STUDIA MATHEMATICA 170 (3) (2005)

Approximate and L^p Peano derivatives of nonintegral order

by

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Abstract. Let n be a nonnegative integer and let $u \in (n, n+1]$. We say that f is u-times Peano bounded in the approximate (resp. L^p, $1 \le p \le \infty$) sense at $x \in \mathbb{R}^m$ if there are numbers $\{f_\alpha(x)\}$, $|\alpha| \le n$, such that $f(x+h) - \sum_{|\alpha|\le n} f_\alpha(x)h^\alpha/\alpha!$ is $O(h^u)$ in the approximate (resp. L^p) sense as $h \to 0$. Suppose f is u-times Peano bounded in either the approximate or L^p sense at each point of a bounded measurable set E. Then for every $\varepsilon > 0$ there is a perfect set $\Pi \subset E$ and a smooth function g such that the Lebesgue measure of $E \setminus \Pi$ is less than ε and $f = g$ on Π. The function g may be chosen to be in C^u when u is integral, and, in any case, to have for every j of order $\le n$ a bounded jth partial derivative that is Lipschitz of order $u - |j|$.

Pointwise boundedness of order u in the L^p sense does not imply pointwise boundedness of the same order in the approximate sense. A classical extension theorem of Calderón and Zygmund is confirmed.

1. Introduction. Throughout this paper n denotes a fixed nonnegative integer, and u a real number in $(n, n+1]$. All functions will be defined on subsets of m-dimensional Euclidean space and will be real-valued.

Definition 1. We say that f is *u-times approximately Peano bounded* at x if f is Lebesgue measurable and for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, all α_i being nonnegative integers, of order $|\alpha| = \sum_{i=1}^n \alpha_i \le n$ there is a number $f_\alpha(x)$ such that

$$f(x+h) = \sum_{|\alpha|\le n} \frac{h^\alpha}{\alpha!} f_\alpha(x) + M_x(x+h)\|h\|^u$$

where $\|h\|$ denotes Euclidean norm in $\mathbb{R}^m$, $h^\alpha = h_1^{\alpha_1} \cdots h_m^{\alpha_m}$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $f_0(x) = f(x)$ and $M_x(x+h)$ remains bounded as $h \to 0$ through a set of

2000 *Mathematics Subject Classification*: Primary 26B05, 26B35; Secondary 26A16, 26A24.

Key words and phrases: Peano derivative, fractional derivative, nonintegral derivative, function decomposition.

This research was partially supported by NSF grant DMS 9707011 and a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University.

density 1 at $h=0$. The set A has *density* 1 at x (equivalently, x is a *point of density* of A) if A is Lebesgue measurable and

$$\lim_{r\to 0}\frac{\lambda(A\cap B(x,r))}{\lambda(B(x,r))}=1,$$

where $B(x,r)$ denotes the closed ball of radius r centered at x and λ denotes Lebesgue measure.

Definition 2. A function $g:\mathbb{R}^m\to\mathbb{R}$ is in the class B_u if for every multi-index j with $0\le |j|\le n$, the derivative $g^j(x)$ is a bounded function of Lipschitz class $u-|j|$. The functions g^j are the ordinary partial derivatives of g, i.e.,

$$g^j(x)=\frac{\partial^{j_1}}{\partial x_1^{j_1}}\cdots\frac{\partial^{j_m}}{\partial x_m^{j_m}}g(x).$$

The main result is this.

Theorem 1. *Suppose f is u-times approximately Peano bounded on a bounded measurable set E. Then for every $\varepsilon>0$ there is a perfect set $\Pi\subset E$ and a B_u function g such that $\lambda(E\setminus\Pi)<\varepsilon$ and $f=g$ on Π. Furthermore if $u=n+1$, then g can be chosen to belong to C^{n+1}.*

A weaker version of this theorem specialized to dimension $m=1$ and $u=n+1\in\mathbb{Z}$ was established by Marcinkiewicz [3], [7, Vol. II, p. 73]. Marcinkiewicz's result has had many applications. In the hope that our theorems will also prove useful, we try to increase visibility by giving an equivalent statement of Theorem 1 using the language of decomposition: Suppose f is u-times approximately Peano bounded at all $x\in E$, where E is a bounded measurable set. Then for every $\varepsilon>0$ there are functions g and h such that

$$f(x)=g(x)+h(x),\qquad g\in B_u,$$

and

$$\lambda(\operatorname{supp} h\cap E)<\varepsilon.$$

The same result holds in L^p norm. Explicitly, for each $p\in[1,\infty]$, we have the following definition.

Definition 3. We say that f is *u-times Peano bounded in the L^p sense* at x if f is L^p in a neighborhood of x and there are numbers $f_s(x)$ such that

$$\left(\frac{1}{h^m}\int_{\|t\|\le h}\left|f(x+t)-\sum_{|s|\le n}\frac{t^s}{s!}f_s(x)\right|^p dt\right)^{1/p}=L_x(h)\|h\|^u, \tag{1.1}$$

where $L_x(h)$ remains bounded as h tends to zero. When $p=\infty$, the left side of (1.1) means, as usual,

$$\operatorname*{ess\,sup}_{\|t\|\le h}\left|f(x+t)-\sum_{|s|\le n}\frac{t^s}{s!}f_s(x)\right|.$$

If, further, $f \in L^p(\mathbb{R}^m)$ and $L_x(h)$ is uniformly bounded for all h, in reference [1] the function f is then said to belong to $T^p_u(x)$.

THEOREM 2. *Suppose f is u-times Peano bounded in the L^p sense on a bounded measurable set E. Then for every $\varepsilon > 0$, there is a perfect set $\Pi \subset E$ and a B_u function g such that $\lambda(E \setminus \Pi) < \varepsilon$ and $f = g$ on Π. Furthermore if $u = n+1$, then g can be chosen to belong to C^{n+1}.*

One might think that this theorem is an immediate consequence of the "folklore fact" that for $p \in [1,\infty)$, if

$$\limsup_{h\to 0} \Big(h^{-m} \int_{\|t\|\le h} |g(t)|^p \, dt\Big)^{1/p} h^{-u} \le M,$$

then the approximate lim sup of $|g(h)|\, \|h\|^{-u}$ is also less than or equal to M. Actually, this is not true, as we will point out in the first part of the next section wherein the relation between L^p and approximate differential behavior is discussed. The failure of this "fact" requires us to adjoin an additional final section for the L^p case.

In the first part of [1, Theorem 9] the authors prove that if $f \in T^p_u(x)$ uniformly for all x in a closed set, then it is a restriction of a B_u function. In this paper we will show that if f is L_p u-times Peano bounded not necessarily uniformly on a compact set E, then E is the union of a sequence of nested closed sets A_k so that on each A_k, f is a restriction of a B_u function. The result from [1] is a special case of our results because under the hypotheses of the corresponding Theorem 9 in [1], we have $E = A_k$ for some integer k.

