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On $C_{0\cdot}$ multi-contractions having a regular dilation

by

DAN POPOVICI (Timişoara)

Abstract. Commuting multi-contractions of class $C_{0\cdot}$ and having a regular isometric dilation are studied. We prove that a polydisc contraction of class $C_{0\cdot}$ is the restriction of a backwards multi-shift to an invariant subspace, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu. A new condition on a commuting multi-operator, which is equivalent to the existence of a regular isometric dilation and improves a recent result of A. Olofsson, is obtained as a consequence.

1. Introduction. The problem of finding an isometric (or a unitary) dilation of a commuting system of contractions was proposed by Sz.-Nagy [8] in the early fifties: for a given commuting system $T = (T_1, \ldots, T_n)$ of bounded operators (called commuting multi-operator; briefly c.m.) on a Hilbert space $\mathscr{H}$ find conditions for the existence of a c.m. $V = (V_1, \ldots, V_n)$ consisting of isometric operators (or, equivalently [5], unitary operators) on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ (as a closed subspace) such that

$$T^m h = P_{\mathscr{H}} V^m h, \quad m \in \mathbb{Z}_+^n, \ h \in \mathscr{H}$$

(if $T = (T_1, \ldots, T_n)$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ we use the notation $T^m = \prod_{i=1}^n T_i^{m_i}$; $P_{\mathscr{H}}$ is just the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$). Then V is said to be an *isometric* (respectively a *unitary*) *dilation* of T. Ando [1] proved that arbitrary pairs of commuting contractions have isometric dilations. Unfortunately, according to a counter-example given by Parrott [7], the existence is not always ensured for systems of at least three commuting contractions. By contrast to the single operator case, for $n \geq 2$, the *minimality* condition

$$\mathscr{K} = \bigvee_{m \in \mathbb{Z}_+^n} V^m \mathscr{H} \tag{1}$$

does not ensure the uniqueness, up to unitary equivalence, of the isometric dilation V (if such a dilation does exist). However, if V is a minimal isometric

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dilation of T then $\mathscr{H}$ is invariant under $V^* := (V_1^*, \ldots, V_n^*)$ and $V_i^*|_{\mathscr{H}} = T_i^*$, $i = 1, \ldots, n$: V^* is said to be a *minimal co-isometric extension* of T^*. In addition, if U is the minimal unitary extension of V then the restriction of U^* to $\bigvee_{m\in\mathbb{Z}_+^n} U^{*m}\mathscr{H}$ is a minimal isometric dilation of T^*. It will be referred to as the *minimal isometric dilation of T^* corresponding to V*.

It was the idea of Brehmer [2] to introduce a special class of isometric or unitary dilations: an isometric or unitary dilation V (on $\mathscr{K}$) of T (on $\mathscr{H}$) is called *regular* if it satisfies

$$(T^{m^-})^* T^{m^+} h = P_{\mathscr{H}} (V^{m^-})^* V^{m^+} h, \qquad m \in \mathbb{Z}^n,\ h \in \mathscr{H}.$$

Brehmer provided necessary and sufficient conditions on T ensuring the existence of a regular isometric (hence also unitary) dilation, namely,

$$\Delta_T^\beta := \sum_{0\le\alpha\le\beta} (-1)^{|\alpha|} T^{*\alpha} T^\alpha \ge 0 \tag{2}$$

for every $\beta \in \mathbb{Z}_+^n$ with $\beta \le e := (1, \ldots, 1)$. If it exists, a minimal regular isometric dilation is unique up to unitary equivalence. A c.m. T is said to be a *polydisc contraction* if $\Delta_T^e \ge 0$. If T is a commuting multi-contraction with $\Delta_T^e = 0$ then T is a *polydisc isometry* ([3]).

For a vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ and a c.m. $T = (T_1, \ldots, T_n)$ let $rT := (r_1T_1, \ldots, r_nT_n)$. It is not hard to observe that, for any multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with $0 \le \beta \le e$,

$$\Delta_{rT}^\beta = \Delta_T^\beta + \sum_{k=1}^n \sum_{i_1,\ldots,i_k=1}^n \Big(\prod_{p=1}^k \beta_{i_p}(1 - r_{i_p}^2) T_{i_p}^*\Big) \Delta_T^{\beta - \sum_{p=1}^k \beta_{i_p} e_{i_p}} \Big(\prod_{p=1}^k T_{i_p}\Big).$$

If β is fixed and $\Delta_T^\alpha \ge 0$ for every α with $0 \le \alpha \le \beta$, we deduce that

$$\Delta_{rT}^\alpha \ge \Delta_T^\alpha, \qquad 0 \le \alpha \le \beta,\ -e \le r \le e.$$

The following result, which can be seen as a particular case of [3, Lemma 3.6] (it was recently rediscovered in [6, Proposition 2.1]), is then obtained:

Proposition 1.1. *If T has a regular isometric dilation then so does rT for every vector r with $-e \le r \le e$.*

It is our aim in the following to study c.m. of *class $C_{0\cdot}$*, namely the class of c.m. $T = (T_1, \ldots, T_n)$ which satisfy $T_i^k h \xrightarrow{k} 0$ ($h \in \mathscr{H},\ i = 1, \ldots, n$). T is of *class $C_{\cdot 0}$* if T^* belongs to the class $C_{0\cdot}$.

