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A nonlinear Banach–Steinhaus theorem and some meager sets in Banach spaces

by

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Dedicated to the memory of my Teacher,
Professor Tadeusz Świątkowski

Abstract. We establish a Banach–Steinhaus type theorem for nonlinear functionals of several variables. As an application, we obtain extensions of the recent results of Balcerzak and Wachowicz on some meager subsets of $L^1(\mu) \times L^1(\mu)$ and $c_0 \times c_0$. As another consequence, we get a Banach–Mazurkiewicz type theorem on some residual subset of $C[0,1]$ involving Kharazishvili's notion of Φ-derivative.

1. Introduction. The Banach–Steinhaus theorem for normed linear spaces is usually given in the following form (see, e.g., [Ze95, p. 173]):

Let X be a Banach space over $\mathbb{K}$ and Y be a normed space over $\mathbb{K}$. If a family $\{T_n : n \in \mathbb{N}\} \subseteq L(X,Y)$ is pointwise bounded on X, then it is uniformly bounded, *i.e.*, $\sup\{\|T_n x\| : n \in \mathbb{N},\ \|x\| \le 1\}$ *is finite.*

This result has numerous applications in functional analysis.

However, in their paper [BS27] Banach and Steinhaus obtained a somewhat more general result. Namely, it suffices that $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded on some set of second category. (This version—even in a more general setting—is given, e.g., in [Ru91, Theorem 2.5].) Hence we get the following equivalent reformulation:

If a family $\{T_n : n \in \mathbb{N}\} \subseteq L(X,Y)$ is not uniformly bounded, *then the set $E := \{x \in X : (T_n x)_{n=1}^{\infty}$ is bounded$\}$ is meager.*

(We use the term "meager set" instead of "set of first category".) Thus the Banach–Steinhaus theorem also gives us a tool to study the Baire category of sets: Given a set $D \subseteq X$, if there exists a normed space Y and a family

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$\{T_n : n \in \mathbb{N}\} \subseteq L(X,Y)$ such that $\limsup_{n\to\infty} \|T_n\| = \infty$ and $D \subseteq E$, then D is meager.

In this paper we are going to present some applications of this approach. However, we will need a more general version of the Banach–Steinhaus theorem in which we allow T_n to be nonlinear. Some results of this kind known in the literature usually deal with homogeneous operators, whereas the additivity of T_n is replaced by weaker conditions like subadditivity of $x \mapsto \|T_n x\|$ (see, e.g., [Ra04, p. 246], [Yo80, p. 69]) or its asymptotic variant [Ga51]. For our purposes, however, both homogeneity and subadditivity conditions are too strong, and moreover, we need to consider functionals of several variables. A Banach–Steinhaus type theorem in this setting is established in Section 2 (Theorem 1). The rest of the paper is devoted to its applications. In Section 3 we generalize a recent result of Balcerzak and Wachowicz [BW01] stating that the set

$$\{(f,g) \in L^1[0,1] \times L^1[0,1] : f \cdot g \in L^1[0,1]\}$$

is meager. Here we consider the Banach space $L^p(\mu)$, where $1 \le p < \infty$ and the measure μ is σ-finite, and we give a list of equivalent conditions for the set

$$\{(f,g) \in L^p(\mu) \times L^p(\mu) : f \cdot g \in L^p(\mu)\}$$

to be of second category (cf. Theorem 2). Some examples of meager subsets of $c_0 \times c_0$ and $c_0 \times l^p$ other than those in [BW01] are given in Section 4. Finally, we establish a Banach–Mazurkiewicz ([Ba31], [Ma31]) type theorem on some residual subset of $C[0,1]$ involving Kharazishvili's [Kh98, p. 147] notion of Φ-derivative.

2. A Banach–Steinhaus theorem for nonlinear functionals. We start with the following version of the uniform boundedness principle which is similar to [Ze95, Theorem 3B], though slightly more general.

PROPOSITION 1. *Let (X,τ) be a topological space and $F_n : X \to \mathbb{R}_+$ be lower semicontinuous for all $n \in \mathbb{N}$. Set*

$$E := \{x \in X : (F_n x)_{n=1}^{\infty} \text{ is bounded}\}. \tag{1}$$

If E is of second category, then the family $\{F_n : n \in \mathbb{N}\}$ is equibounded on some nonempty open subset of X.

Proof. We use a standard argument, observing that $E = \bigcup_{m\in\mathbb{N}} E_m$, where

$$E_m := \bigcap_{n\in\mathbb{N}} \{x \in X : F_n x \le m\}.$$

Since the F_n are lower semicontinuous, all the sets E_m are closed. By hypothesis, at least one of them has a nonempty interior, which yields the assertion. ∎

Let $\mathbb{R}_+$ denote the set of all nonnegative reals. Let A be a subset of a linear space X such that $A + A \subseteq A$. Given $L \geq 1$, we say that a function $\varphi : A \to \mathbb{R}_+$ is *L-subadditive* if

$$\varphi(x+y) \leq L(\varphi(x) + \varphi(y)) \quad \text{for all } x, y \in X.$$

In particular, if $X = \mathbb{R}$, $A := \mathbb{R}_+$ and φ is nondecreasing, then it is easily seen that φ is L-subadditive if and only if φ is *moderated* (see, e.g., [Ra04, p. 235]), i.e., $\varphi(2x) \leq C\varphi(x)$ for some $C > 0$ and all $x \in \mathbb{R}_+$.

The following result is an extension of the classical Banach–Steinhaus [BS27] theorem.

THEOREM 1. *Given $k \in \mathbb{N}$, let $X_1, \ldots, X_k$ be normed linear spaces, $X := X_1$ if $k = 1$, and $X := X_1 \times \cdots \times X_k$ if $k > 1$. Assume that $L \geq 1$, $F_n : X \to \mathbb{R}_+$ $(n \in \mathbb{N})$ are lower semicontinuous and such that all functions $x_i \mapsto F_n(x_1, \ldots, x_k)$ $(i \in \{1, \ldots, k\})$ are L-subadditive and even. Let E be defined by* (1).

(a) *If*

$$\sup\{F_n x : n \in \mathbb{N},\ \|x\| \leq 1\} = \infty, \tag{2}$$

then E is a meager set.

(b) *If* (2) *does not hold, then $E = X$.*

In particular, if $X_1, \ldots, X_k$ are Banach spaces, then the following statements are equivalent:

(i) *E is meager;*
(ii) *$E \neq X$;*
(iii) $\sup\{F_n x : n \in \mathbb{N},\ \|x\| \leq 1\} = \infty$.

