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The Lizorkin–Freitag formula for several weighted L_p spaces and vector-valued interpolation

by

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Abstract. A complete description of the real interpolation space

$$L = (L_{p_0}(\omega_0), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q}$$

is given. An interesting feature of the result is that the whole measure space (Ω, μ) can be divided into disjoint pieces Ω_i $(i \in I)$ such that L is an l_q sum of the restrictions of L to Ω_i, and L on each Ω_i is a result of interpolation of just *two* weighted L_p spaces. The proof is based on a generalization of some recent results of the first two authors concerning real interpolation of vector-valued spaces.

1. Introduction. One of the most important results in real interpolation is the description of the so-called (θ, q) spaces. However, there are not many cases for which a complete and explicit description of these spaces is known. One of them is a couple of weighted L_p spaces which was treated by J. Peetre, J. E. Gilbert, P. L. Lizorkin and D. Freitag. The corresponding description of (θ, q) spaces is usually called the Lizorkin–Freitag formula. This formula seems to be quite fundamental; for example from it and the fact that Besov spaces in wavelet bases are just weighted L_p spaces follows a complete description of (θ, q) spaces for a couple of Besov spaces.

In this paper we will obtain an analog of the Lizorkin–Freitag formula for the case of several weighted L_p spaces. Our interest in this question is connected with the interpolation of several smooth function spaces (see [AKNMP]). The main tool is interpolation of vector-valued spaces for the case of more than two spaces.

In a recent paper (see [AK1]) it was shown that if we have several sequences of spaces $A_i = \{A_i^k\}_{k \in \mathbb{Z}}$, $i = 0, 1, \ldots, n$, then

$$(1.1)\quad (l_{p_0}(\{A_0^{(k)}\}_{k\in\mathbb{Z}}), \ldots, l_{p_n}(\{A_n^{(k)}\}_{k\in\mathbb{Z}}))_{\vec{\theta},q} = l_q(\{(A_0^{(k)}, \ldots, A_n^{(k)})_{\vec{\theta},q}\}_{k\in\mathbb{Z}})$$

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under the condition that

$$A_n^{(k)} = c^k A_{n-1}^{(k)} \quad \text{for all } k \in \mathbb{Z} \text{ and some fixed positive } c \neq 1.$$

Moreover, if for each $k \in \mathbb{Z}$ the spaces $A_0^{(k)}, \ldots, A_{n-1}^{(k)}$ are Banach function lattices on the same measure space (Ω_k, μ_k), then (1.1) holds even when

$$A_n^{(k)} = c^k (A_0^{(k)}, \ldots, A_{n-1}^{(k)})_{\vec{\lambda},r} \tag{1.2}$$

for all $k \in \mathbb{Z}$ and some fixed $\vec{\lambda}, r$ and $c \neq 1$.

In this paper we will generalize these results and use them to obtain a complete description of the space

$$L = (L_{p_0}(\omega_0), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q}.$$

In the diagonal case, i.e. when $1/q = \theta_0/p_0 + \theta_1/p_1 + \cdots + \theta_n/p_n$, the result is known and we have a nice and quite useful formula

$$(L_{p_0}(\omega_0), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q} = L_q(\omega_0^{\theta_0} \cdots \omega_n^{\theta_n}), \tag{1.3}$$

which was obtained for the case of two spaces in 1958 by Stein–Weiss (see [SW]) and for $n > 1$ by G. Sparr (see [S]).

Therefore, the problem is interesting only in the non-diagonal case. In the case of two spaces and $p_0 \neq p_1$ this description is known as the *Lizorkin–Freitag formula* (see [L] and [F]):

$$(L_{p_0}(\omega_0), L_{p_1}(\omega_1))_{\theta,q} = L_{p_\theta,q}(\omega, d\widetilde{\mu}), \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{1.4}$$

where $L_{p_\theta,q}(\omega, d\widetilde{\mu})$ is a weighted Lorentz space with weight ω and measure $d\widetilde{\mu}$ given by the expressions

$$\omega = \frac{\omega_0^{1-\theta}\omega_1^{\theta}}{\left(\frac{\omega_0}{\omega_1}\right)^{\frac{1/p_\theta}{1/p_0 - 1/p_1}}}, \qquad d\widetilde{\mu} = \left(\frac{\omega_0}{\omega_1}\right)^{\frac{1}{1/p_0 - 1/p_1}} d\mu. \tag{1.5}$$

Usually the formula for ω is written in a different way, but for our purpose it is better to rewrite it in the form (1.5). In the case $p_0 = p_1$ the result is different. The first result was obtained by J. Peetre ([P]) in connection with the problem of identification of Beurling's spaces and a general result was obtained by J. E. Gilbert (see [G]), who showed that

$$\begin{gathered} \|f\|_{(L_p(\omega_0), L_p(\omega_1))_{\theta,q}} = \Big(\sum_{k\in\mathbb{Z}} (\|f\chi_{\Omega_k}\|_{L_p(\omega_0^{1-\theta}\omega_1^{\theta})})^q\Big)^{1/q}, \\ \Omega_k = \left\{x : 2^k < \frac{\omega_1(x)}{\omega_0(x)} \leq 2^{k+1}\right\}. \end{gathered} \tag{1.6}$$

In this paper we will show that when we have more than two spaces then the whole set Ω can be divided into some subsets Ω_i $(i \in I)$ such that the space $L = (L_{p_0}(\omega_0), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q}$ on Ω_i can be obtained by interpolation of *just two* weighted L_p spaces and the norm of the function f in L is the l_q norm of the sequence of the norms of the restrictions of f to Ω_i:

$$\|f\|_L = \Big(\sum_i (\|f\chi_{\Omega_i}\|_L)^q\Big)^{1/q}. \tag{1.7}$$

In the particular case of triples (i.e. when $n = 2$ and all weights are powers of one function) this result was obtained in [AKNMP] and it was used for interpolation of several "smooth" function spaces. The proof of (1.7) in [AKNMP] was rather complicated and it was a combination of some calculations, known results and the reiteration theorem for finite collections of Banach function lattices (see [AK2]). In this paper using a completely different proof we will obtain the general result without any restrictions on n and the weights.