For a c.m. of class $C_{0\cdot}$ only one of the Brehmer positivity conditions (2) must a priori be satisfied in order to obtain a regular isometric dilation. More precisely, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu [3, Theorem 3.16], we prove that a polydisc contraction of class $C_{0\cdot}$ is the restriction of a backwards multishift to an invariant subspace (Theorem 2.1). Improving a recent result by A. Olofsson [6] (Corollary 2.5), we deduce a new necessary and sufficient condition on a c.m. in order to ensure the existence of a regular isometric dilation (Corollary 2.4). By contrast to the methods in [6] (where powerful tools of measure theory are used), our proofs are drastically simplified by applying (only) standard operator theory.

2. $C_0.$ multi-contractions and dilations. Recall from [3] that a *multi-shift* is just a doubly commuting tuple of shifts; a *backwards multi-shift* is the adjoint of a multi-shift.

Polydisc contractions of class $C_0.$ are restrictions of backwards multi-shifts. More precisely, extending a particular case of a result by R. E. Curto and F.-H. Vasilescu [3, Theorem 3.16], we have:

THEOREM 2.1. *Let $T = (T_1, \ldots, T_n)$ be a c.m. on $\mathscr{H}$. The following conditions are equivalent*:

(i) *T is a polydisc contraction of class $C_0.$*;
(ii) *T is the restriction of a backwards multi-shift to an invariant subspace*;
(iii) *T has a* (*minimal*) *regular isometric dilation and the corresponding minimal isometric dilation of T^* is a multi-shift.*

Proof. Let m $(1 \le m \le n)$ be a fixed positive integer and suppose that $\Delta_T^\beta \ge 0$ for every $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $\beta \le e$ and $|\beta| := \beta_1 + \cdots + \beta_n = m$.

Easy computations show that

$$\Delta_T^\alpha - T_i^* \Delta_T^\alpha T_i = \Delta_T^{\alpha + e_i}$$

for any $\alpha \in \mathbb{Z}_+^n$ with $\alpha \le e$ and $|\alpha| = m - 1$ and any $i \in \{1, \ldots, n\} \setminus \operatorname{supp} \alpha$. Then

$$T_i^* \Delta_T^\alpha T_i \le \Delta_T^\alpha,$$

since $0 \le \alpha + e_i \le e$ and $|\alpha + e_i| = m$. By iteration we deduce that

$$T_i^{*p} \Delta_T^\alpha T_i^p \le \Delta_T^\alpha \quad \text{for every } p \in \mathbb{Z}_+. \tag{3}$$

If (i) holds true then

$$|\langle T_i^{*p} \Delta_T^\alpha T_i^p h, h\rangle| \le \|\Delta_T^\alpha\| \, \|T_i^p h\|^2 \xrightarrow{p} 0$$

for every $h \in \mathscr{H}$. Hence $\Delta_T^\alpha \ge 0$ by (3). We proceed inductively to prove that a polydisc contraction of class $C_0.$ has a regular isometric dilation. The corresponding minimal isometric dilation $V = (V_1, \ldots, V_n)$ of T^* is then doubly commuting ([4], [9]). Moreover,

$$\|V_i^{*p} V^m h\| = \|V^m V_i^{*p} h\| = \|V_i^{*p} h\| = \|T_i^p h\| \xrightarrow{p} 0$$

for every $i \in \{1, \ldots, n\}$, $m \in \mathbb{Z}_+^n$ and $h \in \mathscr{H}$. The minimality condition (1) forces $V_i \in C_{\cdot 0}$ for any i. Therefore V is a multi-shift, and this completes the proof of (i)⇒(iii).

If V is the minimal isometric dilation of T^* given by (iii) (corresponding to the minimal regular isometric dilation of T) then V^* (which is a backwards multi-shift) is a minimal co-isometric extension of $(T^*)^* = T$. Condition (ii) is obtained.

Finally, observe that a backwards multi-shift is of class $C_{0\cdot}$ and has a regular dilation (being doubly commuting). The same is then true for its restriction to an invariant subspace. The implication (ii)⇒(i) is proved. ∎

The restriction of a c.m. consisting of backwards shifts to an invariant subspace obviously belongs to the class $C_{0\cdot}$, but it is not necessarily a polydisc contraction:

Example 2.2. Let S be the standard (unilateral) shift on $\ell^2_{\mathbb{Z}_+}$, that is,

$$S(c_0, c_1, \ldots) = (0, c_0, c_1, \ldots), \qquad (c_p)_{p\geq 0} \in \ell^2_{\mathbb{Z}_+}.$$

