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Subharmonicity in von Neumann algebras

by

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with unit $1_{\mathcal{M}}$. Let τ be a faithful, normal, semifinite trace on $\mathcal{M}$. Given $x \in \mathcal{M}$, denote by $\mu_t(x)_{t\geq 0}$ the generalized s-numbers of x, defined by

$$\mu_t(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(1_{\mathcal{M}} - e) \leq t\} \quad (t \geq 0).$$

We prove that, if D is a complex domain and $f : D \to \mathcal{M}$ is a holomorphic function, then, for each $t \geq 0$, $\lambda \mapsto \int_0^t \log \mu_s(f(\lambda))\, ds$ is a subharmonic function on D. This generalizes earlier subharmonicity results of White and Aupetit on the singular values of matrices.

1. Introduction. Let D be a domain in $\mathbb{C}$ and let $M_n(\mathbb{C})$ denote the algebra of complex $n \times n$ matrices. It is elementary to show that, if $f : D \to M_n(\mathbb{C})$ is a holomorphic function, then $\lambda \mapsto \log \varrho(f(\lambda))$ is a subharmonic function on D, where $\varrho(x)$ denotes the spectral radius of x. Vesentini [12] extended this result to holomorphic functions $f : D \to A$, where A is a general Banach algebra. Vesentini's theorem has many interesting applications; an account of these can be found in Chapter 5 of [1].

More recently, White [13] and Aupetit [2] independently discovered a subharmonicity theorem for the singular values of matrices. Given $x \in M_n(\mathbb{C})$, let us write $s_1(x), \ldots, s_n(x)$ for the singular values of x, namely the eigenvalues of $|x| := (x^*x)^{1/2}$ listed in decreasing order. White and Aupetit showed that, if $f : D \to M_n(\mathbb{C})$ is holomorphic, then

$$\lambda \mapsto \sum_{j=1}^{k} \log s_j(f(\lambda))$$

is subharmonic on D, for each k with $1 \leq k \leq n$. Notice that, as pointed out in [2], the individual functions $\lambda \mapsto \log s_j(f(\lambda))$ need not be subhar-

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monic if $j \geq 2$. Aupetit proved his theorem in response to a question of O. Nevanlinna, who subsequently exploited the result in his development of a value-distribution theory for matrices (see for example [11, §5]).

The proofs of White and Aupetit (which are essentially the same) rely heavily on the fact that a matrix has a discrete spectrum, and so do not readily generalize to other algebras. There is an alternative method of proof which avoids this problem. The idea is to reduce to the case $k = n$ by considering functions of the form $\lambda \mapsto ef(\lambda)e$, for suitable projections e, and then to observe that $\sum_{j=1}^{n} \log s_j((f(\lambda)) = \log|\det(f(\lambda))|$, which is clearly subharmonic. Our aim in this article is to exploit this general idea to extend the White–Aupetit theorem to the case of holomorphic functions $f : D \to \mathcal{M}$, where $\mathcal{M}$ is an arbitrary semifinite von Neumann algebra.

2. Notation and statement of results. Let $\mathcal{M}$ be a von Neumann algebra. Denote by $1_{\mathcal{M}}$ the unit of $\mathcal{M}$, and by $\mathcal{M}^+$ the cone of positive elements of $\mathcal{M}$. A *trace* on $\mathcal{M}$ is a function $\tau : \mathcal{M}^+ \to [0, \infty]$ with the following properties:

- $\tau(x + y) = \tau(x) + \tau(y) \quad (x, y \in \mathcal{M}^+)$
- $\tau(\alpha x) = \alpha\tau(x) \quad (x \in \mathcal{M}^+,\ \alpha \in \mathbb{R}^+)$
- $\tau(x^*x) = \tau(xx^*) \quad (x \in \mathcal{M})$.

A trace τ is said to be *normal* if, for every bounded, increasing net $\{x_\alpha\}$, we have $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$. It is *finite* if $\tau(1_{\mathcal{M}}) < \infty$, and *semifinite* if $\tau(x) = \sup\{\tau(y) : 0 \leq y \leq x,\ \tau(y) < \infty\}$. Finally, it is *faithful* if $\tau(x) = 0$ implies $x = 0$. A classic reference for von Neumann algebras and traces is [4].