Proof. We start with the proof of (a). Endow X with the max norm if $k > 1$. Assume that (2) holds. Suppose, on the contrary, that E is of second category. Then, by Proposition 1, there exists a closed ball $B(x^0, r)$ such that $\{F_n : n \in \mathbb{N}\}$ is equibounded on $B(x^0, r)$. Let $x^0 = (x_1^0, \ldots, x_k^0)$. Thus we get

$$M := \sup\{F_n(x_1, \ldots, x_k) : \|x_1 - x_1^0\|, \ldots, \|x_k - x_k^0\| \leq r,\ n \in \mathbb{N}\} < \infty.$$

Following the Banach–Steinhaus argument we divide the proof into two steps.

STEP 1. We show that the family $\{F_n : n \in \mathbb{N}\}$ is equibounded on $B(0, r)$. Let $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_k) \in B(0, r)$, i.e., $\|x_i\| \leq r$ for $i \in$

$\{1,\ldots,k\}$. Then, by hypothesis, we get

$$\begin{aligned}F_n x &= F_n(x_1 + x_1^0 - x_1^0, x_2, \ldots, x_k)\\ &\le L\,(F_n(x_1 + x_1^0, x_2, \ldots, x_k) + F_n(-x_1^0, x_2, \ldots, x_k))\\ &= L(F_n(x_1 + x_1^0, x_2 + x_2^0 - x_2^0, x_3, \ldots, x_k)\\ &\quad + F_n(x_1^0, x_2 + x_2^0 - x_2^0, x_3, \ldots, x_k)).\end{aligned}$$

Continuing in this fashion, after k steps we will get the sum of 2^k values of F_n at some points of $B(x_0, r)$ multiplied by L^k. This yields

$$\sup\{F_n x : n \in \mathbb{N},\ \|x\| \le r\} \le 2^k M L^k. \tag{3}$$

STEP 2. Given $n, p \in \mathbb{N}$ and $x \in X$, by L-subadditivity we get

$$F_n x = F_n\Big(\sum_{i=1}^{p} x_1/p, x_2, \ldots, x_k\Big) \le pL^{p-1} F_n(x_1/p, x_2, \ldots, x_k).$$

The same argument applied successively to the coordinates $x_2, \ldots, x_k$ gives

$$F_n x \le p^k L^{k(p-1)} F_n(x/p). \tag{4}$$

Let $p \in \mathbb{N}$ be such that $1/p \le r$. If $\|x\| \le 1$, then $x/p \in B(0, r)$, so (3) and (4) imply that

$$F_n x \le (2p)^k M L^{kp}.$$

This means $\{F_n : n \in \mathbb{N}\}$ is equibounded on the unit ball, contrary to (2). Thus E is meager.

Now we prove (b). So assume that (2) does not hold, i.e.,

$$C := \sup\{F_n x : n \in \mathbb{N},\ \|x\| \le 1\} < \infty.$$

We show that $E = X$. Given $x \in X$, there is a $p \in \mathbb{N}$ such that $\|x/p\| \le 1$. Then, by (4),

$$F_n x \le p^k L^{k(p-1)} C \quad \text{for all } n \in \mathbb{N},$$

which means $x \in E$.

To prove the last statement observe that if E is meager, then $E \ne X$ by Baire's theorem. (ii)$\Rightarrow$(iii) follows from (b), whereas (iii)$\Rightarrow$(i) was stated in (a). ∎

REMARK 1. The referee pointed out that Theorem 1 could be generalized by assuming that $X_1, \ldots, X_k$ are *locally bounded F^*-spaces*. Indeed, by the Aoki–Rolewicz theorem (see, e.g., [Ro84, Theorems 3.2.1 and 3.2.1′]), for some $p > 0$, there are p-homogeneous F-norms on the above spaces equivalent to the original ones. Using these new F-norms, we could rewrite the proof of Theorem 1 with one minor change in the line following (4).

REMARK 2. The assumption of Theorem 1 that all F_n are even in each variable cannot be omitted. Indeed, let $X := \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a nondecreasing function. Then it is easily seen that φ is subadditive iff $\varphi|_{\mathbb{R}_+}$ is subadditive. In particular, if $F_n x := 0$ for $x \le 0$ and $F_n x := nx$ for $x > 0$, then all F_n are subadditive and continuous, the family $\{F_n : n \in \mathbb{N}\}$ is not equibounded on $[-1, 1]$, but E $(= (-\infty, 0])$ is of second category.

If, however, $k = 1$ and a family $\{F_n : n \in \mathbb{N}\}$ of lower semicontinuous, L-subadditive (not necessarily even) and nonnegative functionals is pointwise bounded on some *symmetric* set of second category, then the proof of Theorem 1 shows that $\{F_n : n \in \mathbb{N}\}$ is equibounded on the unit ball. In the case where $L = 1$, this result is given in [Bo02, p. 90].

COROLLARY 1. *Let X, Y and Z be normed linear spaces and let $\mathfrak{B}(X \times Y, Z)$ denote the space of all continuous bilinear mappings from $X \times Y$ into Z. Assume that a family $\{T_n : n \in \mathbb{N}\} \subseteq \mathfrak{B}(X \times Y, Z)$ is pointwise bounded on some set $D \subseteq X \times Y$ of second category. Then the sequence $(\|T_n\|)_{n=1}^{\infty}$ is bounded.*

Proof. Set

$$F_n(x, y) := \|T_n(x, y)\| \quad \text{for } n \in \mathbb{N},\ x \in X \text{ and } y \in Y.$$

Clearly, all F_n are continuous; moreover, they are subadditive and even in each variable. Further, $D \subseteq E$, where E is defined by (1). Since D is of second category, so is E and thus Theorem 1 is applicable. ∎

REMARK 3. It seems that Corollary 1 is not well known since even its particular version with $D = X \times Y$ (which may be found in [Kö79, p. 158] for some wider class of spaces) was rediscovered in [Ge92]. Actually, this version can be obtained via the classical Banach–Steinhaus theorem using the fact that the space $\mathfrak{B}(X \times Y, Z)$ is isometrically isomorphic to $L(X, L(Y, Z))$. In fact, Corollary 1 can also be proved in this way with the help of a converse to the Kuratowski–Ulam theorem (see, e.g., [Ox71, p. 57]; observe that E is an F_σ set, so it has the property of Baire). In this case, however, we need the extra assumption that X or Y is separable.

