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Weak compactness and σ-Asplund generated Banach spaces

by

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Abstract. σ-Asplund generated Banach spaces are used to give new characterizations of subspaces of weakly compactly generated spaces and to prove some results on Radon–Nikodým compacta. We show, typically, that in the framework of weakly Lindelöf determined Banach spaces, subspaces of weakly compactly generated spaces are the same as σ-Asplund generated spaces. For this purpose, we study relationships between quantitative versions of Asplund property, dentability, differentiability, and of weak compactness in Banach spaces. As a consequence, we provide a functional-analytic proof of a result of Arvanitakis: A compact space is Eberlein if (and only if) it is simultaneously Corson and quasi-Radon–Nikodým.

1. Definitions and notation. In [9], the results on quantitative versions of the differentiability of norms and of weak compactness were used to give characterizations of several subclasses of weakly Lindelöf determined spaces.

In the present paper, we study a quantitative version of the Asplund property to obtain new characterizations of subspaces of weakly compactly generated spaces. Recall that a Banach space is called *Asplund* if each separable subspace of it has a separable dual.

Let $(X, \|\cdot\|)$ be a real Banach space with topological dual X^*. The closed unit balls of X and X^* are denoted by B_X and B_{X^*} respectively. Let $\emptyset \neq M \subset B_X$. We define a seminorm $\|\cdot\|_M$ on X^* by

$$\|x^*\|_M := \sup|\langle M, x^*\rangle| := \sup\{|\langle x, x^*\rangle|;\, x \in M\}, \qquad x^* \in X^*.$$

Let $\varepsilon > 0$. Given a convex function $f : X \to \mathbb{R}$, we say that it is *ε-M-differentiable* at $x \in X$ if

$$\lim_{t\downarrow 0} \frac{1}{t} \sup\{f(x+th) + f(x-th) - 2f(x);\, h \in M\} < \varepsilon.$$

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The norm $\|\cdot\|$ is called *ε-M-smooth* if it is ε-M-differentiable at each $0 \neq x \in X$. We say that the norm on X^*, dual to $\|\cdot\|$, and denoted by the same symbol, is *ε-M-LUR* if $\limsup_{n\to\infty} \|x^* - x_n^*\|_M < \varepsilon$ whenever $x^*, x_n^* \in B_{X^*}$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|x^* + x_n^*\| = 2$. We say that the dual norm $\|\cdot\|$ on X^* has the *weak* ε-M-Kadec property* if $\limsup_\tau \|x_\tau^* - x^*\|_M < \varepsilon$ whenever x^* and the net $(x_\tau^*)_{\tau\in T}$ lie in the unit sphere S_{X^*} of X^* and $x_\tau^* \overset{w^*}{\to} x^*$. We note that if we have ε-M-smoothness, ε-M-LUR, or weak* ε-M-Kadec property for every $\varepsilon > 0$, and $M = B_X$, then we get the usual concepts of Fréchet smoothness, LUR, and weak* Kadec property respectively.

Given a non-empty set $A \subset B_X$, we say that the closed dual unit ball B_{X^*} is *$\|\cdot\|_A$-ε-separable* if there exists a countable set $C \subset B_{X^*}$ such that for every $x^* \in B_{X^*}$ there is $c \in C$ so that $\|x^* - c\|_A < \varepsilon$, that is, B_{X^*} can be covered by open balls with centers in C and of $\|\cdot\|_A$-radius ε. We say in this case that C is *$\|\cdot\|_A$-ε-dense* in B_{X^*}. A subset $M \subset B_X$ is said to be *ε-Asplund* if for every countable subset $\emptyset \neq A \subset M$, the dual unit ball B_{X^*} is $\|\cdot\|_A$-ε-separable. Clearly, if a set is ε-Asplund for every $\varepsilon > 0$, then it is an Asplund set (see [7, Definition 1.4.1]). Note that if $A \subset \varepsilon B_X$ then B_{X^*} is $\|\cdot\|_A$-ε'-separable, and hence A is an ε'-Asplund set for every $\varepsilon' > \varepsilon$.