Then $T = (S^*, \ldots, S^*)$ is a c.m. on $\ell^2_{\mathbb{Z}_+}$ consisting of backwards shifts. In addition, if B is the standard bilateral shift on $\ell^2_{\mathbb{Z}}$, namely

$$B(c_p)_{p\in\mathbb{Z}} = (c_{p-1})_{p\in\mathbb{Z}}, \qquad (c_p)_{p\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}},$$

then, under the obvious identification $\ell^2_{\mathbb{Z}_+} \subset \ell^2_{\mathbb{Z}}$, $U = (B^*, \ldots, B^*)$ is a unitary dilation of T. However, T does not have a regular isometric dilation (equivalently, by Theorem 2.1, T is not a polydisc contraction). We only have to observe that

$$\Delta_T^{e_1+e_2} = I - 2SS^* + S^2S^{*2}$$

fails to be a positive operator since, for example,

$$\langle \Delta_T^{e_1+e_2}(0, 1, 0, \ldots), (0, 1, 0, \ldots)\rangle = -1.$$

Corollary 2.3. *A polydisc contraction consisting of strict contractions has a regular isometric dilation.*

A new condition on a c.m. T, equivalent to the existence of a regular isometric dilation, is also obtained:

Corollary 2.4. *T has a regular isometric dilation if and only if there exists a sequence $r_k \in \mathbb{R}^n_+$, $r_k \leq e$, such that $r_k \to e$ as $k \to \infty$ and r_kT is a polydisc contraction of class $C_{0\cdot}$ for all k.*

Proof. If T has a regular isometric dilation then so does rT for every $r \in \mathbb{R}^n_+$ with $r \leq e$ (by Proposition 1.1).

Conversely, if r_kT is a polydisc contraction of class $C_{0\cdot}$ then, by Theorem 2.1, r_kT has a regular isometric dilation. Since $r_k \to e$ as $k \to \infty$ we conclude that T has a regular isometric dilation. ∎

In particular, our methods provide a result of Olofsson [6, Theorem 2.1]:

COROLLARY 2.5. *Let T be a c.m. consisting of contractions. Then T has a regular isometric dilation if and only if there exists a sequence $r_k = (r_1^k, \ldots, r_n^k) \in \mathbb{R}_+^n$, $r_i^k < 1$, such that $r_k \to e$ as $k \to \infty$ and $r_k T$ is a polydisc contraction for all k.*

A similar argument can be applied for polydisc isometries:

PROPOSITION 2.6. *A c.m. of class $C_{0\cdot}$ on a non-null Hilbert space cannot be a polydisc isometry.*

Proof. Let $T = (T_1, \ldots, T_n)$ be a c.m. of class $C_{0\cdot}$ on a Hilbert space $\mathscr{H}$. Suppose that T is a polydisc isometry.

Proceed as in the proof of Theorem 2.1(i)⇒(iii). Suppose that $\Delta_T^\beta = 0$, for every $\beta \in \mathbb{Z}_+^n$ with $\beta \le e$ and $|\beta| = m$ (m is fixed; $1 \le m \le n$). Then, for any $\alpha \in \mathbb{Z}_+^n$ with $\alpha \le e$ and $|\alpha| = m - 1$, and for all $i \in \{1, \ldots, n\} \setminus \operatorname{supp} \alpha$ and $p \in \mathbb{Z}_+$,

$$T_i^{*p} \Delta_T^\alpha T_i^p = \Delta_T^\alpha. \tag{4}$$

We let $p \to \infty$ in (4) to deduce that $\Delta_T^\alpha = 0$. We repeat this step inductively to obtain

$$\Delta_T^{e_i} = I - T_i^* T_i = 0, \qquad i \in \{1, \ldots, n\}.$$

Hence each T_i is isometric, a contradiction. ∎

COROLLARY 2.7. *There are no polydisc isometries consisting of strict contractions.*

References

[1] T. Ando, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged) 24 (1963), 88–90.

[2] S. Brehmer, *Über vertauschbare Kontraktionen des Hilbertschen Raumes*, ibid. 22 (1961), 106–111.

[3] R. E. Curto and F.-H. Vasilescu, *Standard operator models in the polydisc*, Indiana Univ. Math. J. 42 (1993), 791–810.

[4] D. Gaşpar and N. Suciu, *On the intertwinings of regular dilations*, Ann. Polon. Math. 66 (1997), 105–121.

[5] T. Itô, *On the commutative family of subnormal operators*, J. Fac. Sci. Hokkaido 58 (1958), 1–5.

[6] A. Olofsson, *Operator valued n-harmonic measure in the polydisc*, Studia Math. 163 (2004), 203–216.

[7] S. Parrott, *Unitary dilations for commuting contractions*, Pacific J. Math. 34 (1970), 481–490.

[8] B. Sz.-Nagy, *Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe*, Acta Sci. Math. (Szeged) 15 (1953), 104–114.

[9] D. Timotin, *Regular dilations and models for multi-contractions*, Indiana Univ. Math. J. 47 (1998), 671–684.

Department of Mathematics and Computer Science
University of the West Timişoara
Bd. Vasile Pârvan nr. 4
RO-300223 Timişoara, Romania
E-mail: popovici@math.uvt.ro

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