As another simple consequence of Theorem 1, we obtain the following extension of the classical principle of condensation of singularities ([BS27]; see also, e.g., [Yo80, p. 74]), which will be useful in Section 5.

COROLLARY 2. *Let $X_1, \ldots, X_k$ and X be as in Theorem 1. Assume that $L \ge 1$, $F_{n,m} : X \to \mathbb{R}_+$ $(n, m \in \mathbb{N})$ are lower semicontinuous and such that all functions $x_i \mapsto F_{n,m}(x_1, \ldots, x_k)$ $(i \in \{1, \ldots, k\})$ are L-subadditive and even. If*

$$\sup\{F_{n,m} x : n \in \mathbb{N},\ \|x\| \le 1\} = \infty$$

for each $m \in \mathbb{N}$, then the set

$$R := \{x \in X : \limsup_{n\to\infty} F_{n,m}x = \infty \ \textit{for all } m \in \mathbb{N}\}$$

is residual.

Proof. We need to show that $E := X \setminus R$ is meager. Given $m \in \mathbb{N}$, set

$$E_m := \{x \in X : (F_{n,m}x)_{n=1}^{\infty} \text{ is bounded}\}.$$

It is clear that $E = \bigcup_{m\in\mathbb{N}} E_m$. Since, by Theorem 1, each E_m is meager, so is E. ∎

3. A meager set in $L^p(\mu) \times L^p(\mu)$. Let (X, Σ, μ) be a measure space. Following Balcerzak and Wachowicz [BW01], given a real $p \geq 1$, we consider the set

$$E_p := \{(f, g) \in L^p(\mu) \times L^p(\mu) : f \cdot g \in L^p(\mu)\}. \tag{5}$$

It is shown in [BW01] that if $p = 1$, $X := [0, 1]$ and μ is Lebesgue measure, then E is meager.

In what follows we assume that (X, Σ, μ) is σ-finite. Then there exists an ascending sequence $(X_n)_{n=1}^{\infty}$ of measurable sets of finite measure such that $X = \bigcup_{n\in\mathbb{N}} X_n$. Given $n \in \mathbb{N}$ and $f \in L^p(\mu)$, set

$$f^{*n}(x) := \min\{|f(x)|^p, n\}\, \chi_{X_n}(x) \quad \text{for all } x \in X, \tag{6}$$

where χ_{X_n} denotes the characteristic function of X_n. Clearly, $0 \leq f^{*n}(x) \leq n$. Now for $n \in \mathbb{N}$ and $f, g \in L^p(\mu)$, define

$$F_n(f, g) := \int_X f^{*n} g^{*n}\, d\mu. \tag{7}$$

Then we have

$$F_n(f, g) \leq \int n^2 \chi_{X_n} = n^2 \mu(X_n) < \infty,$$

so $F_n : L^p(\mu) \times L^p(\mu) \to \mathbb{R}_+$.

Lemma 1. *Let E_p be defined by* (5). *Then*

$$E_p = \{(f, g) \in L^p(\mu) \times L^p(\mu) : (F_n(f, g))_{n=1}^{\infty} \ \textit{is bounded}\}$$

with F_n as in (7).

Proof. Let $f, g \in L^p(\mu)$. Since $(X_n)_{n=1}^{\infty}$ is ascending and it covers X, we get $\chi_{X_n} \nearrow 1$. Hence and by (6), given $x \in X$, both sequences $(f^{*n}(x))_{n=1}^{\infty}$ and $(g^{*n}(x))_{n=1}^{\infty}$ are nondecreasing and converge to $|f(x)|^p$ and $|g(x)|^p$, respectively. Consequently, $f^{*n} g^{*n} \nearrow |fg|^p$. By Beppo-Levi's theorem, we get

$$F_n(f, g) \to \int |f \cdot g|^p. \tag{8}$$

Now if we assume that $(F_n(f, g))_{n=1}^{\infty}$ is bounded, then (8) implies that $f \cdot g \in L^p(\mu)$, i.e., $(f, g) \in E_p$. Conversely, if $(f, g) \in E_p$, then $\int |f \cdot g|^p < \infty$, and (8) implies that $(F_n(f, g))_{n=1}^{\infty}$ is bounded. This completes the proof. ∎

LEMMA 2. *Let F_n be defined by* (7). *Then F_n is 2^{p-1}-subadditive and even in each variable.*

Proof. Let $f_1, f_2, g \in L^p(\mu)$. Since the function $t \mapsto \min\{t, n\}$ ($t \in \mathbb{R}_+$) is subadditive, with the help of Hölder's inequality we infer that given $x \in X_n$,

$$\begin{aligned}(f_1 + f_2)^{*n}(x) &\le \min\{(|f_1(x)| + |f_2(x)|)^p, n\} \\ &\le \min\{2^{p-1}(|f_1(x)|^p + |f_2(x)|^p), n\} \\ &\le \min\{2^{p-1}|f_1(x)|^p, n\} + \min\{2^{p-1}|f_2(x)|^p, n\}.\end{aligned}$$

As $n \le 2^{p-1}n$, we get

$$(f_1 + f_2)^{*n}(x) \le 2^{p-1}(f_1^{*n}(x) + f_2^{*n}(x)).$$

Since this inequality also holds for $x \in X \setminus X_n$, we obtain

$$(f_1 + f_2)^{*n} \le 2^{p-1}(f_1^{*n} + f_2^{*n}).$$

Hence $f \mapsto F_n(f, g)$ is 2^{p-1}-subadditive. Moreover, since $(-f_1)^{*n} = f_1^{*n}$, we conclude that F_n is even in the first variable. Being symmetric, it has all the properties we need. ∎

LEMMA 3. *The functionals $F_n : L^p(\mu) \times L^p(\mu) \to \mathbb{R}_+$ defined by* (7) *are lower semicontinuous.*

Proof. Fix an $n \in \mathbb{N}$. Given $f_0, g_0 \in L^p(\mu)$, we need to show that

$$M := \liminf_{(f,g)\to(f_0,g_0)} F_n(f, g) \ge F_n(f_0, g_0).$$

Consider a sequence $((f_k, g_k))_{k=1}^{\infty}$ such that $\|f_k - f_0\|_p \to 0$, $\|g_k - g_0\|_p \to 0$ and $F_n(f_k, g_k) \to M$ as $k \to \infty$. By passing to a subsequence if necessary, we may assume that both $(f_k)_{k=1}^{\infty}$ and $(g_k)_{k=1}^{\infty}$ converge a.e. to f and g, respectively. Then it is easily seen that $(f_k^{*n})_{k=1}^{\infty}$ and $(g_k^{*n})_{k=1}^{\infty}$ converge a.e. to f_0^{*n} and g_0^{*n}, respectively. By Fatou's Lemma,

$$\liminf_{k\to\infty} \int f_k^{*n} \cdot g_k^{*n} \ge \int \liminf_{k\to\infty} f_k^{*n} \cdot g_k^{*n},$$

i.e., $M \ge F_n(f_0, g_0)$, which completes the proof. ∎

REMARK 4. In fact, it can be proved that all F_n are continuous (see Appendix), which, however, is somewhat more difficult to show.