The second part of Theorem 9 of [1] asserts that under additional assumptions B_u can be replaced by b_u in the conclusion. (See Subsection 2.2, Definition 6 below for the definition of b_u.) Actually this is not true as we will point out in Subsection 2.2. Our results below show that this was not a very serious defect in the overall program developed in the paper [1]. For example, both [1, Theorem 13] and its given proof are fine if, in the proof, one uses our Theorems 5 and 1 in place of [1, Theorem 9, second part].

Let $h \in [0,1]$. The condition $\limsup_{h\searrow 0} |f(h)| < \infty$ is equivalent to the condition that $\lim_{h\searrow 0} f(h)\varepsilon(h) = 0$ for every nondecreasing function $\varepsilon(h)$ satisfying $\lim_{h\searrow 0} \varepsilon(h) = 0$. In Subsection 2.1 we show that this equivalence fails for approximate limits and that this failure is responsible for the breakdown of the "folklore fact" mentioned above.

2. Two "big oh" and "little oh" comparisons

2.1. *Connections between L^p and approximate behavior.* There has been an idea in the folklore of analysis that approximate behavior is always more general than L^1 behavior. An example on which this notion is based is the fact that if a function is differentiable at a point in the L^1 sense, then it is differentiable in the approximate sense at that point. This section contains three theorems: the first supports the folklore, the second contradicts it, while the third supports it again. The first says that if a function's rate of growth near a point is $o(\|h\|^u)$ in the L^p sense, then its rate of growth must also be $o(\|h\|^u)$ in the approximate sense; the second says that if a function's rate of growth near a point is $O(\|h\|^u)$ in the L^p sense, then its rate of growth is not necessarily $O(\|h\|^u)$ in the approximate sense; the third says that if a function's rate of growth near every point of a set is $O(\|h\|^u)$ in the L^p sense, then at almost every point of that set its rate of growth must also be $O(\|h\|^u)$ in the approximate sense.

Abbreviate $\{x \in \mathbb{R}^m : P(x)\}$ to $\{P(x)\}$.

Definition 4. We say that $\lim \mathrm{ap}_{\|x\|\to 0} f(x) = M$ if there is a set $E \subset \mathbb{R}^m$ so that zero is a point of density of E and $\lim_{\|x\|\to 0,\, x\in E} f(x) = M$. Zero is a *point of dispersion* of a set E if

$$\lim_{h\to 0} \frac{\lambda^*(E \cap B(0,h))}{\lambda(B(0,h))} = 0,$$

where λ^* denotes outer Lebesgue measure. We say that $\limsup \mathrm{ap}_{\|x\|\to 0} f(x) = M$ if for every $N > M$, zero is a point of dispersion of $\{f(x) > N\}$ and M is the infimum of all N with this property. We say that $\liminf \mathrm{ap}_{\|x\|\to 0} f(x) = M$ if for every $N < M$, 0 is a point of dispersion of $\{f(x) < N\}$ and M is the supremum of all N with this property.

The definitions of lim sup ap and lim inf ap can be found on page 218 of [4] and the definition of lim ap can be found on page 323 of [7]. For measurable functions we have $\liminf \mathrm{ap}_{\|x\|\to 0} f(x) = \limsup \mathrm{ap}_{\|x\|\to 0} f(x) = M$ if and only if $\lim \mathrm{ap}_{\|x\|\to 0} f(x) = M$.

Theorem 3. *Let g have an nth L^p Peano derivative at $x \in \mathbb{R}^m$ so that $f(t) = |g(x+t) - \sum_{|j|\le n} g_j(x) t^j|$ satisfies*

$$\frac{1}{h^m} \int_{B(0,h)} f^p = o(h^{np})$$

as $h \searrow 0$. Then g also has an nth approximate derivative at x, in other words, $\lim \mathrm{ap}_{\|t\|\to 0} f(t)/\|t\|^n = 0$.

Proof. We have $\varepsilon_N \to 0$, where ε_N is defined by

$$\frac{1}{2^{-Nm}} \int_{\|x\|\le 2^{-N}} f^p = \varepsilon_N^2 2^{-npN}.$$

Let $I_N = B(0, 2^{-N}) \setminus B(0, 2^{-N-1})$, and let E_N be defined by

$$E_N = \{x \in I_N : f^p(x) \ge \varepsilon_N 2^{-npN}\}.$$

From

$$\varepsilon_N^2 2^{-(np+m)N} \geq \int\limits_{I_N} f^p \geq \int\limits_{E_N} \varepsilon_N 2^{-npN} = \varepsilon_N 2^{-npN} \lambda(E_N),$$

it follows that

$$\varepsilon_N \geq \frac{\lambda(E_N)}{2^{-Nm}} = c_m \frac{\lambda(E_N)}{\lambda(I_N)},$$

so 0 is a point of density of $\bigcup (E_N)^{\mathrm{c}} = G$ and

$$\lim_{\|t\|\to 0,\, t\in G} \frac{f(t)}{\|t\|^n} = 0. \ \blacksquare$$

The above proof is a routine adaptation of a $p = 2$ one-dimensional argument given on page 324 of [7] and is only worth mentioning because of the following example. For the example we specialize to $m = 1$, $p = 1$, and f supported in $[0, 1]$.

THEOREM 4. *There is a function f satisfying*

$$\frac{1}{h}\int\limits_0^h f = O(h)$$

as h goes to 0 *such that for every finite number M*, $\limsup \mathrm{ap}_{x\to 0} f(x)/x > M$.

Proof. We give an example of a nonnegative function f satisfying

$$\frac{1}{h}\int\limits_0^h f = O(h) \tag{2.1}$$

such that $\limsup \mathrm{ap}_{h\to 0} f(h)/h$ is infinite. It is sufficient to prove that for every positive integer j, $\{f(x) \geq jx\}$ does not have 0 as a point of dispersion. Let e_j^k, $j = 1, \ldots, k$, be disjoint subintervals of $[2^{-k-1}, 2^{-k}]$ such that $\lambda(e_j^k) = 2^{-k-j-1}$. Let

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{k} j2^{-k} \chi_{e_j^k}(x).$$

Then

$$\begin{aligned}\int\limits_0^{2^{-n}} f(x)\,dx &= \sum_{k=n}^{\infty}\sum_{j=1}^{k} j2^{-k}\lambda(e_j^k) = \sum_{k=n}^{\infty}\sum_{j=1}^{k} j2^{-k}2^{-k-j-1} \\ &\leq \sum_{k=n}^{\infty} 2^{-2k} \sum_{j=1}^{\infty} j2^{-j-1} \leq \Big(4\sum_{j=1}^{\infty} j2^{-j-1}\Big)(2^{-n})^2.\end{aligned}$$