We say that a Banach space $(X, \|\cdot\|)$ is *σ-Asplund generated* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ where each A_n^ε is an ε-Asplund set. We say that the norm $\|\cdot\|$ on X^*, dual to $\|\cdot\|$, is *σ-LUR* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ such that $\|\cdot\|$ is ε-A_n^ε-LUR for every $n \in \mathbb{N}$. We say that the norm $\|\cdot\|$ on X^*, dual to $\|\cdot\|$, has the *σ-weak* Kadec property* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ such that $\|\cdot\|$ has the weak* ε-A_n^ε-Kadec property for every $n \in \mathbb{N}$. We say that the norm $\|\cdot\|$ on X is *σ-Fréchet smooth* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ such that the norm $\|\cdot\|$ is ε-A_n^ε-differentiable at every $0 \neq x \in X$ for every $n \in \mathbb{N}$.

A Banach space X is called *weakly compactly generated* (*WCG*) if it contains a weakly compact set whose linear span is dense in it. X is called *weakly Lindelöf determined* (*WLD*) if its dual unit ball B_{X^*}, with the weak* topology, is a Corson compact space. A compact space is called *Corson* if it is homeomorphic to a subset of $\Sigma(\Gamma) := \{u \in \mathbb{R}^\Gamma;\ \#\{\gamma \in \Gamma;\ u(\gamma) \neq 0\} \leq \omega\}$, with a suitable set Γ, endowed with the product topology. A compact space K is called *Eberlein* if it is homeomorphic to a weakly compact set of $c_0(\Gamma)$ in its weak topology, for some set Γ. We refer to [4, 7, 11] for the standard notation and results used in this paper.

The paper is organized as follows: Section 2 lists and discusses the main results in this paper. Section 3 studies relationships between the quantitative concepts described above. Most of the statements from this section are then used in Section 4, where the main results are proved.

2. Characterizations of subspaces of WCG spaces. We start with the following well-known

THEOREM 0. *Let X be a Banach space. Then* (o)$\Rightarrow$(i)$\Leftrightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) *where*

- (o) *X is a WCG and Asplund space.*
- (i) *X admits an equivalent norm whose dual norm is LUR.*
- (ii) *X admits an equivalent norm whose dual norm has the weak* Kadec property.*
- (iii) *X admits an equivalent norm which is Fréchet smooth.*
- (iv) *X is an Asplund space.*

Moreover, if X is a WLD space, then all (o)–(iv) *are equivalent.*

As regards its proof, (o)$\Rightarrow$(i) can be found in [4, Theorem VII.1.14]; (i)$\Leftrightarrow$(ii) is from [26, 27]; for (i)$\Rightarrow$(iii)$\Rightarrow$(iv) see, e.g., [4, Proposition II.1.5 and Theorem II.5.3]; and for the last statement, see, e.g., [7, Theorem 8.3.4].

The following analogue of the above pattern is one of the main results of this paper.

THEOREM 1. *Let X be a Banach space. Then* (o)$\Rightarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iv) *and* (o)$\Rightarrow$(i)$\Rightarrow$(iii)$\Rightarrow$(iv), *where*

- (o) *X is a subspace of a WCG space.*
- (i) *X admits an equivalent norm whose dual norm is σ-LUR.*
- (ii) *X admits an equivalent norm whose dual norm has the σ-weak* Kadec property.*
- (iii) *X admits an equivalent norm which is σ-Fréchet smooth.*
- (iv) *X is a σ-Asplund generated space.*

Moreover, if X is a WLD space, then all (o)–(iv) *are equivalent.*

A compact space K is called *quasi-Radon–Nikodým* if there is a lower semicontinuous function $\varrho : K \times K \to [0,1]$ which separates the points of K, and which fragments K, i.e., for every $\varepsilon > 0$ and every $\emptyset \neq M \subset K$ there is an open set $\Omega \subset K$ so that $M \cap \Omega \neq \emptyset$ and $\sup\{\varrho(k,h);\ k,h \in M \cap \Omega\} < \varepsilon$.

THEOREM 2.

- (i) *A compact space K is quasi-Radon–Nikodým if and only if the Banach space $C(K)$ is σ-Asplund generated.*
- (ii) *A Banach space X is σ-Asplund generated if and only if (B_{X^*}, w^*) is a quasi-Radon–Nikodým compact space.*

Here, (i) was proved by Avilés [3, Theorem 20] (see also [8, Theorem 7]), while (ii) follows from [3, Theorem 20], [13, Lemma 4], [8, Theorem 7], and Remark 8 below.