Now Lemmas 1–3 and Theorem 1 immediately yield the following

PROPOSITION 2. *Let (X, Σ, μ) be a σ-finite measure space, $p \ge 1$ and E_p be defined by* (5). *The following statements are equivalent*:

(i) *E_p is meager in $L^p(\mu) \times L^p(\mu)$;*
(ii) *$E_p \ne L^p(\mu) \times L^p(\mu)$.*

In particular, the latter condition holds if $X := [0,1]$ and μ is Lebesgue measure (consider the pair (h,h), where $h(x) := x^{-1/(2p)}$), so Proposition 2 extends [BW01, Theorem 1.2].

REMARK 5. Actually, Proposition 2 is also valid for $p \in (0,1)$. (I owe this result to the referee.) This follows from Remark 1 and the fact that the F-space $L^p(\mu)$ is locally bounded. Moreover, Lemmas 1 and 3 remain valid for such p, whereas the same argument as in the proof of Lemma 2 shows this time that all functionals F_n are subadditive in each variable because of the subadditivity of $t \mapsto t^p$ for $p \in (0,1)$. Proposition 2 can also be extended by substituting the Orlicz space $N(L(\mu))$ for $L^p(\mu)$ if the function N has appropriate properties. The details will be given in a forthcoming paper.

In the rest of this section, we give other conditions equivalent to (ii) of Proposition 2 under some weaker assumptions on the measure space. Recall that (X,Σ,μ) is *semifinite* (cf. [Fr01, 211F]) if whenever $A \in \Sigma$ and $\mu(A) = \infty$, there is a $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$. (X,Σ,μ) is *localizable* (cf. [Fr01, 211G]) if it is semifinite and whenever $\mathfrak{A} \subseteq \Sigma$, there is a $B \in \Sigma$ such that $A \setminus B$ is negligible for every $A \in \mathfrak{A}$; moreover, given $C \in \Sigma$, if $A \setminus C$ is negligible for every $A \in \mathfrak{A}$, then $B \setminus C$ is negligible. Also recall that a measurable function $f : X \to \mathbb{R}$ is *quasi-simple* (cf. [Fr00, 122Y(d)]) if f is μ-integrable and $f(X)$ is countable.

Note first that it is sufficient to examine condition (ii) of Proposition 2 only for $p = 1$ because of the following

LEMMA 4. *Let (X,Σ,μ) be a measure space. The following statements are equivalent*:

(i) $E_1 \neq L^1(\mu) \times L^1(\mu)$;
(ii) $E_r \neq L^r(\mu) \times L^r(\mu)$ *for some* $r \geq 1$;
(iii) $E_p \neq L^p(\mu) \times L^p(\mu)$ *for all* $p \geq 1$.

Proof. (i)⇒(ii) and (iii)⇒(i) are obvious. We show (ii) implies (iii). By (ii), there exist $f, g \in L^r(\mu)$ such that $\int |f \cdot g|^r = \infty$. Given $p \geq 1$, set $f_p := |f|^{r/p}$ and $g_p := |g|^{r/p}$. It is clear that $f_p, g_p \in L^p(\mu)$, but $(f_p, g_p) \notin E_p$, so (iii) holds. ∎

LEMMA 5. *Let (X,Σ,μ) be a measure space and $f : X \to \mathbb{R}_+$ be measurable. Then there exists a sequence $(f_n)_{n=1}^\infty$ of measurable nonnegative functions taking values in some countable set such that $f_n \to f$ uniformly on X and $(f_n(x))_{n=1}^\infty$ is nondecreasing for all $x \in X$.*

Proof. Given $n \in \mathbb{N}$, set

$$A_{i,n} := \left\{ x \in X : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\} \quad \text{for all } i \in \mathbb{N}.$$

Then $\bigcup_{i\in\mathbb{N}} A_{i,n} = X$ since f takes finite values. Hence given $x \in X$, there is an $i_x \in \mathbb{N}$ such that $x \in A_{i_x,n}$, and the i_x is unique since $(A_{i,n})_{i=1}^{\infty}$ is disjoint. We set

$$f_n(x) := \frac{i_x - 1}{2^n} \quad \text{for all } x \in X.$$

Clearly, f_n is measurable and $f_n(X)$ is countable. Since $|f_n(x) - f(x)| < 1/2^n$ for all $x \in X$, it follows that $f_n \to f$ uniformly on X. Finally, it is easily seen that $f_n(x) \le f_{n+1}(x)$ for all $x \in X$. ∎

We will also need the following result concerning sequences of reals, in the proof of which we use Hadamard's trick (cf. [Kn47, §41, 178.1]).

LEMMA 6. *Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive reals such that the series $\sum_{n=1}^{\infty} b_n$ converges. Let $R_{n-1} := \sum_{k=n}^{\infty} b_k$ for $n \in \mathbb{N}$ and assume that the sequence $(R_n/b_n)_{n=1}^{\infty}$ is bounded. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ of positive reals such that*

$$\sum_{n=1}^{\infty} a_n b_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n^2 b_n = \infty.$$

Moreover, $a_n^2 b_n \not\to 0$.