Let $\mathcal{M}$ be a von Neumann algebra, equipped with a faithful, normal, semifinite trace τ. Given $x \in \mathcal{M}$ and $t \in \mathbb{R}^+$, we define

$$\mu_t(x) := \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(1_{\mathcal{M}} - e) \leq t\}. \tag{1}$$

This notion was introduced by Murray and von Neumann [10], and has since been further developed by Fack [5], Fack–Kosaki [6] and Hiai–Nakamura [9]. The μ_t are called *generalized s-numbers*. They generalize the usual notion of singular values in the sense that, if $\mathcal{M}$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$ and τ is the usual trace, then, for a compact operator $x \in \mathcal{M}$, we have $\mu_t(x) = s_j(x)$ for $t \in [j-1, j)$ (where we agree to define $s_j(x) = 0$ if $j > \dim \mathcal{H}$) (see [5, Exemple 1.2.2]). Many results about singular values extend naturally to the context of generalized s-numbers, most notably the Weyl inequalities.

The following result is our main theorem.

THEOREM 2.1. *Let $\mathcal{M}$ be a von Neumann algebra, equipped with a faithful, normal, semifinite trace τ. Let D be a complex domain and let*

$f : D \to \mathcal{M}$ be a holomorphic function. Then, for each $t \geq 0$,

$$\lambda \mapsto \int_0^t \log \mu_s(f(\lambda))\, ds$$

is a subharmonic function on D.

In view of the remarks preceding the theorem, we immediately obtain the following corollary.

Corollary 2.2. *Let $\mathcal{H}$ be a Hilbert space, and let $f : D \to \mathcal{K}(\mathcal{H})$ be a holomorphic function, where $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$. Then, for each positive integer k,*

$$\lambda \mapsto \sum_{j=1}^{k} \log s_j(f(\lambda))$$

is subharmonic on D. ∎

In particular, when $\dim \mathcal{H} = n$, we recover the theorem of White and Aupetit mentioned in the introduction (in fact White proved Corollary 2.2 in its general form).

A further example is furnished by the commutative von Neumann algebra $\mathcal{M} = L^\infty(\Omega, \nu)$, where ν is a σ-finite measure on a set Ω, and $\tau(\phi) = \int \phi\, d\nu$ ($\phi \in \mathcal{M}^+$). In this case, $\mu_s(\phi) = \phi^*(s)$, where ϕ^* denotes the non-decreasing rearrangement of $|\phi|$ (see [5, Exemple 1.2.1]). We thus obtain the following corollary.

Corollary 2.3. *Let ν be a σ-finite measure on Ω, and let $\lambda \mapsto \phi_\lambda : D \to L^\infty(\Omega, \nu)$ be a holomorphic map. Then, for each $t \geq 0$,*

$$\lambda \mapsto \int_0^t \log \phi_\lambda^*(s)\, ds$$

is subharmonic on D, where ϕ_λ^* denotes the non-decreasing rearrangement of $|\phi_\lambda|$. ∎

Theorem 2.1 easily implies the following stronger form of itself.

Corollary 2.4. *Let $\mathcal{M}, \tau, D, f$ be as in Theorem 2.1, and let $g : (-\infty, \infty) \to \mathbb{R}$ be an increasing, convex function. Then, for each $t \geq 0$,*

$$\lambda \mapsto \int_0^t g(\log \mu_s(f(\lambda)))\, ds$$

is subharmonic on D.

Proof. Let $\lambda_0 \in D$, and fix r with $0 < r < \operatorname{dist}(\lambda_0, \partial D)$. For $s \geq 0$, define

$$\phi(s) = \log \mu_s(f(\lambda_0)) \quad \text{and} \quad \psi(s) = \frac{1}{2\pi} \int_0^{2\pi} \log \mu_s(f(\lambda_0 + re^{i\theta}))\, d\theta.$$

By Theorem 2.1, $\int_0^t \phi(s)\,ds \le \int_0^t \psi(s)\,ds$ for all $t \ge 0$. Using [5, Lemme 4.1], we deduce that $\int_0^t g(\phi(s))\,ds \le \int_0^t g(\psi(s))\,ds$ for all $t \ge 0$. This gives the result. ∎