From Theorems 1 and 2, we get, after a mild effort, the following known

THEOREM 3. *For a compact space K the following assertions are equivalent*:

(i) *K is an Eberlein compact space.*
(ii) *K is a Corson compact space and $C(K)$ is σ-Asplund generated.*
(iii) *K is a Corson and simultaneously quasi-Radon–Nikodým compact space.*

The implication (iii)⇒(i) in Theorem 3 is due to Arvanitakis [2]. In his proof, he used purely topological tools.

REMARKS. 1. (i)$\not\Rightarrow$(o) in Theorem 1: Let X be the dual JT* to the James tree space JT [11, pp. 199–201]. Note that JT is separable and that X^* is isomorphic to JT$\oplus\ell_2(\Gamma)$. Thus X^* admits an equivalent dual LUR norm [4, Theorem VII.2.3(ii)]. If X were a subspace of a WCG space, then JT* would admit an equivalent dual LUR norm [4, Theorem VII.2.3(ii)]. But this would imply that JT* is separable, a contradiction. Other counterexamples are the spaces JL_0 or JL_2 constructed by Johnson and Lindenstrauss (see, e.g. [30]).

2. Concerning the implication (ii)⇒(i) in Theorem 1, see the remark at the very end of the paper.

3. (iii)$\not\Rightarrow$(i) in Theorem 1: The space $C([0,\omega_1])$ admits an equivalent Fréchet smooth norm [4, Th. 5.4, Ch. VII]. However, it does not satisfy (i). Indeed, it is easy to check that every dual norm which is σ-LUR is already strictly convex. However, by Talagrand [4, Theorem VII.5.2], $C([0,\omega_1])$ does not admit any equivalent norm whose dual norm would be strictly convex.

4. (iv)$\not\Rightarrow$(iii) in Theorem 1: It is easy to check that the σ-Fréchet smoothness of a norm implies its Gateaux smoothness. Thus any Asplund space which admits no equivalent Gateaux smooth norm provides a counterexample here (see, e.g., [4, Section VII.6]).

5. Every Gateaux smooth norm on a separable Banach space is already σ-Fréchet smooth (so, in particular, every separable Banach space satisfies (iii) in Theorem 1 and hence it is σ-Asplund generated, although not always Asplund). Indeed, assume that $(X,\|\cdot\|)$ is a separable Banach space with Gateaux smooth norm, and let $\{x_n;\, n\in\mathbb{N}\}$ be a dense subset of B_X. Put

$$A_n^\varepsilon = (x_n + (\varepsilon/3)B_X)\cap B_X, \qquad n\in\mathbb{N},\ \varepsilon>0.$$

Then $\bigcup_{n=1}^\infty A_n^\varepsilon = B_X$. Moreover, for every $0\neq x\in X,\ n\in\mathbb{N}$, and $\varepsilon>0$,

$$\lim_{t\downarrow 0}\frac{1}{t}\sup\{\|x+th\|+\|x-th\|-2\|x\|;\, h\in A_n^\varepsilon\} \le \lim_{t\downarrow 0}\frac{1}{t}\{\|x+tx_n\|+\|x-tx_n\|-2\|x\|\}+\frac{2\varepsilon}{3}=\frac{2\varepsilon}{3}<\varepsilon.$$

6. Note that a norm is σ-Fréchet smooth if it is uniformly Gateaux smooth [10], [9, p. 445]. Thus the σ-Fréchet smoothness is a common generalization for both the Fréchet smoothness and the uniform Gateaux smoothness. Klee showed in [20] that any separable Banach space X has an equivalent Gateaux smooth norm $\|\cdot\|$ whose dual norm is not strictly convex. Hence this norm on X is not uniformly Gateaux smooth (see [4, Theorem II.6.7]). If, moreover, X^* is not separable, then X is not an Asplund space and so the norm $\|\cdot\|$ is not Fréchet smooth; however, according to Remark 5, $\|\cdot\|$ is σ-Fréchet smooth.

7. Let X be a weakly $\mathcal{K}$-analytic (and so weakly countably determined) Banach space which is not a subspace of a WCG space—one of such spaces, due to Talagrand, can be found, e.g., in [7, Section 4.3]. Let $\|\cdot\|$ be an equivalent Gateaux smooth norm on X; it exists according to a result of Mercourakis (see, e.g., [4, Theorem VII.1.16]). This norm is not σ-Fréchet smooth by Theorem 1.