To explain the construction of the sets Ω_i in (1.7) and the idea of the proof let us consider the particular case $p_0 < p_1 < \cdots < p_n$. In this case each intermediate p_i $(i = 1, \ldots, n-1)$ can be obtained from the "ends" p_0 and p_n in the following way:

$$\frac{1}{p_i} = \frac{1-\alpha_i}{p_0} + \frac{\alpha_i}{p_n}.$$

If we consider the set

$$\Omega_i^k = \Big\{x : 2^k < \frac{\omega_i}{\omega_0^{1-\alpha_i}\omega_n^{\alpha_i}} \le 2^{k+1}\Big\},$$

then for the restrictions of spaces to this set we will have

$$L_{p_i}(\omega_i; \Omega_i^k) = 2^k (L_{p_0}(\omega_0; \Omega_i^k), L_{p_n}(\omega_n; \Omega_i^k))_{\alpha_i, p_i}. \tag{1.8}$$

Moreover, if we intersect the sets Ω_i^k which correspond to different i and k then we will obtain disjoint sets $\Omega^{(k_1,\ldots,k_{n-1})} = \Omega_1^{k_1} \cap \cdots \cap \Omega_{n-1}^{k_{n-1}}$ such that their union gives the whole measure space Ω and on each of them all spaces $L_{p_i}(\omega_i)$ for intermediate p_i could be obtained by interpolation from the end spaces $L_{p_0}(\omega_0), L_{p_n}(\omega_n)$ (we also have to multiply the norm in the result by 2^{k_i}).

If we denote the restriction of the space $L_{p_i}(\omega_i)$ to $\Omega^{(k_1,\ldots,k_{n-1})}$ by $A_i^{(\vec{k})}$, $\vec{k} = (k_1, \ldots, k_{n-1})$, then $L_{p_i}(\omega_i) = l_{p_i}(\{A_i^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^{n-1}})$. Thus instead of describing the space $L = (L_{p_0}(\omega_0), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q}$ it is sufficient to describe

$$L = (l_{p_0}(\{A_0^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^{n-1}}), \ldots, l_{p_n}(\{A_n^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^{n-1}})_{\vec{\theta},q}. \tag{1.9}$$

Moreover, (1.8) shows that

$$A_i^{(\vec{k})} = 2^{k_i}(A_0^{(\vec{k})}, A_n^{(\vec{k})})_{\alpha_i, p_i}, \qquad i = 1, \ldots, n-1, \tag{1.10}$$

similarly to (1.2). The main difference is that the parameter $\vec{k}$ in (1.9) belongs to $\mathbb{Z}^{n-1}$ and not to $\mathbb{Z}$ as in (1.2), and instead of one condition (1.2) we have $n-1$ conditions (1.10). Nevertheless, as we will show below (see Theorem 4) a formula analogous to (1.1) holds in this situation also. Therefore we will have

$$\begin{aligned} L &= (l_{p_0}(\{A_0^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^{n-1}}), \dots, l_{p_n}(\{A_n^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^{n-1}}))_{\vec{\theta},q} \\ &= l_q(\{(A_0^{(\vec{k})}, \dots, A_n^{(\vec{k})})_{\vec{\theta},q}\}_{\vec{k}\in\mathbb{Z}^{n-1}}) \end{aligned}$$

and we need to calculate the space $(A_0^{(\vec{k})}, \dots, A_n^{(\vec{k})})_{\vec{\theta},q}$. But this is not difficult because $A_i^{(\vec{k})}$ is just $L_{p_i}(\omega_i)$ restricted to $\Omega^{(k_1,\dots,k_{n-1})}$, and on this set the spaces $L_{p_i}(\omega_i)$ $(i = 1, \dots, n-1)$ can be obtained by interpolation of the "end" spaces $L_{p_0}(\omega_0), L_{p_n}(\omega_n)$ restricted to this set (see (1.8)). Moreover, we will do some extra work to give the answer in a symmetrical way closest to the remarkable formula (1.3). This is the content and the idea of the paper.

2. Definitions and some results from real interpolation of several spaces. Let $A_0, A_1, \dots, A_n$ be $n+1$ Banach or quasi-Banach spaces. We will say that they form a *compatible collection* or simply a *collection* $\vec{A} = (A_0, A_1, \dots, A_n)$ if they are linearly and continuously embedded in some (common for all) topological linear space with Hausdorff topology. Then we can, analogously to the case of a couple, define the K-functional (see [S]) by the formula

$$\begin{aligned} K(\vec{t}, a; \vec{A}) &= \inf(\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + \dots + t_n\|a_n\|_{A_n}), \\ \vec{t} &= (t_1, \dots, t_n) \in \mathbb{R}_n^+, \end{aligned} \tag{2.1}$$

where the inf is taken over all decompositions $a = a_0 + a_1 + \dots + a_n$.