This strengthened form of the result has an application to finite von Neumann algebras. Suppose that $\tau(1_{\mathcal{M}}) = 1$. Then, by [5, Proposition 1.11],

$$\int_0^1 \mu_s(x)\,ds = \tau(|x|) \quad (x \in \mathcal{M}), \tag{2}$$

and by [5, Exemple 2.2.2],

$$\int_0^1 \log \mu_s(x)\,ds = \log \Delta(x) \quad (x \in \mathcal{M}), \tag{3}$$

where Δ denotes (the analytic extension of) the Fuglede–Kadison determinant on $\mathcal{M}$ (see [7]).

Corollary 2.5. *Let $\mathcal{M}, \tau, D, f$ be as in Theorem 2.1, and suppose in addition that $\tau(1_{\mathcal{M}}) = 1$. Then*

$$\lambda \mapsto \log \tau(|f(\lambda)|) \quad \textit{and} \quad \lambda \mapsto \log \Delta(f(\lambda))$$

are both subharmonic functions on D.

Proof. The subharmonicity of $\log \Delta(f(\lambda))$ is an immediate consequence of (3) and Theorem 2.1. Similarly, from (2) and Corollary 2.4 (applied with $g(u) = e^u$), it follows that $\tau(|f(\lambda)|)$ is subharmonic on D. Repeating the above with $f(\lambda)$ replaced by $e^{p(\lambda)} f(\lambda)$, where p is an arbitrary complex polynomial, we deduce that $|e^{p(\lambda)}|\tau(|f(\lambda)|)$ is subharmonic on D. This is enough to ensure that $\log \tau(|f(\lambda)|)$ is subharmonic on D (see e.g. [1, Theorem A.1.5]). ∎

In fact, the subharmonicity of $\log \Delta(f(\lambda))$ was already known. It was established by Brown as part of a more general result [3, Theorem 3.3], from which he introduced the so-called spectral distribution measure. The paper of Haagerup and Larsen [8] is a recent source of information on Brown's spectral measure.

3. Proof of the main theorem. We now turn to the proof of Theorem 2.1. Throughout this section, we assume that $\mathcal{M}$ is a von Neumann algebra and that τ is a faithful, normal, semifinite trace on $\mathcal{M}$.

We begin by listing some basic properties of the generalized s-numbers μ_s.

Proposition 3.1. *Let $x, y \in \mathcal{M}$ and $s \in \mathbb{R}^+$. Then*

(a) $\mu_s(x) = \mu_s(x^*) = \mu_s(|x|)$,
(b) $\mu_s(yx) \le \|y\|\mu_s(x)$,

(c) $|\mu_s(x) - \mu_s(y)| \leq \|x - y\|$,
(d) $\mu_s(x + k1_{\mathcal{M}}) \leq \mu_s(x) + k$ *for* $k \in \mathbb{R}^+$, *with equality when* $x \in \mathcal{M}^+$ *and* $s < \tau(1_{\mathcal{M}})$.

Proof. Parts (a)–(c) are in [6, Lemma 2.5]. Part (d) is an easy consequence of [5, Propositions 1.6(iii) and 1.3]. ∎

The next result takes care of upper semicontinuity.

LEMMA 3.2. *Let* $f : D \to \mathcal{M}$ *be a continuous function. Then, for each* $t \geq 0$, *the function* $\lambda \mapsto \int_0^t \log \mu_s(f(\lambda))\, ds$ *is upper semicontinuous on* D.

Proof. Proposition 3.1(c) shows that the function $\lambda \mapsto \mu_s(f(\lambda))$ is continuous. Hence, if $\lambda_n \to \lambda_0$ in D, then, by Fatou's lemma,

$$\limsup_{n\to\infty} \int_0^t \log \mu_s(f(\lambda_n))\, ds \leq \int_0^t \log \mu_s(f(\lambda_0))\, ds.$$

This proves upper semicontinuity. ∎

For the time being, we make the additional assumption that the trace τ is *finite*, i.e. that $\tau(1_{\mathcal{M}}) < \infty$. In this case, τ extends to a positive linear functional on the whole of $\mathcal{M}$.