8. A Banach space X is called *Asplund generated* provided that there are an Asplund space Y and a bounded linear mapping $T : Y \to X$ with TY dense in X. This is equivalent to saying that X contains a linearly dense Asplund subset (see, e.g., [7, Theorem 1.4.4]). In general, this property is not inherited by subspaces: There are examples of subspaces of a WCG Banach space which are not WCG (the first one was given by Rosenthal [28], see also [7, Section 1.6]); use then [7, Theorem 8.3.4]. On the other hand, it is easy to check that *subspaces of Asplund generated Banach spaces are σ-Asplund generated*. Indeed, assume that A is an Asplund set generating a Banach space Z and that X is a subspace of Z. Then the absolutely convex hull of A, say C, is also an Asplund set in Z. Now, the sets

$$A_n^\varepsilon = (nC + (\varepsilon/2)B_Z) \cap B_X, \qquad \varepsilon > 0,\ n \in \mathbb{N},$$

witness that X is σ-Asplund generated. Also, *subspaces of σ-Asplund generated spaces are σ-Asplund generated*. However, we do not know if σ-Asplund generated Banach spaces are already subspaces of Asplund generated Banach spaces.

9. A compact space is called *Radon–Nikodým* if it is homeomorphic to a weak* compact subset of a space that is dual to an Asplund space. A compact space is Radon–Nikodým if and only if it admits a lower semicontinuous `metric` which fragments it. Trivially, a Radon–Nikodým compact space is quasi-Radon–Nikodým, and it is unknown if the opposite is true. It is easy to show that a continuous image of a quasi-Radon–Nikodým compact space is quasi-Radon–Nikodým. Hence, Theorem 3 implies a result of Stegall that *a continuous image of a Radon–Nikodým compact space which is moreover Corson must be Eberlein* (see, e.g., [7, Theorem 8.3.6]). Here we

recall two well-known results. *A compact space is Eberlein if and only if it is simultaneously Radon–Nikodým and Corson* (see, e.g., [7, Theorem 8.3.5] and references therein). *A compact space K is Radon–Nikodým if and only if $C(K)$ is Asplund generated* (see, e.g., [7, Theorem 1.5.4]).

10. The implication (iv)⇒(o) in Theorem 1 for WLD spaces can be deduced once we have Theorem 3 at hand. Indeed, (iv) and [13, Lemma 4] guarantee that $C(B_{X^*}, w^*)$ is σ-Asplund generated. Moreover, if X is WLD then (B_{X^*}, w^*) is a Corson compact space. Therefore, by Theorem 3, (B_{X^*}, w^*) is an Eberlein compact space, and hence X is a subspace of a WCG space (see, e.g. [11, p. 392]).

11. *A Banach space X is a subspace of an Asplund generated space if* (*and only if*) (B_{X^*}, w^*) *is a continuous image of a Radon–Nikodým compact space* [7, Theorem 1.5.6]. It is unknown whether every σ-Asplund generated space is a subspace of an Asplund generated space. Actually, this is equivalent to the question whether every quasi-Radon–Nikodým compact space is a continuous image of a Radon–Nikodým compact space. For more details we refer to [8].

12. Compact spaces K such that $C(K)$ is σ-Asplund generated are called in [13] *countably lower fragmentable.*

We postpone the proofs of Theorems 1 and 3 to Section 4, after we prove, in Section 3, results on ε-versions of several qualitative concepts of Banach space theory.

3. ε-concepts. We collect here quantitative versions of known qualitative results on Asplund property, differentiability, dentability, fragmentability, and weak compactness. We shall use them in the proofs of Theorems 1 and 3. The proofs of such statements are mostly analogous to known proofs of their qualitative counterparts. However, as they are of independent interest, and some more applications of them may be expected in the future, we mostly include their proofs. Sometimes, in the proofs, we are not able to avoid the jump from ε to 2ε or 4ε, and we do not have at hand counterexamples demonstrating the necessity of such growth. Actually, we do not care much about this increase since our main objective is the qualitative essence of the quantitative results.

Let M be a non-empty bounded subset of X. The *M-diameter* of a set $U \subset X^*$ is defined by

$$M\text{-}\operatorname{diam} U := \sup\{\|x_1^* - x_2^*\|_M;\ x_1^*, x_2^* \in U\}.$$

We shall start with a Shmul'yan-like characterization of ε-M-differentiability of a norm; its proof is similar to that of [4, Theorem I.1.4], and hence we omit it.