Let $\vec{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$ be a parameter vector, i.e. $\theta_i > 0$ and $\theta_0 + \theta_1 + \dots + \theta_n = 1$, and let $0 < q \le \infty$. Then the interpolation space $\vec{A}_{\vec{\theta},q} = (A_0, A_1, \dots, A_n)_{\vec{\theta},q}$ (usually denoted by $\vec{A}_{\vec{\theta},q;K}$, but we will omit the index K as we will consider only K-spaces) is defined by the norm (or quasinorm; for simplicity we always say norm)

$$\|a\|_{\vec{\theta},q} = \Big(\int_{\mathbb{R}_+^n} (t_1^{-\theta_1} t_2^{-\theta_2} \cdots t_n^{-\theta_n} K(\vec{t}, a; \vec{A}))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} \Big)^{1/q} \tag{2.2}$$

with the usual modification for $q = \infty$. As the K-functional is a concave function on $\mathbb{R}_+^n$, the norm (2.2) can be written in an equivalent form

$$\|a\|_{\vec{\theta},q} \approx \Big(\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} (2^{-\theta_1 i_1} \cdots 2^{-\theta_n i_n} K(2^{i_1}, \dots, 2^{i_n}, a; \vec{A}))^q \Big)^{1/q}.$$

We will call $\vec{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$ an *extended parameter vector* if $\theta_i \geq 0$ and $\theta_0 + \theta_1 + \dots + \theta_n = 1$, i.e. some of the coordinates θ_i may be zero. In this case in the definition of the K-functional and norms we omit the spaces A_i, parameters t_i and integrate on the set of smaller dimension. In the particular case when $\vec{\theta}$ has one coordinate, say the ith, equal to one, and so all other coordinates are zero, we will mean by $(A_0, A_1, \dots, A_n)_{\vec{\theta},q}$ the space A_i.

We use the so-called "monotonicity" properties of interpolation spaces, which follow easily from the definitions. Namely:

A) if $q_0 \leq q_1$ then

$$(A_0, A_1, \dots, A_n)_{\vec{\theta},q_0} \subset (A_0, A_1, \dots, A_n)_{\vec{\theta},q_1}; \tag{2.3}$$

B) if we have embeddings $A_i \subset B_i$ $(i = 0, \dots, n)$ then

$$(A_0, A_1, \dots, A_n)_{\vec{\theta},q} \subset (B_0, B_1, \dots, B_n)_{\vec{\theta},q}. \tag{2.4}$$

The following theorem will be of importance; it was proved in [AK2] in the Banach case (the proof can be extended after some modification to the quasi-Banach case).

THEOREM 1 (Reiteration Theorem). *Suppose that $A_0, A_1, \dots, A_n$ are Banach or quasi-Banach function lattices on the same measure space (Ω, μ) and the parameter vectors $\vec{\theta}^k$ $(k = 1, \dots, m)$ span $\mathbb{R}^{n+1}$. Then*

$$(\vec{A}_{\vec{\theta}^0,q_0}, \vec{A}_{\vec{\theta}^1,q_1}, \dots, \vec{A}_{\vec{\theta}^m,q_m})_{\vec{\lambda},q} = \vec{A}_{\vec{\theta},q}, \qquad \vec{\theta} = \lambda_0\vec{\theta}^0 + \lambda_1\vec{\theta}^1 + \dots + \lambda_m\vec{\theta}^m, \tag{2.5}$$

where $\vec{\lambda} = (\lambda_0, \dots, \lambda_m)$ is a parameter vector. The formula is also true when $\vec{\theta}^k$ $(k = 1, \dots, m)$ are extended parameter vectors.

The proofs in the next section will be based on the results from [AK1]. For convenience of the reader we will formulate the needed results.

Let $\{A^{(k)}\}_{k\in\mathbb{Z}}$ be a sequence of Banach or quasi-Banach spaces. In the quasi-Banach case we assume that the constants in the triangle inequalities are uniformly bounded. We will denote by $l_p(\{A^{(k)}\})$ the vector-valued space of all sequences $a = \{a^{(k)}\}_{k\in\mathbb{Z}}$, $a^{(k)} \in A^{(k)}$, with the norm

$$\|a\|_{l_p(\{A^{(k)}\})} = \Big(\sum_{k\in\mathbb{Z}} (\|a^{(k)}\|_{A^{(k)}})^p\Big)^{1/p},$$

with the usual modification for $p = \infty$.

Let $\{A_i^{(k)}\}_{k\in\mathbb{Z}}$, $i = 0, \dots, n$, be a family of $n+1$ sequences of Banach or quasi-Banach spaces such that for each $k \in \mathbb{Z}$ the spaces $A_0^{(k)}, \dots, A_n^{(k)}$ form a compatible collection and therefore we can define the spaces $\vec{A}^{(k)}_{\vec{\theta},q} = (A_0^{(k)}, \dots, A_n^{(k)})_{\vec{\theta},q}$. Then we have (see [AK1])

THEOREM 2. *The formula*

$$(l_{p_0}(\{A_0^{(k)}\}_{k\in\mathbb{Z}}),\dots,l_{p_n}(\{A_n^{(k)}\}_{k\in\mathbb{Z}}))_{\vec{\theta},q}=l_q(\{(A_0^{(k)},\dots,A_n^{(k)})_{\vec{\theta},q}\}_{k\in\mathbb{Z}}) \tag{2.6}$$

holds if for some fixed $i\neq n$, fixed positive number $c\neq 1$ and all $k\in\mathbb{Z}$ we have (1)

$$A_n^{(k)}=c^kA_i^{(k)}. \tag{2.7}$$

Formula (2.6) *also holds when the space $A_n^{(k)}$ is obtained from an extended parameter vector $\vec{\lambda}$*:

$$A_n^{(k)}=c^k(A_0^{(k)},\dots,A_{n-1}^{(k)})_{\vec{\lambda},p},\qquad k\in\mathbb{Z},\ c\neq 1, \tag{2.8}$$

under the additional condition that $A_0^{(k)},\dots,A_{n-1}^{(k)}$ are Banach or quasi-Banach function lattices on (Ω_k,μ_k).