LEMMA 3.3. *Let* τ *be a faithful, normal, finite trace on* $\mathcal{M}$. *Then*

$$\int_0^{\tau(1_{\mathcal{M}})} \log \mu_s(\exp x)\, ds = \operatorname{Re} \tau(x) \qquad (x \in \mathcal{M}). \tag{4}$$

Proof. Notice first that the validity of (4) remains unchanged if we work with the trace $c\tau$ ($c > 0$) instead of τ. Thus, we may as well suppose at the outset that $\tau(1_{\mathcal{M}}) = 1$. The left-hand side of (4) is then equal to $\log \Delta(\exp x)$, where Δ is the Fuglede–Kadison determinant (see (3)). From [7, Theorem 1 (2)], we have $\log \Delta(\exp x) = \operatorname{Re} \tau(x)$ for all $x \in \mathcal{M}$. The result follows. ∎

LEMMA 3.4. *Let* τ *be a faithful, normal, finite trace on* $\mathcal{M}$, *and let* $f : D \to \mathcal{M}$ *be a holomorphic function. Then*

$$\lambda \mapsto \int_0^{\tau(1_{\mathcal{M}})} \log \mu_s(f(\lambda))\, ds$$

is a subharmonic function on D.

Proof. For $\varepsilon > 0$, define

$$v_\varepsilon(\lambda) := \int_0^{\tau(1_{\mathcal{M}})} \log(\mu_s(f(\lambda)) + \varepsilon)\, ds \qquad (\lambda \in D).$$

We prove that each v_ε is a subharmonic function on D. The result then follows upon letting $\varepsilon \to 0$.

Fix $\varepsilon > 0$. The upper semicontinuity of v_ε is proved in the same way as in Lemma 3.2. It remains to check the mean-value inequality. Let $\lambda_0 \in D$. Fix a partial isometry $u_0 \in \mathcal{M}$ such that $u_0 f(\lambda_0) = |f(\lambda_0)|$, and define

$$w(\lambda) := \int_0^{\tau(1_{\mathcal{M}})} \log \mu_s(u_0 f(\lambda) + \varepsilon 1_{\mathcal{M}})\, ds \qquad (\lambda \in D).$$

Using parts (b) and (d) of Proposition 3.1, we see that $w \le v_\varepsilon$ on D, with equality at λ_0. We claim that w is actually harmonic in a neighbourhood of λ_0. If so, then, for all small enough $r > 0$, we have

$$v_\varepsilon(\lambda_0) = w(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} w(\lambda_0 + re^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} v_\varepsilon(\lambda_0 + re^{i\theta})\, d\theta,$$

thereby proving the mean-value inequality.

It remains to justify the claim about w. Notice that the spectrum of $u_0 f(\lambda_0) + \varepsilon 1_{\mathcal{M}}$ is a subset of $[\varepsilon, \infty)$. By the upper semicontinuity of the spectrum, there exists an open disk D_0 around λ_0 such that the spectrum of $u_0 f(\lambda) + \varepsilon 1_{\mathcal{M}}$ lies in the open right half-plane for every $\lambda \in D_0$. Using the holomorphic functional calculus, we can define $h(\lambda) := \log(u_0 f(\lambda) + \varepsilon 1_{\mathcal{M}})$ $(\lambda \in D_0)$. It follows that

$$w(\lambda) = \int_0^{\tau(1_{\mathcal{M}})} \log \mu_s(\exp(h(\lambda)))\, ds = \operatorname{Re} \tau(h(\lambda)) \qquad (\lambda \in D_0),$$

where the second equality is from Lemma 3.3. Since h is holomorphic on D_0, we deduce that w is harmonic on D_0, justifying the claim. ∎

The last ingredient is a technique from [5] which permits us to reduce the general case to the case of finite trace. Given a projection $e \in \mathcal{M}$ with $\tau(e) < \infty$, the restriction of τ to the von Neumann algebra $e\mathcal{M}e$ gives a normal, faithful, finite trace (since the unit of $e\mathcal{M}e$ is e). We will use the symbol μ_s^e to denote the generalized s-numbers in $e\mathcal{M}e$.

