On $C_0$ multi-contractions having a regular dilation

by

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Abstract. Commuting multi-contractions of class $C_0$ and having a regular isometric dilation are studied. We prove that a polydisc contraction of class $C_0$ is the restriction of a backwards multi-shift to an invariant subspace, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu. A new condition on a commuting multi-operator, which is equivalent to the existence of a regular isometric dilation and improves a recent result of A. Olofsson, is obtained as a consequence.

1. Introduction. The problem of finding an isometric (or a unitary) dilation of a commuting system of contractions was proposed by Sz.-Nagy [8] in the early fifties: for a given commuting system $T = (T_1, \ldots, T_n)$ of bounded operators (called commuting multi-operator; briefly c.m.) on a Hilbert space $\mathcal{H}$ find conditions for the existence of a c.m. $V = (V_1, \ldots, V_n)$ consisting of isometric operators (or, equivalently [5], unitary operators) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ (as a closed subspace) such that

$$T^m h = P_{\mathcal{H}} V^m h, \quad m \in \mathbb{Z}_+^n, \quad h \in \mathcal{H}$$

(if $T = (T_1, \ldots, T_n)$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ we use the notation $T^m = \prod_{i=1}^n T_i^{m_i}$; $P_{\mathcal{H}}$ is just the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$).

Then $V$ is said to be an isometric (respectively a unitary) dilation of $T$. Ando [1] proved that arbitrary pairs of commuting contractions have isometric dilations. Unfortunately, according to a counter-example given by Parrott [7], the existence is not always ensured for systems of at least three commuting contractions. By contrast to the single operator case, for $n \geq 2$, the minimality condition

$$\mathcal{H} = \bigvee_{m \in \mathbb{Z}_+^n} V^m \mathcal{H}$$

(1)

does not ensure the uniqueness, up to unitary equivalence, of the isometric dilation $V$ (if such a dilation does exist). However, if $V$ is a minimal isometric

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dilation of $T$ then $\mathcal{H}$ is invariant under $V^* := (V_1^*, \ldots, V_n^*)$ and $V_i^*|_\mathcal{H} = T_i^*$, $i = 1, \ldots, n$: $V^*$ is said to be a *minimal co-isometric extension* of $T^*$. In addition, if $U$ is the minimal unitary extension of $V$ then the restriction of $U^*$ to $\bigvee_{m \in \mathbb{Z}_+^n} U^m \mathcal{H}$ is a minimal isometric dilation of $T^*$. It will be referred to as the *minimal isometric dilation of $T^*$ corresponding to $V$*.

It was the idea of Brehmer [2] to introduce a special class of isometric or unitary dilations: an isometric or unitary dilation $V$ (on $\mathcal{H}$) of $T$ (on $\mathcal{H}$) is called *regular* if it satisfies

$$(T^{m^-})^* T^{m^+} h = P_{\mathcal{H}} (V^{m^-})^* V^{m^+} h, \quad m \in \mathbb{Z}^n, \ h \in \mathcal{H}.$$ 

Brehmer provided necessary and sufficient conditions on $T$ ensuring the existence of a regular isometric (hence also unitary) dilation, namely,

$$(2) \quad \Delta_T^\beta := \sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} T^* \alpha T^\alpha \geq 0$$

for every $\beta \in \mathbb{Z}_+^n$ with $\beta \leq e := (1, \ldots, 1)$. If it exists, a minimal regular isometric dilation is unique up to unitary equivalence. A c.m. $T$ is said to be a *polydisc contraction* if $\Delta^e_T \geq 0$. If $T$ is a commuting multi-contraction with $\Delta_T^e = 0$ then $T$ is a *polydisc isometry* ([3]).

For a vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ and a c.m. $T = (T_1, \ldots, T_n)$ let $rT := (r_1 T_1, \ldots, r_n T_n)$. It is not hard to observe that, for any multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with $0 \leq \beta \leq e$,

$$\Delta_{rT}^\beta = \Delta_T^\beta + \sum_{k=1}^n \sum_{i_1, \ldots, i_k}^{n} \left( \prod_{p=1}^k \beta_{i_p} (1 - r_{i_p}^2) T_{i_p}^* \right) \Delta_T^{-\sum_{p=1}^k \beta_{i_p} e_{i_p}} \left( \prod_{p=1}^k T_{i_p} \right).$$

If $\beta$ is fixed and $\Delta_T^\alpha \geq 0$ for every $\alpha$ with $0 \leq \alpha \leq \beta$, we deduce that

$$\Delta_{rT}^\alpha \geq \Delta_T^\alpha, \quad 0 \leq \alpha \leq \beta, \ -e \leq r \leq e.$$

The following result, which can be seen as a particular case of [3, Lemma 3.6] (it was recently rediscovered in [6, Proposition 2.1]), is then obtained:

**PROPOSITION 1.1.** If $T$ has a regular isometric dilation then so does $rT$ for every vector $r$ with $-e \leq r \leq e$.

