &\le \limsup_{s\to\infty} \max_{|\alpha|=s} p(a_1^{\alpha_1} \cdots a_n^{\alpha_n})^{1/s} \\ &\le \widehat{r}(a_1, \ldots, a_n), \end{aligned}$$

which gives (16).

To conclude the proof we have to show

$$r_*(a_1, \ldots, a_n) \le r'(a_1, \ldots, a_n). \tag{17}$$

Let $\lambda > r'(a_1, \ldots, a_n)$. Notice that in view of (16), (14), and (10) such a λ exists. Take any $f \in A'$. Then there exists s_0 such that for every $s \ge s_0$ and every multi-index α with $|\alpha| = s$ we have

$$|f(a_1^{\alpha_1} \cdots a_n^{\alpha_n})|^{1/s} < \lambda.$$

In particular for every $j \in \{1, \ldots, n\}$ and all $s \ge s_0$ we get

$$|f(a_j^s)|^{1/s} < \lambda.$$

Hence

$$|f((\lambda^{-1} a_j)^s)| < 1.$$

This implies that the sequence $((\lambda^{-1} a_j)^s)_{s=1}^{\infty}$ is weakly bounded, therefore bounded and $\lambda \ge \beta(a_j)$. Finally,

$$\lambda \ge \max_{j=1,\ldots,n} \beta(a_j) = r_*(a_1, \ldots, a_n).$$

Since $\lambda > r'(a_1, \ldots, a_n)$ was arbitrary we get (17). ■

REMARK 4. From the proof of the Theorem it follows that the equalities

$$\max_{j=1,\dots,n} \beta(a_j) = r'(a) = \widehat{r}(a) \tag{18}$$

hold true in any locally convex algebra. Since without the assumption of completeness there is no satisfactory spectral theory for locally convex algebras one cannot expect equalities in (7) in that case.

To see that (7) need not to be true in a non-complete algebra it is enough to consider the algebra $\mathcal{H}(\mathbb{C})$ of entire functions on the complex plane with the norm $\|f\| = \max\{|f(z)|: |z| \le 1\}$. It is a commutative normed algebra in which

$$r(f) = \max_{j=1,\dots,n} r(f_j) = r_*(f) = \infty$$

for any n-tuple $f = (f_1, \dots, f_n)$ containing a non-constant function but (18) is finite for such an n-tuple.

REMARK 5. V. Müller ([8], [9]) also considered formulas for the joint spectral radii of elements of a Banach algebra in which the ℓ_∞ norm in $\mathbb{C}^n$ is replaced by an ℓ_p norm $(1 \le p < \infty)$.

It is also possible to give analogous formulas in the locally convex case. We conjecture that a counterpart of Müller's result ([8, Thm. 3]) in a locally convex case is also true.

References

[1] G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. (3) 15 (1965), 399–421.

[2] G. R. Allan, H. G. Dales, and J. P. McClure, *Pseudo-Banach algebras*, Studia Math. 40 (1971), 55–69.

[3] H. Arizmendi-Peimbert and K. Jarosz, *Extended spectral radius in topological algebras*, Rocky Mountain J. Math. 23 (1993), 1179–1195.

[4] M. A. Berger and Y. Wang, *Bounded semigroups of matrices*, Linear Algebra Appl. 166 (1992), 21–27.

[5] M. Chō and W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. 21 (1992), 251–258.

[6] E. Yu. Emel'yanov and Z. Ercan, *A formula for the joint local spectral radius*, Proc. Amer. Math. Soc. 132 (2003), 1449–1451.

[7] A. Fernández and V. Müller, *Renormalization of Banach and locally convex algebras*, Studia Math. 96 (1990), 237–242.

[8] V. Müller, *On the joint spectral radius*, Ann. Polon. Math. 66 (1997), 173–182.

[9] —, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Birkhäuser, Basel, 2003.

[10] L. Oubbi, *Further radii in topological algebras*, Bull. Belg. Math. Soc. 9 (2002), 279–292.

[11] P. Rosenthal and A. Sołtysiak, *Formulas for the joint spectral radius of non-commuting Banach algebra elements*, Proc. Amer. Math. Soc. 123 (1995), 2705–2708.

[12] G.-C. Rota and W. G. Strang, *A note on the joint spectral radius*, Indag. Math. 22 (1960), 379–381.

[13] V. S. Shulman and Yu. V. Turovskii, *Joint spectral radius, operator semigroups, and a problem of W. Wojtyński*, J. Funct. Anal. 177 (2000), 383–441.

[14] A. Sołtysiak, *On a certain class of subspectra*, Comment. Math. Univ. Carolin. 20 (1991), 715–721.

[15] —, *On the joint spectral radii of commuting Banach algebra elements*, Studia Math. 105 (1993), 93–99.

[16] W. Żelazko, *Metric generalizations of Banach algebras*, Dissertationes Math. (Rozprawy Mat.) 47 (1965).

Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87
61-614 Poznań, Poland
E-mail: asoltys@amu.edu.pl

Received June 3, 2005
Revised version April 12, 2006 (5668)