PROPOSITION 4. *Let* $(X, \|\cdot\|)$ *be a Banach space*, $M \subset B_X$ *a non-empty subset*, $\varepsilon > 0$, *and* x_0 *an element in the unit sphere* S_X *of* X. *Then the following statements are equivalent*:

(i) *The norm* $\|\cdot\|$ *is* ε-M-*differentiable at* x_0.
(ii) *For all sequences* (x_n^*) *and* (y_n^*) *in* B_{X^*} *such that* $\langle x_0, x_n^*\rangle \to 1$ *and* $\langle x_0, y_n^*\rangle \to 1$ *as* $n \to \infty$, *we have* $\|x_n^* - y_n^*\|_M < \varepsilon$ *for all sufficiently large* $n \in \mathbb{N}$.
(iii) *The slice* $\{x^* \in B_{X^*}; \langle x_0, x^*\rangle > 1 - \delta\}$, *with a suitable* $\delta > 0$, *has* M-*diameter less than* ε.

Let $\varepsilon > 0$ and $\emptyset \neq M \subset B_X$ be given. We say that B_{X^*} is *weak** ε-M-*dentable* if for every non-empty set $U \subset B_{X^*}$ there are $x \in X$ and $\alpha \in \mathbb{R}$ such that the slice $\{x^* \in U; \langle x, x^*\rangle > \alpha\}$ is non-empty and has M-diameter less than ε. We say that B_{X^*} is *weak** ε-M-*fragmentable* if for every non-empty set $U \subset B_{X^*}$ there exists a weak* open set $\Omega \subset X^*$ such that the intersection $U \cap \Omega$ is non-empty and has M-diameter less than ε. Clearly, the weak* ε-B_X-dentability (the weak* ε-B_X-fragmentability) valid for every $\varepsilon > 0$ yields the classical concept of the weak* dentability (resp. fragmentability) by the norm.

The following proposition shows connections between ε-versions of differentiability of functions on the space and the dentability and fragmentability in the dual.

PROPOSITION 5. *Let* $(X, \|\cdot\|)$ *be a Banach space*, $M \subset B_X$ *be a non-empty set*, *and let* $\varepsilon > 0$ *be given. Then the following assertions are equivalent*:

(i) B_{X^*} *is weak** ε-M-*dentable.*
(ii) B_{X^*} *is weak** ε-M-*fragmentable.*
(iii) *Every convex* 1-*Lipschitzian function on* X *is* ε-M-*differentiable at every point of an open dense subset of* X.
(iv) *Every convex* 1-*Lipschitzian function on* X *is* ε-M-*differentiable at least at one point.*

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii). Let $f : X \to \mathbb{R}$ be a convex 1-Lipschitzian function. Let $U \subset X$ be any non-empty open set. Let $\partial f : X \to 2^{X^*}$ be the Moreau–Rockafellar subdifferential of f (see [25, p. 6]). ∂f is norm-to-weak* upper semicontinuous [25, Proposition 2.5]. Let $F : X \to 2^{X^*}$ be a minimal norm-to-weak* upper semicontinuous mapping such that $F(x) \subset \partial f(x)$ for every $x \in X$. Since f is 1-Lipschitzian, $F(U)$ is a subset of B_{X^*}. Now, (ii) yields a weak* open set $W \subset X^*$ such that the set $F(U) \cap W$ is non-empty and has M-diameter less than ε. By [7, Lemma 3.1.2], there exists a non-empty open set $\Omega \subset U$ so that $F(\Omega) \subset W$. Then M-diam $F(\Omega) < \varepsilon$. We shall show that

f is ε-M-differentiable at each point of Ω. So fix any $x \in \Omega$. Find $t > 0$ so small that $x \pm tM \subset \Omega$. Then for all $h \in M$ we have

$$\frac{1}{t}\left(f(x+th)+f(x-th)-2f(x)\right) \le \langle h, \xi-\eta\rangle \le M\text{-}\operatorname{diam} F(\Omega) < \varepsilon,$$

where $\xi \in F(x+th)$ and $\eta \in F(x-th)$. We have thus proved that any open set $U \subset X$ contains an open subset $\Omega \subset U$ such that f is ε-M-differentiable at each point of Ω, which is (iii).