LEMMA 3.5. *Let e be a projection in $\mathcal{M}$ with $\tau(e) < \infty$. Then*

$$\mu_s^e(exe) \le \mu_s(x) \qquad (x \in \mathcal{M},\ s \in \mathbb{R}^+).$$

Proof. For $x \in \mathcal{M}^+$, this is proved in [5, Proposition 1.5(i)]. The general case follows upon observing that, for general $x \in \mathcal{M}$,

$$\mu_s^e(exe) = \mu_s^e(|exe|) \le \mu_s^e(e|x|e) \le \mu_s(|x|) = \mu_s(x). \ ∎$$

LEMMA 3.6. *Assume that $\mathcal{M}$ has no minimal projections. Let $x \in \mathcal{M}^+$ and $t > 0$. Suppose that $\mu_s(x) > 0$ for $0 \le s < t$. Then there exists* *a projection $e \in \mathcal{M}$, commuting with x, such that $\tau(e) = t$ and $\mu_s^e(exe) = \mu_s(x)$ for $0 \le s < t$.*

Proof. This is proved in [5, Lemme 1.13], under the additional assumption that $\lim_{s\to\infty} \mu_s(x) = 0$. But in fact it is true without this restriction. Indeed, just as in the proof of [6, Lemma 4.1], one can reduce to the case $\mathcal{M} = L^\infty(\Omega, \nu)$, where ν is a non-atomic measure, and $\tau(\phi) = \int \phi\, d\nu$ ($\phi \in \mathcal{M}$) (see also [6, Remarks 2.3]). In this case $\mu_s(\phi) = \phi^*(s)$, where ϕ^* denotes the non-decreasing rearrangement of $|\phi|$.

Given x, t as in the lemma, set $E = \{\omega \in \Omega : |x(\omega)| > x^*(t)\}$ and $F = \{\omega \in \Omega : |x(\omega)| \ge x^*(t)\}$. Then $\nu(E) \le t \le \nu(F)$. As ν is non-atomic, there exists a measurable set G with $E \subset G \subset F$ and $\nu(G) = t$. Let e be the characteristic function of G; this satisfies all the conclusions of the lemma. ∎

As remarked in both [5] and [6], the assumption that $\mathcal{M}$ have no minimal projections is not a very serious restriction, because we can always embed $\mathcal{M}$ into $\mathcal{M} \otimes L^\infty[0,1]$, extending τ by taking its tensor product with the trace $\phi \mapsto \int_0^1 \phi$ on $L^\infty[0,1]$. The s-numbers $\widetilde{\mu}_s$ in this new algebra satisfy $\widetilde{\mu}_s(x \otimes 1) = \mu_s(x)$ ($x \in \mathcal{M}$).

Completion of the proof of Theorem 2.1. Upper semicontinuity has been proved in Lemma 3.2, so it remains to check the mean-value inequality. Let $\lambda_0 \in D$ and $0 < r < \operatorname{dist}(\lambda_0, \partial D)$. We shall show that

$$\int_0^t \log \mu_s(f(\lambda_0))\, ds \le \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \log \mu_s(f(\lambda_0 + re^{i\theta}))\, ds\, d\theta.$$

If $\mu_s(f(\lambda_0)) = 0$ for some $s < t$, then the left-hand side is $-\infty$, and the inequality is trivially satisfied. So we can suppose that $\mu_s(f(\lambda_0)) > 0$ for $0 \le s < t$. By the remark above, we may also assume that the algebra $\mathcal{M}$ has no minimal projections. Hence, by Lemma 3.6, there exists a projection $e \in \mathcal{M}$, commuting with $|f(\lambda_0)|$, such that $\tau(e) = t$ and $\mu_s^e(ef(\lambda_0)e) = \mu_s(f(\lambda_0))$ for $0 \le s < t$. Therefore

$$\begin{aligned}\int_0^t \log \mu_s(f(\lambda_0))\, ds &= \int_0^t \log \mu_s^e(ef(\lambda_0)e)\, ds \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \log \mu_s^e(ef(\lambda_0 + re^{i\theta})e)\, ds\, d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \int_0^t \log \mu_s(f(\lambda_0 + re^{i\theta}))\, ds\, d\theta,\end{aligned}$$

where the first inequality is by Lemma 3.4 and the second by Lemma 3.5. This completes the proof. ∎

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