Proof. Set

$$a_n := \frac{\sqrt{R_{n-1}} - \sqrt{R_n}}{b_n} \quad \text{for all } n \in \mathbb{N}.$$

Then $\sum_{k=1}^{n} a_k b_k = \sqrt{R_0} - \sqrt{R_n}$. Since $R_n \to 0$, we conclude that

$$\sum_{n=1}^{\infty} a_n b_n = \Big(\sum_{n=1}^{\infty} b_n\Big)^{1/2},$$

in particular, the series $\sum_{n=1}^{\infty} a_n b_n$ converges. On the other hand,

$$\begin{aligned} a_n^2 b_n &= \bigg(\frac{\sqrt{R_{n-1}} - \sqrt{R_n}}{\sqrt{b_n}}\bigg)^2 = \bigg(\frac{\sqrt{b_n}}{\sqrt{b_n + R_n} + \sqrt{R_n}}\bigg)^2 \\ &= \frac{1}{(\sqrt{1 + R_n/b_n} + \sqrt{R_n/b_n})^2} \ge \frac{1}{4(1 + R_n/b_n)}. \end{aligned}$$

By hypothesis, there is an $M > 0$ such that $R_n/b_n \le M$ for all $n \in \mathbb{N}$. Hence $a_n^2 b_n \not\to 0$, so the series $\sum_{n=1}^{\infty} a_n^2 b_n$ diverges. ∎

PROPOSITION 3. *Let (X, Σ, μ) be a measure space and E_1 be defined by* (5)*. The following statements are equivalent*:

(i) $E_1 \ne L^1(\mu) \times L^1(\mu)$;
(ii) $L^1(\mu) \setminus L^2(\mu) \ne \emptyset$;
(iii) *there is a quasi-simple function g such that $g \notin L^2(\mu)$*;
(iv) $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} = 0$;

(v) *there is a disjoint sequence* $(A_n)_{n=1}^{\infty}$ *of measurable sets of positive measure such that* $\mu(A_n) \to 0$.

Proof. (i)⇒(ii): By hypothesis, there are $f, g \in L^1(\mu)$ such that $f \cdot g \notin L^1(\mu)$. Then f and g are finite a.e., so

$$f(x)g(x) = ((f(x)+g(x))^2 - (f(x)-g(x))^2)/4 \quad \text{a.e.}$$

This implies that $f+g \notin L^2(\mu)$ or $f-g \notin L^2(\mu)$, so $L^1(\mu) \setminus L^2(\mu) \neq \emptyset$.

(ii)⇒(iii): By hypothesis, there is an $f \in L^1(\mu) \setminus L^2(\mu)$. Without loss of generality we may assume that $f \geq 0$ and it takes finite values on X.

Step 1. Assume that μ is finite. By Lemma 5, there is a measurable function $h : X \to \mathbb{R}_+$ such that $h(X)$ is countable, $h \leq f^2$ and

$$f^2(x) - h(x) \leq 1 \quad \text{for all } x \in X. \tag{9}$$

Set $g := \sqrt{h}$. Then $g \leq f$ so $g \in L^1(\mu)$. Moreover,

$$\infty = \int f^2 = \int (f^2 - g^2) + \int g^2.$$

Since by (9), $\int (f^2 - g^2) \leq \mu(X) < \infty$, we infer that $\int g^2 = \infty$. Thus g is the desired function.

Step 2. Now assume that $\mu(X) = \infty$ and μ is σ-finite. Then there is a disjoint cover $(X_n)_{n=1}^{\infty}$ of X by measurable sets of finite positive measure. By Lemma 5, given $n \in \mathbb{N}$, there is a measurable function $h_n : X_n \to \mathbb{R}_+$ such that $h_n \leq f^2|_{X_n}$,

$$f^2(x) - h_n(x) \leq \frac{1}{2^n \mu(X_n)} \quad \text{for all } x \in X_n,$$

and $h_n(X_n)$ is countable. Define $h := \bigcup_{n \in \mathbb{N}} h_n$. Then $h : X \to \mathbb{R}_+$, $h(X)$ is countable and $h \leq f^2$. Set $g := \sqrt{h}$. Since

$$\int_X (f^2 - g^2) = \sum_{n=1}^{\infty} \int_{X_n} (f^2 - h_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

we infer as in Step 1 that $g \notin L^2(\mu)$. Moreover, g is quasi-simple.

Step 3. Finally, we consider the case of arbitrary measure. Set

$$X' := \{x \in X : f(x) > 0\}.$$

Since $f \in L^1(\mu)$, the subspace measure $\mu_{X'}$ is σ-finite. Clearly, $f|_{X'} \in L^1(\mu_{X'}) \setminus L^2(\mu_{X'})$, so by Steps 1 and 2, there is a quasi-simple function $g : X' \to \mathbb{R}_+$ such that $g \in L^1(\mu_{X'}) \setminus L^2(\mu_{X'})$. It suffices to extend g onto X by setting $g(x) := 0$ for $x \in X \setminus X'$.

(iii)⇒(iv): Since $g \in L^1(\mu) \setminus L^2(\mu)$, it follows that $g(X)$ is infinite. Thus there is a sequence $(a_n)_{n=1}^{\infty}$ of distinct reals such that $g(X) = \{a_n : n \in \mathbb{N}\}$ and we may assume $a_n \geq 0$. Let $A_n := g^{-1}(\{a_n\})$. Then $g = \sum_{n=1}^{\infty} a_n \chi_{A_n}$.

Suppose, on the contrary, that

$$r := \inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} > 0.$$

Let $K := \{n \in \mathbb{N} : \mu(A_n) > 0\}$. Then $\int g = \sum_{n\in K} a_n\mu(A_n)$ and $a_n\mu(A_n) \geq ra_n$ for all $n \in K$. Hence $\sum_{n\in K} a_n < \infty$; in particular, $(a_n)_{n\in K}$ is bounded. On the other hand, we have

$$\infty = \int g^2 = \sum_{n\in K} a_n^2\mu(A_n) \leq \sup_{n\in K} a_n \cdot \int g < \infty,$$

which is a contradiction.

(iv)⇒(v): By hypothesis, there is a sequence $(B_n)_{n=1}^{\infty}$ of measurable sets of positive measure such that $\mu(B_n) \to 0$. By passing to a subsequence if necessary, we may assume that $\mu(B_{n+1}) < \mu(B_n)/2$. Set

$$A_n := B_n \setminus \bigcup_{i\in\mathbb{N}} B_{n+i} \quad \text{for all } n \in \mathbb{N}.$$

Since $\mu(B_{n+i}) < \mu(B_n)/2^i$, we get

$$\mu\Big(\bigcup_{i\in\mathbb{N}} B_{n+i}\Big) < \sum_{i=1}^{\infty} \frac{\mu(B_n)}{2^i} = \mu(B_n),$$

which implies $\mu(A_n) > 0$. Clearly, the sets A_n are disjoint and $\mu(A_n) \to 0$ since $\mu(A_n) \leq \mu(B_n)$.