Now, let $j \geq 1$ be fixed. For any $k \geq j$, if $x \in e_j^k$ then

$$\frac{f(x)}{x} \geq \frac{2^{-k} j}{2^{-k}} = j.$$

Thus,

$$e_j^k \subset \{x : f(x) \geq jx\} \cap [0, 2^{-k}].$$

Hence

$$\lambda(\{x : f(x) \geq jx\} \cap [0, 2^{-k}]) \geq \lambda(e_j^k) = 2^{-k-j-1}.$$

Thus, for any positive integer j and for any $k \geq j$,

$$\frac{\lambda(\{x : f(x) \geq jx\} \cap [0, 2^{-k}])}{\lambda([0, 2^{-k}])} \geq 2^{-j-1}.$$

Hence, for any positive integer j, zero is not a point of dispersion for the set $\{f(x) \geq jx\}$. ∎

Let $h \in [0, 1]$. The condition $\limsup_{h \searrow 0} |f(h)| < \infty$ is equivalent to the condition that $\lim_{h \searrow 0} f(h)\varepsilon(h) = 0$ for every nondecreasing function $\varepsilon(h)$ satisfying $\lim_{h \searrow 0} \varepsilon(h) = 0$. We use the example from the previous theorem to show that this equivalence fails for approximate limits:

Proposition 1. *Assume that for any nondecreasing function $\varepsilon(x)$ on $(0, 1]$ such that $\lim_{h \to 0} \varepsilon(h) = 0$, we have*

$$\operatorname*{lim\,ap}_{x \to 0} f(x)\varepsilon(x) = 0. \tag{2.2}$$

Then it does not follow that there must exist a constant M so that 0 is a point of dispersion of $\{x : f(x) \geq M\}$. Consequently, f need not have a finite $\limsup \text{ap}$ *at $x = 0$.*

Proof. Let f be the example function just above and let $g(x) := f(x)/x$. We have already shown that there does not exist a constant M so that $\{x : g(x) \geq M\}$ has 0 as a point of dispersion. Let $\varepsilon(x)$ be a nondecreasing function on $(0, 1]$ such that $\lim_{h \to 0} \varepsilon(h) = 0$. Let $\zeta, \eta > 0$. Pick k so large that $\sum_{i=1}^{k} 2^{-i} > 1 - \zeta$. Then pick N so large that $\varepsilon(2^{-N}) < \eta/2k$. For every $M \geq N$,

$$\{x \in [2^{-M-1}, 2^{-M}] : g(x)\varepsilon(x) > \eta\} \subset \bigcup_{i=k+1}^{M} e_i^M$$

so that the relative density of $\{g > \eta\}$ in $[2^{-M-1}, 2^{-M}]$ is less than ζ. Hence the relative density of $\{g\varepsilon > \eta\}$ in $[0, 2^{-N}]$ is less than ζ. Since ζ and η were arbitrary, relation (2.2) holds for g and $\varepsilon(x)$. Since $\varepsilon(x)$ was arbitrary, relation (2.2) holds for g and every such $\varepsilon(x)$. ∎

As Theorem 4 shows we cannot prove a pointwise analogue of Theorem 3 in the "big oh" case. The following corollary of Theorem 8 of the final section is a substitute.

THEOREM 5. *Suppose that $\lambda(E) > 0$ and at each $x \in E$ we have*

$$\left(\frac{1}{|B(0,h)|}\int_{B(0,h)}\Big|g(x+t)-\sum_{|j|\le n} g_j(x)t^j\Big|^p dt\right)^{1/p} = O(h^u). \tag{2.3}$$

Then at a.e. $x \in E$,

$$\limsup_{\|h\|\to 0}\operatorname{ap}\frac{|g(x+h)-\sum_{|j|\le n} g_j(x)h^j|}{\|h\|^u} < \infty.$$

2.2. *A T_u^p extension theorem without a t_u^p analogue.* Roughly speaking, Theorem 9 of [1] asserts that a uniformly T_u^p function can be extended to a B_u function and that "similarly" a t_u^p function can be extended to a b_u function. The proof of the former statement appearing in [1] is correct. The latter statement is false when u is an integer. To make this assertion precise, we first give Calderón and Zygmund's definitions of $t_u^p(x_0)$, $B_u(Q)$ and $b_u(Q)$, where x_0 is a point of $\mathbb{R}^m$ and Q is a closed set in $\mathbb{R}^m$.

DEFINITION 5. Let f be a function in $T_u^p(x_0)$. We say that $f \in t_u^p(x_0)$ if there exists a polynomial $P(x - x_0)$ of degree $\le u$ such that

$$\left(\varrho^{-m}\int_{|x-x_0|\le\varrho}|f(x)-P(x-x_0)|^p\,dx\right)^{1/p} = o(\varrho^u) \quad \text{as } \varrho \to 0.$$

Here $1 \le p < \infty$.

DEFINITION 6. Let Q be a closed set. For a bounded function f we say that $f \in B_u(Q)$, $u > 0$, if there exist bounded functions f_α, $|\alpha| < u$, such that

$$f_\alpha(x+h) = \sum_{|\beta|<u-|\alpha|} f_{\alpha+\beta}(x)\,\frac{h^\beta}{\beta!} + R_\alpha(x,h)$$

for all x and $x+h$ in Q, with $|R_\alpha(x,h)| \le C\|h\|^{u-|\alpha|}$. We say that $f \in b_u(Q)$, $u \ge 0$, if there exist functions f_α, $|\alpha| \le u$, such that

$$f_\alpha(x+h) = \sum_{|\beta|\le u-|\alpha|} f_{\alpha+\beta}(x)\,\frac{h^\beta}{\beta!} + R_\alpha(x,h)$$

for all x and $x + h$ in Q, with $|R_\alpha(x,h)| \le C\,\|h\|^{u-|\alpha|}$ and, in addition, $R_\alpha(x,h) = o(\|h\|^{u-|\alpha|})$ as $h \to 0$, uniformly in $x \in Q$. The connection between $B_u(Q)$ and the previously defined B_u is that $f \in B_u$ if $f \in B_u(\mathbb{R}^m)$ and additionally for all j with $|j| \le n$ the Peano derivative f_j is equal to f^j, the ordinary partial derivative of f. Similarly we define $f \in b_u$ to mean $f \in b_u(\mathbb{R}^m)$ and $f^j = f_j$ whenever $|j| \le u$.