3. Multiparameter vector-valued interpolation. Let

$$A=\{A^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}$$

be an "m-dimensional" sequence of spaces. The space $l_p(A)$ is defined by the norm (quasinorm if $p<1$)

$$\|\{a^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}\|_{l_p(A)}=\Big(\sum_{\vec{k}\in\mathbb{Z}^m}(\|a^{(\vec{k})}\|_{A^{(\vec{k})}})^p\Big)^{1/p},$$

with the usual modification for $p=\infty$.

Let $\{A_i^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}$, $i=0,\dots,n$, be a collection of $n+1$ sequences of Banach or quasi-Banach spaces. In the quasi-Banach case we assume that the constants in the triangle inequalities are uniformly bounded (do not depend on $\vec{k}$). We also assume that for each $\vec{k}\in\mathbb{Z}^m$ the spaces $A_0^{(\vec{k})},\dots,A_n^{(\vec{k})}$ form a compatible collection and therefore we can define the spaces $\vec{A}_{\vec{\theta},q}^{(\vec{k})}=(A_0^{(\vec{k})},\dots,A_n^{(\vec{k})})_{\vec{\theta},q}$.

Then we can consider the collection

$$(l_{p_0}(\{A_0^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}),\dots,l_{p_n}(\{A_n^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}))$$

and look for conditions under which

$$(l_{p_0}(\{A_0^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}),\dots,l_{p_n}(\{A_n^{(\vec{k})}\}_{\vec{k}\in\mathbb{Z}^m}))_{\vec{\theta},q}=l_q(\{(A_0^{(\vec{k})},\dots,A_n^{(\vec{k})})_{\vec{\theta},q}\}_{\vec{k}\in\mathbb{Z}^m}). \tag{3.1}$$

We will prove the following theorem:

(1) The equality $A_i=B_i$ ($i\in I$) of sequences of Banach or quasi-Banach spaces here and below means that the spaces A_i,B_i coincide as sets, their norms are equivalent for each i and the constants of equivalence *do not depend* on i.

THEOREM 3. *Suppose that $m \le n$ and for $i = 1, \dots, m$,*

$$A^{(\vec{k})}_{n-m+i} = (c_i)^{k_i} A^{(\vec{k})}_{s(i)}, \qquad c_i \neq 1,\ 0 \le s(i) \le n-m, \tag{3.2}$$

for all $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$. Then the formula (3.1) *is valid.*

Proof. The right-hand side in (3.1) does not depend on p_i $(i = 0, \dots, n)$. Therefore, from monotonicity properties (see (2.3)–(2.4)) it follows that it is enough to prove (3.1) for the case when $p_i = p$ for all i.

Everywhere below for any $\vec{k} = (k_1, \dots, k_m)$ we denote by $\vec{k}^{(m-1)}$ the vector $(k_1, \dots, k_{m-1})$. The proof is by induction on m. For $m = 1$ the assertion was proved in [AK1]. For the induction step, consider the spaces $B^{(r)}_i = l_p(\{A^{(k_1,\dots,k_{m-1},r)}_i\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}})$ ($r \in \mathbb{Z}$ is fixed and the space l_p is constructed over $\mathbb{Z}^{m-1}$). Then we can write

$$l_p(\{A^{(\vec{k})}_i\}_{\vec{k} \in \mathbb{Z}^m}) = l_p(\{B^{(r)}_i\}_{r \in \mathbb{Z}}), \qquad i = 0, \dots, n.$$

Moreover,

$$B^{(r)}_n = (c_m)^r B^{(r)}_{s(m)} \qquad \text{for all } r \in \mathbb{Z}.$$

Indeed, for a fixed r we have $A^{(k_1,\dots,k_{m-1},r)}_n = (c_m)^r A^{(k_1,\dots,k_{m-1},r)}_{s(m)}$ and so

$$\begin{aligned} B^{(r)}_n &= l_p(\{A^{(k_1,\dots,k_{m-1},r)}_n\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}}) \\ &= (c_m)^r l_p(\{A^{(k_1,\dots,k_{m-1},r)}_{s(m)}\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}}) = (c_m)^r B^{(r)}_{s(m)}. \end{aligned}$$

Therefore from Theorem 2 for the spaces $B^{(r)}_i$ we get

$$\begin{aligned} &(l_p(\{A^{(\vec{k})}_0\}_{\vec{k} \in \mathbb{Z}^m}), l_p(\{A^{(\vec{k})}_1\}_{\vec{k} \in \mathbb{Z}^m}), \dots, l_p(\{A^{(\vec{k})}_n\}_{\vec{k} \in \mathbb{Z}^m}))_{\vec{\theta},q} \\ &\qquad = (l_p(\{B^{(r)}_0\}_{r \in \mathbb{Z}}), l_p(\{B^{(r)}_1\}_{r \in \mathbb{Z}}), \dots, l_p(\{B^{(r)}_n\}_{r \in \mathbb{Z}}))_{\vec{\theta},q} \\ &\qquad = l_q(\{(B^{(r)}_0, \dots, B^{(r)}_n)_{\vec{\theta},q}\}_{r \in \mathbb{Z}}). \end{aligned} \tag{3.3}$$

From (3.2) with $i = 1, \dots, m-1$, fixed r and the induction hypothesis, it follows that

$$\begin{aligned} &(B^{(r)}_0, \dots, B^{(r)}_n)_{\vec{\theta},q} \\ &\quad = (l_p(\{A^{(\vec{k}^{(m-1)},r)}_0\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}}, \dots, l_p(\{A^{(\vec{k}^{(m-1)},r)}_n\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}})_{\vec{\theta},q} \\ &\quad = l_q(\{(A^{(\vec{k}^{(m-1)},r)}_0, \dots, A^{(\vec{k}^{(m-1)},r)}_n)_{\vec{\theta},q}\}_{\vec{k}^{(m-1)} \in \mathbb{Z}^{m-1}}). \end{aligned}$$

Therefore the assertion follows from (3.3) and the property $l_q(l_q) = l_q$. ∎

Now we will consider the "intermediate" result.