(v)⇒(i): Set $b_n := \mu(A_n)$ for all $n \in \mathbb{N}$. Again, as in the proof of (iv)⇒(v), we may assume that $b_{n+1} < b_n/2$, so that $b_{n+i} < b_n/2^i$ for all $i, n \in \mathbb{N}$. Hence if R_n is as in Lemma 6, then $R_n \leq b_n$, so $R_n/b_n \leq 1$. By Lemma 6, there is a sequence $(a_n)_{n=1}^{\infty}$ of positive reals such that $\sum_{n=1}^{\infty} a_nb_n < \infty$ and $\sum_{n=1}^{\infty} a_n^2b_n = \infty$. Hence if

$$f := \sum_{n=1}^{\infty} a_n\chi_{A_n},$$

then $f \in L^1(\mu) \setminus L^2(\mu)$, i.e., $(f, f) \in L^1(\mu) \times L^1(\mu) \setminus E_1$. ∎

Lemma 7. *Let (X, Σ, μ) be a localizable measure space. If μ is not purely atomic, then there exists a $\delta > 0$ such that $\mu(\Sigma) \supseteq [0, \delta]$.*

Proof. By the Saks theorem (see, e.g., [Fr01, 214X(a)]), there is an $X' \in \Sigma$ such that the subspace measure $\mu_{X'}$ is atomless and $\mu_{X\setminus X'}$ is purely atomic. Then $\mu(X') > 0$; otherwise, if $A \in \Sigma$ and $\mu(A) > 0$, then $\mu(A \setminus X') > 0$ so $A \setminus X'$ (hence A) contains an atom. This means μ is purely atomic, a contradiction. If $\mu(X')$ is finite, then $\mu(\Sigma) \supseteq [0, \mu(X')]$ by the Sierpiński–Fichtenholz theorem (cf. [Fr01, 215D]) since $\mu_{X'}$ is atomless. If $\mu(X') = \infty$, then there is an $A \subseteq X'$ such that $A \in \Sigma$ and $0 < \mu(A) < \infty$ since, in particular, μ is semifinite. Then as before we infer that $\mu(\Sigma) \supseteq [0, \mu(A)]$. ∎

PROPOSITION 4. *Let (X, Σ, μ) be a localizable measure space and E_1 be defined by* (5). *The following statements are equivalent*:

(i) $E_1 \neq L^1(\mu) \times L^1(\mu)$;
(ii) *if μ is purely atomic*, *then* $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} = 0$.

Proof. (i)⇒(ii) is obvious since (i) implies (iv) of Proposition 3.

(ii)⇒(i): It is enough to show that (ii) implies (iv) of Proposition 3. If μ is purely atomic, then we are done; if not, then (iv) holds in virtue of Lemma 7. ∎

PROPOSITION 5. *Let (X, Σ, μ) be a measure space and E_1 be defined by* (5). *The following statements are equivalent*:

(i) $E_1 \neq L^1(\mu) \times L^1(\mu)$;
(ii) $L^1(\mu) \setminus L^\infty(\mu) \neq \emptyset$.

Proof. (i)⇒(ii): Suppose, on the contrary, that $L^1(\mu) \subseteq L^\infty(\mu)$. Let $f, g \in L^1(\mu)$. Then there is an $M > 0$ such that $|f| \le M$ a.e. on X. Hence $|f \cdot g| \le M \cdot |g|$, which implies $f \cdot g \in L^1(\mu)$. Thus $E_1 = L^1(\mu) \times L^1(\mu)$, a contradiction.

(ii)⇒(i): It suffices to show that (ii) implies (iv) of Proposition 3. Let $f \in L^1(\mu) \setminus L^\infty(\mu)$. Suppose, on the contrary, that

$$r := \inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} > 0.$$

For $n \in \mathbb{N}$, set

$$A_n := \{x \in X : |f(x)| \ge n\}.$$

Since $f \notin L^\infty(\mu)$, we infer that $\mu(A_n) > 0$, which implies $\mu(A_n) \ge r$. Hence we get

$$\infty > \int_X |f| \ge \int_{A_n} |f| \ge n\mu(A_n) \ge nr$$

for all $n \in \mathbb{N}$, which yields a contradiction. ∎

Summing up the results of this section, we have the following

THEOREM 2. *Let (X, Σ, μ) be a σ-finite measure space and*

$$E := \left\{(f, g) \in L^1(\mu) \times L^1(\mu) : f \cdot g \in L^1(\mu)\right\}.$$

The following statements are equivalent:

(i) *E is of second category*;
(ii) $E = L^1(\mu) \times L^1(\mu)$, *i.e.*, $(L^1(\mu), \cdot)$ *is a semigroup*, *where "$\cdot$" denotes multiplication of functions*;
(iii) $(L^p(\mu), \cdot)$ *is a semigroup for some* $p \ge 1$;
(iv) $(L^p(\mu), \cdot)$ *is a semigroup for all* $p \ge 1$;
(v) $L^1(\mu) \subseteq L^2(\mu)$;
(vi) *every quasi-simple function is in* $L^2(\mu)$;

(vii) $L^1(\mu) \subseteq L^\infty(\mu)$;
(viii) *μ is purely atomic and* $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} > 0$;
(ix) $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} > 0$;
(x) *every disjoint sequence $(A_n)_{n=1}^\infty$ of measurable sets of positive measure has the property that $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \infty$.*

Proof. Since a σ-finite measure space is localizable (cf. [Fr01, 211L]), the equivalence of conditions (i)–(ix) follows from Lemma 4 and Propositions 2–5. (ix)$\Rightarrow$(x) is obvious. To prove the converse suppose, on the contrary, that $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) > 0\} = 0$. By Proposition 3, there is a disjoint sequence $(A_n)_{n=1}^\infty$ of measurable sets of positive measure such that $\mu(A_n) \to 0$. By passing to a subsequence if necessary, we may assume that $\mu(A_n) < 1/2^n$. Then $\mu(\bigcup_{n\in\mathbb{N}} A_n) < 1$, which contradicts (x). ∎

4. Some meager sets in products of Banach sequence spaces. As another application of Theorem 1, we give the following generalization of [BW01, Theorem 1.1]. We consider here the space c_0 of all real sequences convergent to 0, endowed with the sup norm.

THEOREM 3. *Let $\alpha \in \mathbb{R}^\mathbb{N}$ and set*

$$E_\alpha := \Big\{((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \in c_0 \times c_0 : \Big(\sum_{i=1}^n \alpha_i x_i y_i\Big)_{n=1}^\infty \text{ is bounded}\Big\}.$$

Then E_α is meager in $c_0 \times c_0$ if and only if $\alpha \notin l^1$, i.e., $\sum_{n=1}^\infty |\alpha_n| = \infty$.

REMARK 6. It is shown in [BW01] that the set $E_{(1,\dots)}$ is meager.