Now the second part of Theorem 9 of [1] asserts that if $f \in B_u(Q)$ and in addition $f \in t_u^p(x_0)$ for all x_0 in the closed set Q, then $\overline{f}$ can be chosen to be in $b_u(\mathbb{R}^m)$ in such a way that $(\partial/\partial x)^\beta \overline{f}(x_0) = f_\beta(x_0)$ for $|\beta| \le u$, and all $x_0 \in Q$.

To see what the problem is, let $u = m = p = 1$. Then let $Q = \mathbb{R}^1$ be the entire space, so that the original function f and the extension function $\overline{f}$ coincide. Suppose that f is compactly supported and has a uniformly bounded derivative which is not continuous. Then $f \in t_1^1(x_0)$ for every real x_0, but $f \notin b_1(\mathbb{R}^1)$ since f' is not continuous.

3. Proofs: the approximate case. Recall that n is the fixed nonnegative integer $\lceil u \rceil - 1$ so that $n < u \le n+1$.

Some additional notation and simple facts about multi-indices will be needed. If α and β are two multi-indices, then $\alpha+\beta = (\alpha_1+\beta_1, \dots, \alpha_m+\beta_m)$. Moreover $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for each $i = 1, \dots, m$. For $\beta \le \alpha$ we set $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_m - \beta_m)$ and $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. With this notation the following version of the Binomial Theorem holds. If $x, y \in \mathbb{R}^m$, then $(x+y)^\alpha = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha-\beta}$. Recall that g^j means the ordinary jth partial derivative.

The first part of Theorem 1 is immediate from the following result which may be worthwhile on its own.

Theorem 6. *Suppose f is u-times approximately Peano bounded on a compact set E. Then there is a decomposition of E into a nested sequence $\{A_k\}$ of closed sets such that on A_k the function f is a restriction of a B_u function.*

In order to define the sets A_k of Theorem 6 we need some additional notation and the following lemma. For the rest of this paper we will set

$$S = \sum_{s=0}^{n} \binom{s+m-1}{s} = \binom{n+m}{n};$$

S denotes the number of multi-indices less than or equal to n.

Lemma 1. *Let $P_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, S'$, be polynomials that are independent vectors over $\mathbb{R}$. For any S' points $h_1, \dots, h_{S'}$ of $\mathbb{R}^m$, set*

$$M = \begin{pmatrix} P_1(h_1) & P_2(h_1) & \cdots & P_{S'}(h_1) \\ \vdots & \vdots & \cdots & \vdots \\ P_1(h_{S'}) & P_2(h_{S'}) & \cdots & P_{S'}(h_{S'}) \end{pmatrix}.$$

Then one can choose the h_i such that $\det M \ne 0$.

Proof. Independence means that if $a_1, \dots, a_{S'}$ are real numbers such that

$$a_1 P_1(h) + \cdots + a_{S'} P_{S'}(h) = 0 \quad \text{for all } h \in \mathbb{R}^m,$$

then $a_1 = \cdots = a_{S'} = 0$.

If $S' = 1$, i.e. $M = [P_1(h)]$, then $\det M = 1 \cdot P_1(h)$, so there is an h for which $P_1(h) \neq 0$ by definition of independence.

Assume the lemma has been proved for $S' - 1$ and let M be as above. Expanding along the first row, we have

$$\det M = P_1(h_1)C_{11}(h_2, \ldots, h_{S'}) + P_2(h_1)C_{12}(h_2, \ldots, h_{S'}) + \cdots . \tag{3.1}$$

By induction there are points $h_2, \ldots, h_{S'}$ such that $C_{11}(h_2, \ldots, h_{S'}) \neq 0$. Now fixing such a choice $h_2, \ldots, h_{S'}$ and thinking of h_1 as a variable, if $\det M$ were identically 0, then equation (3.1) would contradict independence. ■

We will apply this lemma with $S' = S$ and $P_i(h) = h^i$ for $0 \leq |i| \leq n$ (the monomials h^i are independent over $\mathbb{R}$) to obtain S points $\{h_i\}$ such that the determinant of the corresponding M is not zero. Since scaling a column scales the value of the determinant, there is no loss of generality in assuming that $\|h_i\| < 1$ for each multi-index $0 \leq |i| \leq n$. Continuity of $\det M$ allows us to find a positive number $\delta < 1$ such that for every i, $B(h_i, \delta) \subset B(0, 1)$ and such that $|\det M| \geq \delta$ for any S points $q_i \in B(h_i, \delta)$. For the rest of this section δ and $B(h_i, \delta)$ denote the number and the balls respectively that were just introduced.

Let $M_x(y)$ be the real number defined by

$$f(y) - \sum_{|\alpha| \leq n} \frac{(y - x)^\alpha}{\alpha!} f_\alpha(x) = M_x(y)\|y - x\|^u. \tag{3.2}$$

For a positive integer k let $N_k(x, r) = \{y \in B(x, r) : |M_x(y)| \leq k\}$. Since f is a measurable function, $N_k(x, r)$ is a measurable set. We define

$$A_k = \left\{x \in E : \lambda(N_k(x, r)) \geq \left(1 - \frac{1}{3}\frac{\delta^m}{2^m}\right)\lambda(B(x, r)) \text{ for all } r < \frac{1}{k}\right\}.$$

Clearly the sets A_k are nested and since f is u-times approximately Peano bounded on E, we have $E = \bigcup_{k=1}^{\infty} A_k$. The proof of Theorem 6 follows from the theorem below and the Extension Theorem 4 from [5, p. 177].

THEOREM 7. *The sets A_k are closed and there is a constant M such that for all x and y from A_k we have*

$$\left| f_s(y) - \sum_{i=s}^{n} \frac{(y - x)^{i-s}}{(i - s)!} f_i(x) \right| \leq M\|y - x\|^{u-|s|} \quad \text{for } 0 \leq |s| \leq n.$$

Before we prove this theorem we will need several lemmas, the first of which is a several variables version of Lemma 5 in [2].