THEOREM 4. *Suppose that for each $\vec{k} \in \mathbb{Z}^m$ the collection $A^{(\vec{k})}_0, A^{(\vec{k})}_1, \dots, A^{(\vec{k})}_{n-m}$ consists of Banach or quasi-Banach function lattices on some measure* *space* $(\Omega_{\vec{k}}, \mu_{\vec{k}})$ *and the last* m $(m \le n)$ *spaces have the form*

$$A^{(\vec{k})}_{n-m+i} = (c_i)^{k_i}(A^{(\vec{k})}_0, A^{(\vec{k})}_1, \dots, A^{(\vec{k})}_{n-m})_{\vec{\lambda}_i, q_i}, \tag{3.4}$$

where $\vec{\lambda}_i$ *are extended parameter vectors and* $c_i \neq 1$, $i = 1, \dots, m$. *Then*

$$\begin{aligned}(l_{p_0}(\{A^{(\vec{k})}_0\}_{\vec{k}\in\mathbb{Z}^m}), l_{p_1}(\{A^{(\vec{k})}_1\}_{\vec{k}\in\mathbb{Z}^m}), \dots, l_{p_n}(\{A^{(\vec{k})}_n\}_{\vec{k}\in\mathbb{Z}^m}))_{\vec{\theta},q}& \\ = l_q(\{(A^{(\vec{k})}_0, A^{(\vec{k})}_1, \dots, A^{(\vec{k})}_n)_{\vec{\theta},q}\}_{\vec{k}\in\mathbb{Z}^m}).&\end{aligned} \tag{3.5}$$

REMARK 1. In the theorem, for simplicity, we required that the "last m spaces" can be obtained by interpolation from the "first $n+1-m$" spaces, but the same proof shows that the assertion is valid if we require that some m spaces can be obtained by interpolation from the others.

Proof. The proof is analogous to the proof of the previous theorem, except some details which we will explain below. As in the previous theorem, from the monotonicity property and reiteration theorem it follows that it is enough to prove the theorem under the additional condition that all p_i and q_j (see (3.4)) are equal to some p. Now we argue by induction. For $m = 1$ the theorem was proved in [AK1]. The induction step can be done in the following way. Consider the spaces $B^{(r)}_i = l_p(\{A^{(k_1,\dots,k_{m-1},r)}_i\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}})$ ($r \in \mathbb{Z}$ is fixed and l_p is constructed over $\mathbb{Z}^{m-1}$). Then we can write

$$l_p(\{A^{(\vec{k})}_i\}_{\vec{k}\in\mathbb{Z}^m}) = l_p(\{B^{(r)}_i\}_{r\in\mathbb{Z}}), \qquad i = 0, \dots, n.$$

To use Theorem 2 we need the property

$$B^{(r)}_n = (c_m)^r(B^{(r)}_0, \dots, B^{(r)}_{n-m})_{\vec{\lambda}_m,p} \qquad \text{for all } r \in \mathbb{Z}.$$

But from the condition (3.4) for $i = m$ and vector-valued interpolation in the diagonal case (see [S]) it follows that

$$\begin{aligned}
B^{(r)}_n &= l_p(\{A^{(k_1,\dots,k_{m-1},r)}_n\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}}) \\
&= l_p(\{(c_m)^r(A^{(k_1,\dots,k_{m-1},r)}_0, \dots, A^{(k_1,\dots,k_{m-1},r)}_{n-m})_{\vec{\lambda}_m,p}\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}}) \\
&= l_p(\{(c_m)^r(A^{(k_1,\dots,k_{m-1},r)}_0, \dots, A^{(k_1,\dots,k_{m-1},r)}_{n-m})_{\vec{\lambda}_m,p}\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}}) \\
&= (c_m)^r \\
&\cdot (l_p(\{A^{(k_1,\dots,k_{m-1},r)}_0\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}}), \dots, l_p(\{A^{(k_1,\dots,k_{m-1},r)}_n\}_{\vec{k}^{(m-1)}\in\mathbb{Z}^{m-1}}))_{\vec{\lambda}_m,p} \\
&= (c_m)^r(B^{(r)}_0, \dots, B^{(r)}_{n-m})_{\vec{\lambda}_m,p}.
\end{aligned}$$

Now everything is ready to apply Theorem 2. Using it we obtain

$$\begin{aligned}(l_p(\{A^{(\vec{k})}_0\}_{\vec{k}\in\mathbb{Z}^m}), l_p(\{A^{(\vec{k})}_1\}_{\vec{k}\in\mathbb{Z}^m}), \dots, l_p(\{A^{(\vec{k})}_n\}_{\vec{k}\in\mathbb{Z}^m}))_{\vec{\theta},q}& \\ = l_q(\{(B^{(r)}_0, \dots, B^{(r)}_n)_{\vec{\theta},q}\}_{r\in\mathbb{Z}}).&\end{aligned}$$