Proof of Theorem 3. Set

$$T_n(x,y) := \sum_{i=1}^n \alpha_i x_i y_i \quad \text{for all } n \in \mathbb{N} \text{ and } x, y \in c_0.$$

Clearly, all T_n are continuous bilinear functionals on $c_0 \times c_0$ and

$$E_\alpha := \{(x,y) \in c_0 \times c_0 : (T_n(x,y))_{n=1}^\infty \text{ is bounded}\}.$$

Thus by Theorem 1 (with F_n defined as in the proof of Corollary 1), E_α is meager iff $\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty$. It is an easy exercise to show that $\|T_n\| = \sum_{i=1}^n |\alpha_i|$. Thus E_α is meager iff $\sum_{n=1}^\infty |\alpha_n| = \infty$. ∎

In a similar way, with the help of theorems on representation of dual spaces, we may obtain results of the above type for other products of Banach sequence spaces. For example, we give the following theorem, the proof of which is left to the reader.

THEOREM 4. *Let* $\alpha \in \mathbb{R}^{\mathbb{N}}$, $1 \le p \le \infty$, *and let* q *be such that* $1/p + 1/q = 1$. *Set*

$$E_\alpha := \Big\{ ((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \in c_0 \times l^p : \Big(\sum_{i=1}^n \alpha_i x_i y_i \Big)_{n=1}^\infty \text{ is bounded} \Big\}.$$

Then E_α *is meager iff* $\alpha \notin l^q$.

5. A residual set in $C[0,1]$**.** Assume that a mapping $\Phi : [0,1] \to c_0$ is given, so that $\Phi = (\varphi_n)_{n=1}^\infty$, where $\varphi_n : [0,1] \to \mathbb{R}$ and $\varphi_n(x) \to 0$ for all $x \in X$. Suppose that

(10) $\quad \varphi_n(1) < 0 < \varphi_n(0) \quad$ and $\quad \varphi_n(x) \ne 0$ for all $n \in \mathbb{N}$ and $x \in (0,1)$.

Given $f : [0,1] \to \mathbb{R}$ and $x \in [0,1]$, we say that f has a *Φ-derivative* at x (see [Kh98, p. 147]) if the following limit (possibly infinite) exists:

$$f'_\Phi(x) := \lim_{n\to\infty} \frac{f(x+\varphi_n(x)) - f(x)}{\varphi_n(x)}.$$

In what follows we consider the Banach space $C[0,1]$ endowed with the sup norm. The following result was established in [Kh98, p. 148].

THEOREM 5 (Kharazishvili). *Assume that* Φ *has the Baire property. Define* $R_1 \subseteq C[0,1]$ *as follows*: $f \in R_1$ *if there is a residual set* A *in* $[0,1]$ *such that for all* $x \in A$, *the finite* Φ*-derivative* $f'_\Phi(x)$ *does not exist. Then* R_1 *is residual in* $C[0,1]$.

Using the notion of Φ-derivative, the classical theorem of Banach and Mazurkiewicz ([Ba31], [Ma31]) can be written in the following way.

THEOREM 6 (Banach–Mazurkiewicz). *Define the set* $R_2 \subseteq C[0,1]$ *as follows*: $f \in R_2$ *if there is a mapping* $\Phi : [0,1] \to c_0$ *such that* $\varphi_n(x) > 0$ *for all* $n \in \mathbb{N}$ *and* $x \in [0,1)$, *and* $f'_\Phi(x)$ *is infinite for all* $x \in [0,1)$. *Then* R_2 *is residual in* $C[0,1]$.

Note that R_2 can also be defined as follows: $f \in R_2$ if for all $x \in [0,1)$, $D^+f(x)$ or $D_+f(x)$ is infinite, where $D^+f(x)$ (resp., $D_+f(x)$) denotes the upper (resp., lower) right Dini derivative of f at x (in Bruckner's [Br94] notation).

Given a mapping Φ satisfying (10), we define the *upper Dini Φ-derivative* of $f : [0,1] \to \mathbb{R}$ at $x \in [0,1]$ as follows:

$$\overline{f'_\Phi}(x) := \limsup_{n\to\infty} \frac{f(x+\varphi_n(x)) - f(x)}{\varphi_n(x)}.$$

By substituting lim inf for lim sup, we get the *lower Dini Φ-derivative* $\underline{f'_\Phi}(x)$. Using these notions and Corollary 2, we will prove the following

THEOREM 7. *Assume that $\Phi : [0,1] \to c_0$ satisfies* (10) *and A is a countable dense subset of* $[0,1]$. *Define $R_3 \subseteq C[0,1]$ as follows*: $f \in R_3$ *if for all $x \in A$, $\overline{f'_\Phi}(x)$ or $\underline{f'_\Phi}(x)$ is infinite. Then R_3 is residual in $C[0,1]$.*

Proof. Let $A = \{x_m : m \in \mathbb{N}\}$. Condition (10) implies that given $x \in [0,1]$, $x + \varphi_n(x) \in [0,1]$ for sufficiently large n. Hence given $m \in \mathbb{N}$, there is a $k_m \in \mathbb{N}$ such that $x_m + \varphi_{n+k_m}(x_m) \in [0,1]$ for all $n \in \mathbb{N}$. Thus we may define functionals $F_{n,m}$ by

$$F_{n,m}(f) := \left| \frac{f(x_m + \varphi_{n+k_m}(x_m)) - f(x_m)}{\varphi_{n+k_m}(x_m)} \right|$$

for all $n, m \in \mathbb{N}$ and $f \in C[0,1]$. We show the assumptions of Corollary 2 are satisfied with $k := 1$ and $X_1 := C[0,1]$. It is easily seen that all $F_{n,m}$ are subadditive and even. Set

$$h_{n,m} := \varphi_{n+k_m}(x_m) \quad \text{for all } n, m \in \mathbb{N}.$$

Given $f, g \in C[0,1]$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned} |F_{n,m}(f) - F_{n,m}(g)| &\le |f(x_m + h_{n,m}) - g(x_m + h_{n,m})|/|h_{n,m}| \\ &\quad + |g(x_m) - f(x_m)|/|h_{n,m}| \\ &\le 2\|f - g\|/|h_{n,m}|, \end{aligned}$$

which shows $F_{n,m}$ is Lipschitzian, hence continuous. Fix an $m \in \mathbb{N}$. We show

$$M := \sup\{F_{n,m}(f) : n \in \mathbb{N},\ \|f\| \le 1\} = \infty.$$

Suppose that, on the contrary, M is finite. The following three cases are possible.