LEMMA 2. *Let $x, y, h \in \mathbb{R}^m$. Suppose that f is u-times approximately Peano bounded at x and y. Then*

$$\sum_{|\alpha|\le n} \frac{h^\alpha}{\alpha!}\Bigl(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!}\, f_{\alpha+\beta}(x)\Bigr) = \|y-x+h\|^u M_x(y+h) - \|h\|^u M_y(y+h). \tag{3.3}$$

Proof. This identity is obtained by writing $f(y+h)$ in two ways as follows. First we have

$$f(y+h) = \sum_{0\le|\alpha|\le n} \frac{h^\alpha}{\alpha!}\, f_\alpha(y) + M_y(y+h)\|h\|^u$$

by expanding about the point y. Then write $y+h$ as $x+(y-x+h)$ and expand about x to get

$$\begin{aligned} f(y+h) &= f(x+y-x+h) \\ &= \sum_{|\beta|\le n} \frac{(y-x+h)^\beta}{\beta!}\, f_\beta(x) + M_x(y+h)\|y-x+h\|^u \\ &= \sum_{|\beta|\le n}\sum_{\alpha\le\beta} \frac{(y-x)^{\beta-\alpha}h^\alpha}{\alpha!(\beta-\alpha)!}\, f_\beta(x) + \|y-x+h\|^u M_x(y+h). \end{aligned}$$

Change the order of summation to obtain

$$f(y+h) = \sum_{|\alpha|\le n}\sum_{\substack{\beta\ge\alpha\\ |\beta|\le n}} \frac{(y-x)^{\beta-\alpha}h^\alpha}{\alpha!(\beta-\alpha)!}\, f_\beta(x) + \|y-x+h\|^u M_x(y+h).$$

Substitute for $\beta-\alpha$ a new positive multi-index β to obtain

$$\begin{aligned} f(y+h) &= \sum_{|\alpha|\le n}\ \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta h^\alpha}{\alpha!\beta!}\, f_{\alpha+\beta}(x) + \|y-x+h\|^u M_x(y+h) \\ &= \sum_{|\alpha|\le n} \frac{h^\alpha}{\alpha!}\Bigl(\sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!}\, f_{\alpha+\beta}(x)\Bigr) + \|y-x+h\|^u M_x(y+h). \end{aligned}$$

Equating these two expansions gives the desired result. ∎

LEMMA 3. *Let $y,b,c\in\mathbb{R}^m$. Suppose that f is u-times approximately Peano bounded at b and c. Then*

$$\begin{aligned} &f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-b)^\beta}{\beta!}\, f_{\alpha+\beta}(b) \\ &\quad = f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-c)^\beta}{\beta!}\, f_{\alpha+\beta}(c) \\ &\qquad + \sum_{|\alpha+\beta|\le n} \frac{(y-c)^\beta}{\beta!}\Bigl(f_{\alpha+\beta}(c) - \sum_{|\alpha+\beta+\eta|\le n} \frac{(c-b)^\eta}{\eta!}\, f_{\alpha+\beta+\eta}(b)\Bigr). \end{aligned}$$

Proof. It is enough to check that

$$\sum_{|\alpha+\beta|\le n}\ \sum_{|\alpha+\beta+\eta|\le n} \frac{(y-c)^\beta}{\beta!}\,\frac{(c-b)^\eta}{\eta!} f_{\alpha+\beta+\eta}(b) = \sum_{|\alpha+\beta|\le n} \frac{(y-b)^\beta}{\beta!}\, f_{\alpha+\beta}(b).$$

Indeed, if we introduce a new positive multi-index $\kappa = \beta+\eta$, then the left side is

$$\sum_{|\alpha+\beta|\le n}\ \sum_{\substack{\beta\le\kappa\\ |\alpha+\kappa|\le n}} \frac{(y-c)^\beta}{\beta!}\,\frac{(c-b)^{\kappa-\beta}}{(\kappa-\beta)!}\, f_{\alpha+\kappa}(b)$$

$$= \sum_{|\alpha+\kappa|\le n} \biggl(\sum_{\beta\le\kappa} \frac{(y-c)^\beta}{\beta!}\,\frac{(c-b)^{\kappa-\beta}}{(\kappa-\beta)!}\biggr) f_{\alpha+\kappa}(b) = \sum_{|\alpha+\kappa|\le n} \frac{(y-b)^\kappa}{\kappa!}\, f_{\alpha+\kappa}(b). \ \blacksquare$$

LEMMA 4. *Let $x \in A_k$ and $r < 1/k$. If I is a ball inside $B(x,r)$ and such that $\lambda(I) \ge \delta^m 2^{-m}\lambda(B)$, then $\lambda(N_k(x,r)\cap I) \ge \frac{2}{3}\lambda(I)$.*

Proof. Indeed, $\lambda(B(x,r)) - \lambda(I) + \lambda(N_k(x,r)\cap I) \ge \lambda(N_k(x,r)\cap B(x,r)) \ge (1-\frac13\delta^m 2^{-m})\lambda(B(x,r))$. Hence $\lambda(N_k(x,r)\cap I) \ge \lambda(I) - \frac13\delta^m 2^{-m}\lambda(B(x,r)) \ge \lambda(I) - \frac13\lambda(I) = \frac23\lambda(I)$. $\blacksquare$

Proof of Theorem 7. Let $x \in \overline{A}_k$, and let $K \ge k$ be such that $x \in A_K$. Now let $y \in A_k$ be such that $\|y-x\| < 1/2K$. For a multi-index i, $0 \le |i| \le n$, let $I_i = y + \|y-x\| B(h_i,\delta)$. Then I_i is inside $B(y,\|y-x\|) \subset B(x, 2\|y-x\|)$, and $\lambda(I_i) = \delta^m\lambda(B(y,\|y-x\|) = \delta^m 2^{-m}\lambda(B(x,2\|y-x\|))$. By Lemma 4 we have $\lambda(N_k(y,\|y-x\|)\cap I_i) \ge \frac23\lambda(I_i)$ and $\lambda(N_K(x,2\|y-x\|)\cap I_i) \ge \frac23\lambda(I_i)$. Therefore for each $i \in \{\alpha : |\alpha| \le n\}$ there are points $y_i \in N_k(y,\|y-x\|)\cap N_K(x,2\|y-x\|)\cap I_i$. Notice that $q_i = (y_i - y)/\|y-x\| \in B(h_i,\delta)$ so that $|\det M| \ge \delta$ where $\det M$ was evaluated at $\{q_i\}_{0\le|i|\le n}$.

By replacing h in (3.3) with $y_j - y$ for $j \in \{\alpha : |\alpha| \le n\}$, we obtain a system of S linear equations in the S unknowns $X_{00\ldots0}, \ldots, X_{0\ldots0n}$,

$$\sum_{0\le|s|\le n} (y_j - y)^s X_s = b_j, \qquad j \in \{\alpha : |\alpha| \le n\}, \tag{3.4}$$

where

$$X_s = \frac{1}{s!}\biggl(f_s(y) - \sum_{|s+i|\le n} \frac{(y-x)^i}{i!}\, f_{s+i}(x)\biggr)$$

and

$$b_j = \|y_j - x\|^u M_x(y_j) - \|y_j - y\|^u M_y(y_j).$$

(In order to apply standard matrix methods such as Cramer's rule, we assume that the set $\{\alpha : |\alpha| \le n\}$ of S elements is linearly ordered.) The main determinant $\det\Delta$ of the system (3.4) is $\|y-x\|^T \det M$, where $T = \sum_{|\alpha|\le n}|\alpha|$. Hence $|\det\Delta| \ge \|y-x\|^T\delta$. On the other hand, if $\det\Delta_s$ is the determinant obtained by replacing the sth column of Δ with the values