Now we can calculate the space $(B_0^{(r)},\dots,B_n^{(r)})_{\vec\theta,q}$ by using the induction hypothesis (see end of proof of Theorem 3) and the theorem follows from the property $l_q(l_q)=l_q$. ∎

4. Interpolation of weighted L_p spaces. Let (Ω,μ) be a measure space with σ-finite measure μ. Let $\omega_0,\omega_1,\dots,\omega_n$ be some positive functions defined on the set Ω. We will consider the collection

$$L_{\vec p}(\vec\omega)=(L_{p_0}(\omega_0),L_{p_1}(\omega_1),\dots,L_{p_n}(\omega_n)),$$

where $0<p_i\le\infty$, $i=0,1,\dots,n$, and the norm (quasinorm) in $L_{p_i}(\omega_i)$ is defined by the formula

$$\|f\|_{L_{p_i}(\omega_i)}=\Big(\int_\Omega|f\omega_i|^{p_i}\,d\mu\Big)^{1/p_i} \tag{4.1}$$

with the usual modification when $p_i=\infty$.

Without loss of generality we can restrict ourselves to the case $p_0\le p_1\le\dots\le p_n$. If $p_n\ne p_0$, then we can define the numbers $\alpha_i\in[0,1]$ such that

$$\frac{1}{p_i}=\frac{1-\alpha_i}{p_0}+\frac{\alpha_i}{p_n},\qquad i=1,\dots,n-1,$$

and the sets

$$\Omega^{(\vec k)}=\Big\{x\in\Omega:2^{k_i}<\frac{\omega_i(x)}{\omega_0^{1-\alpha_i}(x)\omega_n^{\alpha_i}(x)}\le2^{k_i+1},\ i=1,\dots,n-1\Big\}, \tag{4.2}$$

where $\vec k=(k_1,\dots,k_{n-1})\in\mathbb Z^{n-1}$.

If $p_n=p_0$ we take

$$\Omega^{(\vec k)}=\Big\{x\in\Omega:2^{k_i}<\frac{\omega_i(x)}{\omega_0(x)}\le2^{k_i+1},\ i=1,\dots,n\Big\}; \tag{4.3}$$

notice that here $\vec k$ is in $\mathbb Z^n$ (and not in $\mathbb Z^{n-1}$ as in the case $p_n\ne p_0$).

It is clear that in both cases the sets $\Omega^{(\vec k)}$ are disjoint for different $\vec k$ and their union gives all Ω.

Our main result reads:

THEOREM 5. (a) *If $p_n\ne p_0$ then*

$$\|f\|_{(L_{p_0}(\omega_0),L_{p_1}(\omega_1),\dots,L_{p_n}(\omega_n))_{\vec\theta,q}}\approx\Big(\sum_{\vec k}(\|f\omega\chi_{\Omega^{(\vec k)}}\|_L)^q\Big)^{1/q}, \tag{4.4}$$

where L is a Lorentz space $L_{p_\theta,q}$ on Ω with measure

$$d\widetilde\mu=\Big(\frac{\omega_0}{\omega_n}\Big)^{\frac{1}{1/p_0-1/p_n}}d\mu,$$

$\Omega^{(\vec{k})}$ is defined by (4.2), *and*

$$\omega = \frac{\omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}}{(\frac{\omega_0}{\omega_n})^{\frac{1/p_\theta}{1/p_0-1/p_n}}}, \quad \textit{where} \quad \frac{1}{p_\theta} = \frac{\theta_0}{p_0} + \frac{\theta_1}{p_1} + \cdots + \frac{\theta_n}{p_n}. \tag{4.5}$$

(b) *If $p_n = p_0$ then formula* (4.4) *is also valid, but we have to take $d\widetilde{\mu} = d\mu$, $\omega = \omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}$ and $\Omega^{(\vec{k})}$ are defined by* (4.3).

Proof. (a) Denote by $L_{p_i}^{(\vec{k})}(\omega_i)$ the restriction of the space $L_{p_i}(\omega_i)$ to the set $\Omega^{(\vec{k})}$. Then $L_{p_i}(\omega_i)$ can be written as

$$L_{p_i}(\omega_i) = l_{p_i}(\{L_{p_i}^{(\vec{k})}(\omega_i)\}_{\vec{k}\in\mathbb{Z}^{n-1}}).$$

Therefore

$$\begin{aligned}(L_{p_0}(\omega_0), L_{p_1}(\omega_1), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q} \\ = (l_{p_0}(\{L_{p_0}^{(\vec{k})}(\omega_0)\}_{\vec{k}\in\mathbb{Z}^{n-1}}), \ldots, l_{p_n}(\{L_{p_n}^{(\vec{k})}(\omega_n)\}_{\vec{k}\in\mathbb{Z}^{n-1}}))_{\vec{\theta},q}.\end{aligned} \tag{4.6}$$

Moreover, on $\Omega^{(\vec{k})}$ given by (4.2) we have, for $\alpha_i \in (0,1)$,

$$L_{p_i}^{(\vec{k})}(\omega_i) = 2^{k_i}(L_{p_0}^{(\vec{k})}(\omega_0), L_{p_n}^{(\vec{k})}(\omega_n))_{\alpha_i,p_i} \tag{4.7}$$

and

$$L_{p_i}^{(\vec{k})}(\omega_i) = 2^{k_i}L_{p_0}^{(\vec{k})}(\omega_0) \quad \text{when } \alpha_i = 0, \tag{4.8}$$

$$L_{p_i}^{(\vec{k})}(\omega_i) = 2^{k_i}L_{p_n}^{(\vec{k})}(\omega_n) \quad \text{when } \alpha_i = 1. \tag{4.9}$$