1) $x_m \in (0,1)$. Let $\delta > 0$ be such that $(x_m - \delta, x_m + \delta) \subseteq [0,1]$ and $\delta(M+1) \le 1$. Set

$$f(x) := \begin{cases} (M+1)(x - x_m) & \text{for } x \in (x_m - \delta, x_m + \delta), \\ -\delta(M+1) & \text{for } x \in [0, x_m - \delta], \\ \delta(M+1) & \text{for } x \in [x_m + \delta, 1]. \end{cases}$$

Then f is continuous and $\|f\| \le 1$. Since $\lim_{n\to\infty} h_{n,m} = 0$, there is an $n \in \mathbb{N}$ such that $h_{n,m} < \delta$. Then $F_{n,m}(f) = M + 1$, which is a contradiction.

2) $x_m = 0$. Set

$$f(x) := \begin{cases} (M+1)x & \text{for } x \in [0, 1/(M+1)), \\ 1 & \text{for } x \in [1/(M+1), 1]. \end{cases}$$

If $n \in \mathbb{N}$ is such that $h_{n,m} < 1/(M+1)$, then $F_{n,m}(f) = M + 1$, a contradiction.

3) $x_m = 1$. Use a similar argument to item 2) setting

$$f(x) := \begin{cases} -1 & \text{for } x \in [0, M/(M+1)], \\ (M+1)(x-1) & \text{for } x \in (M/(M+1), 1]. \end{cases}$$

Thus $M=\infty$ and by Corollary 2, the set

$$R=\{f\in C[0,1]:\limsup_{n\to\infty}F_{n,m}(f)=\infty \text{ for all } m\in\mathbb{N}\}$$

is residual. Clearly, R coincides with R_3, so the proof is complete. ∎

Remark 7. Actually, the assumption of Theorem 7 that A is dense is superfluous, but the result seems to be more interesting for such sets. Moreover, Theorem 7 yields the more general version. Indeed, if A is a countable subset of $[0,1]$ and $\widetilde{A}:=A\cup(\mathbb{Q}\cap[0,1])$, then $R_3(\widetilde{A})\subseteq R_3(A)$, and since by Theorem 7, $R_3(\widetilde{A})$ is residual, so is $R_3(A)$.

6. Appendix: Continuity of integral functionals. Our purpose here is to show that the functionals F_n defined by (7) are continuous.

Lemma 8. *Given $1\le p<\infty$, the operator $f\mapsto|f|^p$ from $L^p(\mu)$ into $L^1(\mu)$ is continuous.*

Proof. This follows immediately from the fact (cf. [Ru74, Chapter 3, Exercise 24]) that if $f,g\in L^p(\mu)$ and $M\ge 0$ is such that $\|f\|_p\le M$ and $\|g\|_p\le M$, then

$$\int \big|\,|f|^p-|g|^p\big|\le 2pM^{p-1}\|f-g\|_p. \quad\blacksquare \tag{11}$$

Remark 8. Inequality (11) may be sharpened. Indeed, by Lagrange's theorem we have

$$|s^p-t^p|\le p|s-t|(s^{p-1}+t^{p-1})\qquad\text{for all } s,t\ge 0,$$

which yields

$$\big|\,|f|^p-|g|^p\big|\le p\big|\,|f|-|g|\,\big|(|f|^{p-1}+|g|^{p-1}). \tag{12}$$

Let $p>1$ and $q:=p/(p-1)$. Since $|f|^{p-1}$ and $|g|^{p-1}$ are in $L^q(\mu)$, and $|f|-|g|\in L^p(\mu)$, we get, by (12) and using Hölder's inequality twice,

$$\begin{aligned}\int\big|\,|f|^p-|g|^p\big| &\le p\|\,|f|-|g|\,\|_p(\|f\|_p^{p/q}+\|g\|_p^{p/q})\\ &= p\|\,|f|-|g|\,\|_p(\|f\|_p^{p-1}+\|g\|_p^{p-1}),\end{aligned}$$

which sharpens (11).

Lemma 9. *The operator $(\cdot)^{*n}:L^p(\mu)\to L^1(\mu)$ defined by (6) is continuous.*

Proof. Let $f_k\in L^p(\mu)$ for all $k\in\mathbb{N}\cup\{0\}$, and $\|f_k-f_0\|_p\to 0$. Since there is a conegligible subset of X on which all functions f_k have finite values, we may assume that f_k are real-valued. Then, given $k\in\mathbb{N}$, we have

$$f_k^{*n}=\chi_{X_n}(|f_k|^p+n-\big|\,|f_k|^p-n\big|)/2.$$

By continuity of the operators $f \mapsto |f|^p$ from $L^p(\mu)$ into $L^1(\mu)$ and $(f,g) \mapsto f+g$ from $L^1(\mu) \times L^1(\mu)$ into $L^1(\mu)$, we now infer that $\|f_k^{*n} - f_0^{*n}\|_1 \to 0$ as $k \to \infty$. ∎

LEMMA 10. *The functional $F_n : L^p(\mu) \times L^p(\mu) \to \mathbb{R}_+$ defined by* (7) *is continuous for all $n \in \mathbb{N}$.*

Proof. Fix an $n \in \mathbb{N}$. Assume that $\|f_k - f\|_p \to 0$ and $\|g_k - g\|_p \to 0$ as $k \to \infty$. By Lemma 9, we have $\|f_k^{*n} - f^{*n}\|_1 \to 0$ and $\|g_k^{*n} - g^{*n}\|_1 \to 0$ as $k \to \infty$. By the inequalities $0 \le f_k^{*n} \le n$, $0 \le g_k^{*n} \le n$, we hence get

$$\begin{aligned}|F_n(f_k,g_k) - F_n(f,g)| &\le \int_X |f_k^{*n} g_k^{*n} - f^{*n} g^{*n}| \chi_{X_n} \\ &= \int_{X_n} |f_k^{*n}(g_k^{*n} - g^{*n}) + g^{*n}(f_k^{*n} - f^{*n})| \\ &\le n(\|g_k^{*n} - g^{*n}\|_1 + \|f_k^{*n} - f^{*n}\|_1) \to 0 \text{ as } k \to \infty. \ ∎\end{aligned}$$

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Added in proof. Recently W. Wojdowski informed me that (as I expected) Lemma 5 is known and can be found on p. 151 of: G. P. Tolstov, *Measure and Integral*, Nauka, Moscow, 1976 (in Russian).

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