It is our aim in the following to study c.m. of *class $C_0$*, namely the class of c.m. $T = (T_1, \ldots, T_n)$ which satisfy $T_i^k h \xrightarrow{k \to \infty} 0$ ($h \in \mathcal{H}$, $i = 1, \ldots, n$). $T$ is of *class $C_0$* if $T^*$ belongs to the class $C_0$.

For a c.m. of class $C_0$, only one of the Brehmer positivity conditions (2) must a priori be satisfied in order to obtain a regular isometric dilation. More precisely, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu [3, Theorem 3.16], we prove that a polydisc contraction of class $C_0$ is the restriction of a backwards multishift to an invariant subspace (Theorem 2.1). Improving a recent result by A. Olofsson [6] (Corollary 2.5),
we deduce a new necessary and sufficient condition on a c.m. in order to ensure the existence of a regular isometric dilation (Corollary 2.4). By contrast to the methods in [6] (where powerful tools of measure theory are used), our proofs are drastically simplified by applying (only) standard operator theory.

2. $C_0$ multi-contractions and dilations. Recall from [3] that a multi-shift is just a doubly commuting tuple of shifts; a backwards multi-shift is the adjoint of a multi-shift.

Polydisc contractions of class $C_0$ are restrictions of backwards multi-shifts. More precisely, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu [3, Theorem 3.16], we have:

**Theorem 2.1.** Let $T = (T_1, \ldots, T_n)$ be a c.m. on $\mathcal{H}$. The following conditions are equivalent:

(i) $T$ is a polydisc contraction of class $C_0$;

(ii) $T$ is the restriction of a backwards multi-shift to an invariant subspace;

(iii) $T$ has a (minimal) regular isometric dilation and the corresponding minimal isometric dilation of $T^*$ is a multi-shift.

**Proof.** Let $m$ ($1 \leq m \leq n$) be a fixed positive integer and suppose that $\Delta^\beta T \geq 0$ for every $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $\beta \leq e$ and $|\beta| := \beta_1 + \cdots + \beta_n = m$.

Easy computations show that

$$\Delta^\alpha_T - T_{i^*}^\alpha \Delta^\alpha_{T^*} T_i = \Delta^\alpha_{T^*} + e_i$$

for any $\alpha \in \mathbb{Z}_+^n$ with $\alpha \leq e$ and $|\alpha| = m - 1$ and any $i \in \{1, \ldots, n\} \setminus \text{supp} \alpha$. Then

$$T_{i^*}^\alpha \Delta^\alpha_{T^*} T_i \leq \Delta^\alpha_T,$$

since $0 \leq \alpha + e_i \leq e$ and $|\alpha + e_i| = m$. By iteration we deduce that

$$T_{i^*}^p \Delta^\alpha_{T^*} T_i^p \leq \Delta^\alpha_{T^*}$$

for every $p \in \mathbb{Z}_+$. If (i) holds true then

$$|\langle T_{i^*}^p \Delta^\alpha_{T^*} T_i^p h, h \rangle| \leq \|\Delta^\alpha_{T^*}\| \|T_i^p h\|^2 \xrightarrow{p \to 0} 0$$

for every $h \in \mathcal{H}$. Hence $\Delta^\alpha_{T^*} \geq 0$ by (3). We proceed inductively to prove that a polydisc contraction of class $C_0$ has a regular isometric dilation. The corresponding minimal isometric dilation $V = (V_1, \ldots, V_n)$ of $T^*$ is then doubly commuting ([4], [9]). Moreover,

$$\|V_{i^*}^p V^m h\| = \|V^m V_{i^*}^p h\| = \|V_{i^*}^p h\| = \|T_{i^*}^p h\| \xrightarrow{p \to 0} 0$$

for every $i \in \{1, \ldots, n\}$, $m \in \mathbb{Z}_+^n$ and $h \in \mathcal{H}$. The minimality condition (1)
forces \( V_i \in C_0 \) for any \( i \). Therefore \( V \) is a multi-shift, and this completes the proof of (i)\( \Rightarrow \) (iii).

If \( V \) is the minimal isometric dilation of \( T^\ast \) given by (iii) (corresponding to the minimal regular isometric dilation of \( T \)) then \( V^\ast \) (which is a backwards multi-shift) is a minimal co-isometric extension of \((T^\ast)^\ast = T\). Condition (ii) is obtained.