So the conditions of Theorem 4 are fulfilled with $m = n-1$ and $A_i^{(\vec{k})} = L_{p_i}^{(\vec{k})}(\omega_i)$, $i = 0, \ldots, n$. Applying this theorem in the form of Remark 1 we obtain

$$\begin{aligned}(L_{p_0}(\omega_0), L_{p_1}(\omega_1), \ldots, L_{p_n}(\omega_n))_{\vec{\theta},q} \\ = l_q(\{(L_{p_0}^{(\vec{k})}(\omega_0), \ldots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q}\}_{\vec{k}\in\mathbb{Z}^{n-1}})\end{aligned} \tag{4.10}$$

and we only need to calculate the space $(L_{p_0}^{(\vec{k})}(\omega_0), \ldots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q}$. But from the relations (4.7)–(4.9) and the theorem of G. Sparr (see [S, Theorem 9.2]) it follows that

$$\begin{aligned}(L_{p_0}^{(\vec{k})}(\omega_0), \ldots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q} \\ = 2^{k_1\theta_1+\cdots+k_{n-1}\theta_{n-1}}(L_{p_0}^{(\vec{k})}(\omega_0), L_{p_n}^{(\vec{k})}(\omega_n))_{\gamma,q},\end{aligned} \tag{4.11}$$

where

$$\gamma = \alpha_1\theta_1 + \cdots + \alpha_{n-1}\theta_{n-1} + \theta_n. \tag{4.12}$$

Moreover, from (4.2) we have

$$2^{k_1\theta_1+\cdots+k_{n-1}\theta_{n-1}} \approx \left(\frac{\omega_1(x)}{\omega_0^{1-\alpha_1}(x)\omega_n^{\alpha_1}(x)}\right)^{\theta_1}\cdots\left(\frac{\omega_{n-1}(x)}{\omega_0^{1-\alpha_{n-1}}(x)\omega_n^{\alpha_{n-1}}(x)}\right)^{\theta_{n-1}}. \tag{4.13}$$

As

$$\begin{aligned}\frac{1}{p_\theta} &= \frac{\theta_0}{p_0}+\frac{\theta_1}{p_1}+\cdots+\frac{\theta_n}{p_n}\\ &= \frac{\theta_0}{p_0}+\theta_1\left(\frac{1-\alpha_1}{p_0}+\frac{\alpha_1}{p_n}\right)+\cdots+\theta_{n-1}\left(\frac{1-\alpha_{n-1}}{p_0}+\frac{\alpha_{n-1}}{p_n}\right)+\frac{\theta_n}{p_n}\\ &= \frac{\theta_0+\theta_1(1-\alpha_1)+\cdots+\theta_{n-1}(1-\alpha_{n-1})}{p_0}+\frac{\theta_1\alpha_1+\cdots+\theta_{n-1}\alpha_{n-1}+\theta_n}{p_n}\\ &= \frac{1-\gamma}{p_0}+\frac{\gamma}{p_n},\end{aligned}$$

from (4.12) and (4.13) it follows that

$$\begin{aligned}2^{k_1\theta_1+\cdots+k_{n-1}\theta_{n-1}} &\approx \frac{\omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}}{\omega_0^{\theta_0+\theta_1(1-\alpha_1)+\cdots+\theta_{n-1}(1-\alpha_{n-1})}\omega_n^{\theta_1\alpha_1+\cdots+\theta_{n-1}\alpha_{n-1}+\theta_n}}\\ &= \frac{\omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}}{\omega_0^{1-\gamma}\omega_n^{\gamma}}.\end{aligned}$$

Now by using the Lizorkin–Freitag formula (see (1.4)–(1.5)) we infer that $(L_{p_0}^{(\vec{k})}(\omega_0), L_{p_n}^{(\vec{k})}(\omega_n))_{\gamma,q}$ is $L_{p_\gamma,q}$ on the set $\Omega_{\vec{k}}$ with the measure $d\widetilde{\mu} = (\omega_0/\omega_n)^{\frac{1}{1/p_0-1/p_n}}\,d\mu$ and weight

$$\frac{\omega_0^{1-\gamma}\omega_n^{\gamma}}{\left(\frac{\omega_0}{\omega_n}\right)^{\frac{1/p_\gamma}{1/p_0-1/p_n}}}$$

so

$$(L_{p_0}^{(\vec{k})}(\omega_0), L_{p_n}^{(\vec{k})}(\omega_n))_{\gamma,q} = L_{p_\gamma,q}\left(\frac{\omega_0^{1-\gamma}\omega_n^{\gamma}}{(\omega_0/\omega_n)^{\frac{1/p_\gamma}{1/p_0-1/p_n}}}, \left(\frac{\omega_0}{\omega_n}\right)^{\frac{1}{1/p_0-1/p_n}}d\mu, \Omega_{\vec{k}}\right),$$

where $1/p_\gamma = (1-\gamma)/p_0+\gamma/p_n = 1/p_\theta$.