$b_1, \dots, b_S$, then in the expansion of Δ_s about the sth column, each minor is the sum of $(S-1)!$ terms of the form $\pm \prod_{j\neq j_0,\, |\alpha|\le n,\, \alpha\neq s}(y_j - y)^\alpha$. Since $\|y_j - y\| \le \|y-x\|$, $\|y_j - x\| \le 2\|y-x\|$ and $|M_x(y_j)| \le K$, $|M_y(y_j)| \le k$ we have $|b_j| \le \|y-x\|^u(k+2^uK)$. Hence

$$|\Delta_s| \le \sum_{j\in\{k\,:\,|k|\le n\}} |b_j|(S-1)!\,\|y-x\|^{T-|s|} \le S!\,\|y-x\|^{T+u-|s|}(k+2^uK).$$

By Cramer's rule

$$|X_s| = \left|\frac{\det \Delta_s}{\det \Delta}\right| \le S!\,\|y-x\|^{u-|s|}\,\frac{k+2^uK}{\delta}. \tag{3.5}$$

In particular, for $0\le |s|\le n$, $\lim_{y\to x,\, y\in A_k} f_s(y) = f_s(x)$.

Next let $x_j \in A_k$ be a sequence converging to x. Fix $r < 1/k$. Let $y \in \bigcap_{j=1}^\infty \bigcup_{s=j}^\infty N_k(x_s, r)$. Then y is in $N_k(x_j, r)$ for infinitely many j and thus continuity of f_s establishes

$$\left| f(y) - \sum_{|i|\le n} \frac{(y-x)^i}{i!}\, f_i(x)\right| \le k\|y-x\|^u.$$

Therefore $y\in N_k(x,r)$ and thus $\bigcap_{j=1}^\infty \bigcup_{s=j}^\infty N_k(x_s,r) \subset N_k(x,r)$. Since

$$\lambda\Big(\bigcup_{s=j}^{\infty} N_k(x_s,r)\Big) \ge \left(1 - \frac{1}{3}\,\frac{\delta^m}{2^m}\right)\lambda(B(x,r))$$

we have

$$\lambda(N_k(x,r)) \ge \left(1 - \frac{1}{3}\,\frac{\delta^m}{2^m}\right)\lambda(B(x,r)).$$

Thus $x\in A_k$ and A_k is closed. Hence in the proof of the theorem we could take $K = k$ to obtain

$$\left| f_s(y) - \sum_{|i+s|\le n} \frac{(y-x)^i}{i!}\, f_{i+s}(x)\right| \le M\|y-x\|^{u-|s|} \tag{3.6}$$

for $0\le |s|\le n$, where $M = \max_{|\alpha|\le n}\{\alpha!\}(1+2^u)k/\delta$ (in (3.5) replace $S!$ by $\max_{|\alpha|\le n}\{\alpha!\}$ and K by k) whenever x, y are in A_k such that $\|y-x\| < 1/2k$.

We would like to get these inequalities for any x and y from A_k. To that end let $\{I_j\}$ be a finite open cover of A_k with centers from A_k and radii equal to $1/2k$. Since J, the set of all centers of the balls $\{I_j\}$, is finite, there is a constant W such that

$$\max_{b,c\in J}\left| f_s(c) - \sum_{|r+s|\le n} \frac{(c-b)^r}{r!}\, f_{r+s}(b)\right| \le W\|c-b\|^{u-|s|}, \qquad 0\le |s|\le n. \tag{3.7}$$

We may assume that x and y are in two different balls centered at b and c respectively, and that $\|y-x\| > 1/2k$. We first show that there is a constant

M' such that

$$\left|f_s(y) - \sum_{|i+s|\le n} \frac{(y-b)^i}{i!}\, f_{i+s}(b)\right| \le M'\|y-b\|^{u-|s|}, \qquad 0 \le |s| \le n. \tag{3.8}$$

By Lemma 3,

$$\begin{aligned}
&\left|f_s(y) - \sum_{|i+s|\le n} \frac{(y-b)^i}{i!}\, f_{i+s}(b)\right| \\
&\qquad = \Bigg|f_s(y) - \sum_{|i+s|\le n} \frac{(y-c)^i}{i!}\, f_{i+s}(c) \\
&\qquad\qquad + \sum_{|i+s|\le n} \frac{(y-c)^i}{i!}\left[f_{i+s}(c) - \sum_{|i+s+r|\le n} \frac{(c-b)^r}{r!}\, f_{i+s+r}(b)\right]\Bigg| \\
&\qquad \le M\|y-c\|^{u-|s|} + \sum_{|i+s|\le n} \frac{\|y-c\|^{|i|}}{i!}\, W\|c-b\|^{u-|i+s|}.
\end{aligned}$$

Since $\|y-c\| \le \|y-b\|$ and $\|c-b\| \le 2\|y-b\|$, the last quantity is

$$\le M\|y-b\|^{u-|s|} + \sum_{|i+s|\le n} \frac{\|y-b\|^{|i|}}{i!}\, 2^{u-|s|} W\|y-b\|^{u-|s|} = M_s\|y-b\|^{u-|s|}.$$

Setting $M' = \max_{|s|\le n}\{M_s\}$ establishes (3.8).

We use Lemma 3 again but this time applied to y, x, and b:

$$\begin{aligned}
&\left|f_s(y) - \sum_{|i+s|\le n} \frac{(y-x)^i}{i!}\, f_{i+s}(x)\right| \\
&\qquad = \Bigg|f_s(y) - \sum_{|i+s|\le n} \frac{(y-b)^i}{i!}\, f_{i+s}(b) \\
&\qquad\qquad + \sum_{|i+s|\le n} \frac{(y-b)^i}{i!}\left[f_{i+s}(b) - \sum_{|i+s+r|\le n} \frac{(b-x)^r}{r!} f_{i+s+r}(x)\right]\Bigg|.
\end{aligned}$$

Inequalities (3.8) and (3.6) applied to the right hand side yield

$$\le M'\|y-b\|^{u-|s|} + \sum_{|i+s|\le n} \frac{\|y-b\|^{|i|}}{i!}\, M\|b-x\|^{u-|i+s|}.$$

Finally since $\|y-b\| \le 2\|y-x\|$ and $\|b-x\| \le \|y-x\|$, the last quantity is

$$\le 2^{u-|s|} M'\|y-x\|^{u-|s|} + \sum_{|i+s|\le n} \frac{2^{|i|}\|y-x\|^{|i|}}{i!}\, M\|y-x\|^{u-|i+s|}$$

$$= \left[2^{u-|s|}M' + \sum_{|i+s|\le n} \frac{2^{|i|}}{i!} M\right] \|y-x\|^{u-|s|}$$

$$\le \left[2^{u}M' + \sum_{|i|\le n} \frac{2^{|i|}}{i!} M\right] \|y-x\|^{u-|s|}. \ \blacksquare$$

In the special case $u = n+1$, we can improve Theorem 6, as in the following corollary, which is also the second part of Theorem 1.