Finally, observe that a backwards multi-shift is of class \( C_0 \) and has a regular dilation (being doubly commuting). The same is then true for its restriction to an invariant subspace. The implication (ii)\( \Rightarrow \) (i) is proved. ■

The restriction of a c.m. consisting of backwards shifts to an invariant subspace obviously belongs to the class \( C_0 \), but it is not necessarily a polydisc contraction:

**Example 2.2.** Let \( S \) be the standard (unilateral) shift on \( \ell^2_{\mathbb{Z}^+} \), that is, 
\[
S(c_0, c_1, \ldots) = (0, c_0, c_1, \ldots), \quad (c_p)_{p \geq 0} \in \ell^2_{\mathbb{Z}^+}.
\]
Then \( T = (S^\ast, \ldots, S^\ast) \) is a c.m. on \( \ell^2_{\mathbb{Z}^+} \) consisting of backwards shifts. In addition, if \( B \) is the standard bilateral shift on \( \ell^2_{\mathbb{Z}} \), namely
\[
B(c_p)_{p \in \mathbb{Z}} = (c_{p-1})_{p \in \mathbb{Z}}, \quad (c_p)_{p \in \mathbb{Z}} \in \ell^2_{\mathbb{Z}},
\]
then, under the obvious identification \( \ell^2_{\mathbb{Z}^+} \subset \ell^2_{\mathbb{Z}} \), \( U = (B^\ast, \ldots, B^\ast) \) is a unitary dilation of \( T \). However, \( T \) does not have a regular isometric dilation (equivalently, by Theorem 2.1, \( T \) is not a polydisc contraction). We only have to observe that
\[
\Delta_T^{e_1+e_2} = I - 2SS^\ast + S^2S^2
\]
fails to be a positive operator since, for example,
\[
\langle \Delta_T^{e_1+e_2}(0, 1, 0, \ldots), (0, 1, 0, \ldots) \rangle = -1.
\]

**Corollary 2.3.** A polydisc contraction consisting of strict contractions has a regular isometric dilation.

A new condition on a c.m. \( T \), equivalent to the existence of a regular isometric dilation, is also obtained:

**Corollary 2.4.** \( T \) has a regular isometric dilation if and only if there exists a sequence \( r_k \in \mathbb{R}^n_+ \), \( r_k \leq e \), such that \( r_k \to e \) as \( k \to \infty \) and \( r_kT \) is a polydisc contraction of class \( C_0 \) for all \( k \).

**Proof.** If \( T \) has a regular isometric dilation then so does \( rT \) for every \( r \in \mathbb{R}^n_+ \) with \( r \leq e \) (by Proposition 1.1).

Conversely, if \( r_kT \) is a polydisc contraction of class \( C_0 \), then, by Theorem 2.1, \( r_kT \) has a regular isometric dilation. Since \( r_k \to e \) as \( k \to \infty \) we conclude that \( T \) has a regular isometric dilation. ■
In particular, our methods provide a result of Olofsson [6, Theorem 2.1]:

**Corollary 2.5.** Let \( T \) be a c.m. consisting of contractions. Then \( T \) has a regular isometric dilation if and only if there exists a sequence \( r_k = (r^k_1, \ldots, r^k_n) \in \mathbb{R}^n_+ \), \( r^k_i < 1 \), such that \( r_k \to e \) as \( k \to \infty \) and \( r_k T \) is a polydisc contraction for all \( k \).

A similar argument can be applied for polydisc isometries:

**Proposition 2.6.** A c.m. of class \( C_0 \cdot \) on a non-null Hilbert space cannot be a polydisc isometry.

**Proof.** Let \( T = (T_1, \ldots, T_n) \) be a c.m. of class \( C_0 \cdot \) on a Hilbert space \( \mathcal{H} \).

Suppose that \( T \) is a polydisc isometry.

Proceed as in the proof of Theorem 2.1(i)\( \Rightarrow \)(iii). Suppose that \( \Delta_T^\beta = 0 \), for every \( \beta \in \mathbb{Z}^n_+ \) with \( \beta \leq e \) and \( |\beta| = m \) (\( m \) is fixed; \( 1 \leq m \leq n \)). Then, for any \( \alpha \in \mathbb{Z}^n_+ \) with \( \alpha \leq e \) and \( |\alpha| = m - 1 \), and for all \( i \in \{1, \ldots, n\} \setminus \text{supp} \alpha \) and \( p \in \mathbb{Z}_+ \),

\[
T_i^p \Delta_T^\alpha T_i^{-p} = \Delta_T^\alpha.
\]

We let \( p \to \infty \) in (4) to deduce that \( \Delta_T^\alpha = 0 \). We repeat this step inductively to obtain

\[
\Delta_T^\alpha = I - T_i^p T_i^{-p} = 0, \quad i \in \{1, \ldots, n\}.
\]

Hence each \( T_i \) is isometric, a contradiction.

**Corollary 2.7.** There are no polydisc isometries consisting of strict contractions.

References


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