The constant $2^{k_1\theta_1+\cdots+k_{n-1}\theta_{n-1}}$ on the right-hand side of (4.11) can be moved to the weight, so the new weight will be

$$\begin{aligned}&2^{k_1\theta_1+\cdots+k_{n-1}\theta_{n-1}}\cdot\frac{\omega_0^{1-\gamma}\omega_n^{\gamma}}{\left(\frac{\omega_0}{\omega_n}\right)^{\frac{1/p_\gamma}{1/p_0-1/p_n}}}\\ &\qquad\approx \frac{\omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}}{\omega_0^{1-\gamma}\omega_n^{\gamma}}\cdot\frac{\omega_0^{1-\gamma}\omega_n^{\gamma}}{\left(\frac{\omega_0}{\omega_n}\right)^{\frac{1/p_\theta}{1/p_0-1/p_n}}} = \frac{\omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}}{\left(\frac{\omega_0}{\omega_n}\right)^{\frac{1/p_\theta}{1/p_0-1/p_n}}} = \omega.\end{aligned}$$

So, from (4.11) it follows that on $\Omega^{(\vec{k})}$ we have

$$(L_{p_0}^{(\vec{k})}(\omega_0), \dots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q} = L_{p_\theta,q}\left(\omega, \left(\frac{\omega_0}{\omega_n}\right)^{\frac{1}{1/p_0-1/p_n}} d\mu,\ \Omega^{(\vec{k})}\right),$$

and from (4.10) we have the desired result.

(b) The proof is analogous and even simpler, because on each set $\Omega^{(\vec{k})}$ (see (4.3), where $\vec{k}$ is in $\mathbb{Z}^n$ and not in $\mathbb{Z}^{n-1}$ as in the case $p_n \neq p_0$) we have

$$L_{p_i}^{(\vec{k})}(\omega_i) = 2^{k_i} L_{p_0}^{(\vec{k})}(\omega_0) \quad \text{for } i = 1, \dots, n.$$

Therefore we can apply Theorem 4 to obtain

$$(L_{p_0}(\omega_0), L_{p_1}(\omega_1), \dots, L_{p_n}(\omega_n))_{\vec{\theta},q} = l_q(\{(L_{p_0}^{(\vec{k})}(\omega_0), \dots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q}\}_{\vec{k}\in\mathbb{Z}^n}).$$

Moreover we can calculate the space $(L_{p_0}^{(\vec{k})}(\omega_0), \dots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q}$ directly without using the Lizorkin–Freitag formula. Indeed,

$$\begin{aligned}(L_{p_0}^{(\vec{k})}(\omega_0), \dots, L_{p_n}^{(\vec{k})}(\omega_n))_{\vec{\theta},q} &= (L_{p_0}^{(\vec{k})}(\omega_0), 2^{k_1} L_{p_0}^{(\vec{k})}(\omega_0), \dots, 2^{k_n} L_{p_0}^{(\vec{k})}(\omega_0))_{\vec{\theta},q} \\ &= 2^{k_1\theta_1+\dots+k_n\theta_n} L_{p_0}^{(\vec{k})}(\omega_0) = L_{p_0}^{(\vec{k})}(2^{k_1\theta_1+\dots+k_n\theta_n}\omega_0)\end{aligned}$$

and it remains to notice that

$$2^{k_1\theta_1+\dots+k_n\theta_n}\omega_0 \approx \left(\frac{\omega_1(x)}{\omega_0(x)}\right)^{\theta_1} \cdots \left(\frac{\omega_n(x)}{\omega_0(x)}\right)^{\theta_n} \omega_0 = \omega_0^{\theta_0}\omega_1^{\theta_1}\cdots\omega_n^{\theta_n}. \ \blacksquare$$

Remark 2. Formulas for ω and $d\widetilde{\mu}$ in the theorem in the case when $p_0 \neq p_n$ depend on the parameters $p_0, p_n, \omega_0, \omega_n$ of the "end" spaces $L_{p_0}(\omega_0)$, $L_{p_n}(\omega_n)$ and therefore it seems that the result is not symmetric with respect to other spaces. However, it is possible to rewrite the result by using the parameters of the spaces $L_{p_i}(\omega_i), L_{p_j}(\omega_j)$ in the case when $p_i \neq p_j$. Indeed, from the equality

$$(\alpha_j - \alpha_i)\left(\frac{1}{p_0} - \frac{1}{p_n}\right) = \frac{1-\alpha_i}{p_0} + \frac{\alpha_i}{p_n} - \left(\frac{1-\alpha_j}{p_0} + \frac{\alpha_j}{p_n}\right) = \frac{1}{p_i} - \frac{1}{p_j}$$

it follows that on $\Omega^{(\vec{k})}$ we have

$$\begin{aligned}\left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{1/p_i-1/p_j}} &\approx \left(2^{k_i-k_j}\,\frac{\omega_0^{1-\alpha_i}(x)\omega_n^{\alpha_i}(x)}{\omega_0^{1-\alpha_j}(x)\omega_n^{\alpha_j}(x)}\right)^{\frac{1}{1/p_i-1/p_j}} \\ &= (2^{k_i-k_j})^{\frac{1}{1/p_i-1/p_j}}\left(\frac{\omega_0}{\omega_n}\right)^{\frac{\alpha_j-\alpha_i}{1/p_i-1/p_j}} \\ &= (2^{k_i-k_j})^{\frac{1}{1/p_i-1/p_j}}\left(\frac{\omega_0}{\omega_n}\right)^{\frac{1}{1/p_0-1/p_n}}.\end{aligned}$$

Therefore, from the well known property of the norm of the Lorentz space $L_{p,q}$:

$$\|cf\|_{L_{p,q(d\mu)}} = \|f\|_{L_{p,q(c^p d\mu)}} \tag{4.14}$$

it follows that if in the formulas for ω and $d\widetilde{\mu}$ instead of $(\omega_0/\omega_n)^{\frac{1}{1/p_0-1/p_n}}$ we take $(\omega_i/\omega_j)^{\frac{1}{1/p_i-1/p_j}}$ then the norm $\|f\omega\chi_{\Omega^{(\vec{k})}}\|_L$ will be changed to an equivalent one, with the constant of equivalence not depending on $\Omega^{(\vec{k})}$.

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