COROLLARY 1. *Suppose f is $n+1$-times approximately Peano bounded on a bounded measurable set E. Then for every $\varepsilon > 0$ there is a closed set $\varPi$ with $\lambda(E - \varPi) < \varepsilon$ and a C^{n+1} function h such that on $\varPi$ the function f and its partial derivatives agree with h and the corresponding partial derivatives of h.*

Proof. Let A_k and g_k be from Theorem 6 such that $\lambda(E - A_k) < \varepsilon/3$. By Theorem 6, for each multi-index j with $|j| = n$, the function g_k^j is Lipschitz. Hence by a theorem of H. Rademacher g_k^j is totally differentiable at almost every $x \in \mathbb{R}^m$. Now let $P \supset A_k$ be a bounded open set such that $\lambda(P - A_k) < \varepsilon/3$. The function g_k on P satisfies the conditions of Theorem 4 from [6], so by that theorem, there is a closed set $Q \subset P$ such that $\lambda(P - Q) < \varepsilon/3$ and a C^{n+1} function h that agrees with g_k on Q. Notice that $\lambda(E - A_k \cap Q) < \varepsilon$ and that $h = f$ on the closed set $\varPi = A_k \cap Q$. $\blacksquare$

Corollary 1 in the case $n = 0$ was proved by H. Whitney. (See Theorem 1 in [6].) The proof of this result from [6] uses the fact that approximate partial derivatives of a measurable function are measurable. (See Theorem 11.2, page 299 of [4].) The proof of Corollary 1 does not require measurability of the approximate partial derivatives f_s for $|s| \ge 1$. However measurability of the f_s is an immediate consequence of this corollary and Luzin's theorem.

COROLLARY 2. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. Suppose f is $n+1$-times approximately Peano differentiable on a measurable set E. Then for every multi-index $|s| = n+1$, the partials f_s are measurable.*

Proof. By the phrase "f is n-times approximately Peano differentiable at a point x" we mean that the left hand side of expression (3.2) is $o(\|y - x\|^n)$ as $y \to x$ through a set of density 1 at x. If f is $n+1$-times approximately Peano differentiable on a measurable set E, then clearly f is $n+1$-times approximately Peano bounded with

$$|M_x(y)| \le \sum_{|s|=n+1} \frac{1}{s!} \max_{|s|=n+1} |f_s(x)| + 1.$$

For an integer i let $E_i = E \cap B(0, i)$. Then E is a countable union of bounded measurable sets E_i. By Corollary 1, for each $\varepsilon > 0$, f_s agrees with a continuous function on a set F_i with $\lambda(E_i - F_i) < \varepsilon$. Hence by Luzin's theorem f_s is measurable on each E_i. Since $f_s = \lim_{i\to\infty} f_s\chi_{E_i}$ where χ_{E_i} denotes the characteristic function of E_i we see that f_s is measurable. ∎

4. Proofs: the L^p case

DEFINITION 7. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. We say that f is *locally L^p u-times Peano bounded* at x if for each multi-index α with $|\alpha| \le n$, there is a number $f_\alpha(x)$ such that

$$(4.1)\qquad \Big(\frac{1}{\varrho^m}\int\limits_{\|y-x\|\le\varrho}\Big|f(y)-\sum_{0\le|\alpha|\le n}\frac{(y-x)^\alpha}{\alpha!}\,f_\alpha(x)\Big|^p\,dy\Big)^{1/p} = L_x(\varrho)\varrho^u,$$

where $L_x(\varrho)$ remains bounded as $\varrho \to 0$. In this definition we will assume that $f_{(0,0,\ldots,0)}(x) = f(x)$.

Recall S denotes the number of multi-indices less than or equal to $n = \lceil u\rceil - 1$.

The main result of this section is this.

THEOREM 8. *Suppose f is locally L^p u-times Peano bounded on a compact set E. Then there is a decomposition of E into a nested sequence $\{A_k\}$ of closed sets such that on A_k the function f is a restriction of a B_u function.*

Proof. Let

$$M_x(y) = f(y) - \sum_{0\le|\alpha|\le n}\frac{(y-x)^\alpha}{\alpha!}\,f_\alpha(x).$$

Then f being locally L^p u-times Peano bounded at x means that there is $\delta > 0$ and a constant M such that

$$\Big(\int\limits_{\|y-x\|\le\varrho}|M_x(y)|^p\,dy\Big)^{1/p} \le M\varrho^{u+m/p}\qquad\text{for all } 0<\varrho<\delta.$$

For a positive integer k let

$$A_k = \Big\{x\in E : \int\limits_{\|y-x\|\le\varrho}|M_x(y)|^p\,dy \le k\varrho^{u+m/p}\text{ for all } 0<\varrho<\frac{1}{k}\Big\}.$$

Clearly the sets A_k are nested and since f is locally L^p u-times Peano bounded on E, we have $E = \bigcup_{k=1}^\infty A_k$. The proof of Theorem 8 follows from Theorem 9 below and the Extension Theorem 4 from [5, p. 177]. ∎

THEOREM 9. *The sets A_k are closed and there is a constant M such that for all x and y from A_k we have*

$$\Big|f_s(y) - \sum_{|s+i|\le n}\frac{(y-x)^i}{i!}\,f_{s+i}(x)\Big| \le M|y-x|^{u-|s|}\qquad\textit{for } 0\le|s|\le n.$$

In the proof of this theorem we will use the following two lemmas.

Lemma 5. *Let $x, y, h \in \mathbb{R}^m$. Then*

$$\sum_{|\alpha|\le n} \frac{h^\alpha}{\alpha!}\Big(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big) = M_x(y+h) - M_y(y+h). \tag{4.2}$$

The proof of this lemma is implicit in the proof of Lemma 2 of the last section. We will also need the following version of Lemma 2.6 from p. 182 of [1].

Lemma 6. *Let C denote the vector space of continuous functions defined on the closed ball $\overline{B}(0,1) \subset \mathbb{R}^m$. Then the linear map $T : C \to \mathbb{R}^S$ defined by $T(\varphi) = (\int_{\|h\|\le 1} \varphi(h) h^\alpha\, dh : |\alpha| \le n)$ is onto.*

Proof. Indeed, if T were not onto then there are S numbers $\{c_s\}$ not all zero such that for every $\varphi \in C$ we have

$$\sum_{|s|\le n} c_s \int_{\|h\|\le 1} \varphi(h) h^s\, dh = 0. \tag{4.3}$$

In particular this would be true for $\varphi(h) = \sum_{|s|\le n} c_s h^s$. Substitution in (4.3) yields $\int_{\|h\|\le 1} (\sum_{|s|\le n} c_s h^s)^2\, dh = 0$; thus $c_s = 0$ for all s. This is a contradiction. ∎

Proof of Theorem 9. Let $x \in \overline{A}_k$, and let $K \ge k$ be such that $x \in A_K$. Now let $y \in A_k$ be such that $\|y - x\| < 1/2K$. By Lemma 6 for every multi-index $|\alpha| \le n$ there is $\varphi_\alpha \in C$ such that $T(\varphi_\alpha) = (0, \ldots, 0, \alpha!, 0, \ldots, 0)$, where in the vector $(0, \ldots, 0, \alpha!, 0, \ldots, 0) \in \mathbb{R}^S$, the only nonzero entry is the one that corresponds to α. By Lemma 5,

$$\begin{aligned} &\int_{\|h\|\le\|y-x\|} \varphi_\alpha\Big(\frac{h}{\|y-x\|}\Big) \sum_{|\alpha|\le n} \frac{h^\alpha}{\alpha!}\Big(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big)\, dh \\ &\qquad = \int_{\|h\|\le\|y-x\|} \varphi_\alpha\Big(\frac{h}{\|y-x\|}\Big)(M_x(y+h) - M_y(y+h))\, dh. \end{aligned} \tag{4.4}$$

The left hand side of (4.4) reduces to

$$\Big(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big) \int_{\|h|\le\|y-x\|} \varphi_\alpha\Big(\frac{h}{\|y-x\|}\Big) \frac{h^\alpha}{\alpha!}\, dh.$$

The change of variable $k = h/\|y - x\|$ yields

$$\begin{aligned} &\Big(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big) \|y-x\|^{|\alpha|+m} \int_{\|k\|\le 1} \varphi_\alpha(k) \frac{k^\alpha}{\alpha!}\, dk \\ &\qquad = \Big(f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big) \|y-x\|^{|\alpha|+m} \end{aligned} \tag{4.5}$$

by the choice of φ_α. On the other hand, if N is a bound for $\{|\varphi_\alpha| : |\alpha| \le n\}$, the right hand side of (4.4) is bounded by

$$N\Big(\int_{\|h\|\le\|y-x\|} |M_x(y+h)|\,dh + \int_{\|h\|\le\|y-x\|} |M_y(y+h)|\,dh\Big)$$
$$\le N\Big(\int_{\|h\|\le 2\|y-x\|} |M_x(x+h)|\,dh + \int_{\|h\|\le\|y-x\|} |M_y(y+h)|\,dh\Big).$$

By Hölder's inequality this is bounded by

$$\begin{aligned} N\Big(&\int_{\|h\|\le 2\|y-x\|} |M_x(x+h)|^p\,dh\Big)^{1/p} \lambda(B(0,2\|y-x\|))^{1/q} \\ &+ N\Big(\int_{\|h\|\le\|y-x\|} |M_y(y+h)|^p\,dh\Big)^{1/p} \lambda(B(0,\|y-x\|))^{1/q} \\ &\le N(K(2\|y-x\|)^{u+m/p} 2^{m/q}\lambda(B(0,\|y-x\|))^{1/q} \\ &\quad + k\|y-x\|^{u+m/p}\lambda(B(0,\|y-x\|))^{1/q}) \\ &= N(K\|y-x\|^{u+m/p}2^{u+m}\lambda(B(0,\|y-x\|))^{1/q} \\ &\quad + k\|y-x\|^{u+m/p}\lambda(B(0,\|y-x\|))^{1/q}) \\ &= N'\|y-x\|^{u+m}(K2^{u+m}+k), \end{aligned}$$

where N' is independent of K and k. Combining this with (4.5) we find that whenever $\|y-x\| \le 1/2K$,

$$\Big|f_\alpha(y) - \sum_{|\alpha+\beta|\le n} \frac{(y-x)^\beta}{\beta!} f_{\alpha+\beta}(x)\Big| \le M\|y-x\|^{u-|\alpha|} \quad \text{for all } |\alpha| \le n.$$

In particular, for $0 \le |s| \le n$, $\lim_{y\to x,\, y\in A_k} f_s(y) = f_s(x)$.

Next let $x_j \in A_k$ be a sequence converging to x. Fix $\varrho < 1/k$. Then for infinitely many j's we can find $\varrho \le \varrho_j < 1/k$ such that $B(x,\varrho) \subset B(x_j,\varrho_j)$ and $\varrho_j \to \varrho$ as $j \to \infty$. Then

$$\begin{aligned} \text{(4.6)} \qquad &\int_{\|y-x\|\le\varrho} \Big|f(y) - \sum_{|i|\le n} \frac{(y-x_j)^i}{i!} f_i(x_j)\Big|^p dy \\ &\le \int_{\|y-x_j\|\le\varrho_j} \Big|f(y) - \sum_{|i|\le n} \frac{(y-x_j)^i}{i!} f_i(x_j)\Big|^p dy \le k\varrho_j^{up+m}. \end{aligned}$$

Letting $j \to \infty$ in (4.6) gives

$$\int_{\|y-x\|\le\varrho} \Big|f(y) - \sum_{|i|\le n} \frac{(y-x)^i}{i!} f_i(x)\Big|^p dy \le k\varrho^{up+m}.$$

Therefore $x \in A_k$. Hence A_k is closed and we may take $K = k$ to obtain

$$(4.7) \qquad \Big| f_s(y) - \sum_{|i+s| \le n} \frac{(y-x)^i}{i!} f_{i+s}(x) \Big| \le M|y-x|^{u-|s|}$$

for $0 \le |s| \le n$, where $M = N'(k2^{u+m} + k)$ and whenever x, y are in A_k such that $\|y - x\| < 1/2k$.

To complete the proof of Theorem 9, these inequalities must be shown to also hold for any x and y in A_k. The argument for this appears above in the last part of the proof of Theorem 7. ∎

It is now easy to see that Theorem 5 is a corollary of Theorem 8. In fact by Theorem 8 there are sets A_k and a B_u function that agrees with f on A_k. Thus if x is a density point of A_k we have

$$\limsup_{y \to x} \operatorname{ap} \Big| f_\beta(y) - \sum_{\beta \le |\alpha| \le n} \frac{(y-x)^\alpha}{\alpha!} f_\alpha(x) \Big| / \|y - x\|^{u-|\beta|} < \infty.$$

Acknowledgements. We thank the referee for a painstaking reading which enabled us to make many improvements.

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Received May 31, 2004
Revised version March 15, 2005 (5431)