(iii)⇒(iv) is trivial.

(iv)⇒(i). Fix any non-empty set $U \subset B_{X^*}$. Put $f = \sup\langle\cdot, U\rangle$; this is a convex 1-Lipschitzian function on X. Find $x_0 \in X$, $t > 0$, and $\delta > 0$ so small that $(1/t)\sup\{f(x_0+th)+f(x_0-th)-2f(x_0);\ h \in M\} + 3\delta/t < \varepsilon$. Consider any x_1^*, x_2^* in the slice $\{x^* \in U;\ \langle x_0, x^*\rangle > f(x_0)-\delta\}$. Then for all $h \in M$ we have

$$\langle x_0 \pm th, x_1^*\rangle + \langle x_0 \mp th, x_2^*\rangle - 2f(x_0) \le f(x_0 \pm th) + f(x_0 \mp th) - 2f(x_0) < t\varepsilon - 3\delta,$$

and hence,

$$\pm t\langle h, x_1^*-x_2^*\rangle < t\varepsilon - 3\delta + 2f(x_0) - \langle x_0, x_1^*\rangle - \langle x_0, x_2^*\rangle < t\varepsilon - 3\delta + 2\delta = t\varepsilon - \delta,$$

and so $\|x_1^* - x_2^*\|_M < \varepsilon - \delta/t$. Therefore the slice $\{x^* \in U;\ \langle x_0, x^*\rangle > f(x_0)-\delta\}$ has M-diameter not greater than $\varepsilon - \delta/t < \varepsilon$. ∎

Remark. There exists a direct proof of the implication (ii)⇒(i). Indeed, it is enough to follow carefully the argument in [24, p. 742].

Let $\varepsilon > 0$ and $\emptyset \ne M \subset B_X$ be given. We say that the dual ball B_{X^*} is *ε-M-dentable* if for every non-empty set $U \subset B_{X^*}$ there are $x^{**} \in X^{**}$ and $\alpha \in \mathbb{R}$ such that the slice $\{x^* \in U;\ \langle x^{**}, x^*\rangle > \alpha\}$ is non-empty and has M-diameter less than ε. If B_{X^*} is ε-B_X-dentable for every $\varepsilon > 0$, then we get the usual concept of dentability of X^*, equivalent, as is well known, to the Radon–Nikodým property of X^*. Trivially, weak* ε-M-dentability implies ε-M-dentability.

Proposition 6. *Let X be a Banach space, $\emptyset \ne M \subset B_X$, $\varepsilon > 0$, and assume that the dual unit ball B_{X^*} is ε-M-dentable. Then B_{X^*} is weak* $2\varepsilon'$-M-dentable for every $\varepsilon' > \varepsilon$.*

Proof. We follow the proof of [5, Proposition 2]. Fix any $\varepsilon' > \varepsilon$. Assume that B_{X^*} is not weak* $2\varepsilon'$-M-fragmentable. Find a set $S \subset B_{X^*}$ whose non-empty weak* relatively open subsets have M-diameter at least $2\varepsilon'$ each. Take $\varepsilon'' \in (\varepsilon, \varepsilon')$. Set $\mathcal{D} = \{\emptyset\} \cup \{0,1\} \cup \{0,1\}^2 \cup \cdots$. For $d \in \mathcal{D}$ we shall construct weak* relatively open sets $U_d \subset S$ and vectors $h_d \in M$ such that $U_{d0} \cup U_{d1} \subset U_d$ and $\inf\langle h_d, U_{d0} - U_{d1}\rangle > 2\varepsilon''$; here and further we put $di = (d_1, \ldots, d_n, i)$ if $d = (d_1, \ldots, d_n) \in \mathcal{D}$ and $i \in \{0,1\}$. Put $U_\emptyset = S$. Consider any $d \in \mathcal{D}$ and assume that U_d has already been constructed. We know that $\sup\langle M, U_d - U_d\rangle > 2\varepsilon''$. Find $h_d \in M$ and $\xi_0, \xi_1 \in U_d$ such that $\langle h_d, \xi_0 - \xi_1\rangle > 2\varepsilon''$. Then find weak* relatively open sets $U_{di} \subset U_d$ such that $\xi_i \in U_{di}$, $i = 0, 1$, and $\inf\langle h_d, U_{d0} - U_{d1}\rangle > 2\varepsilon''$. This finishes the induction step.

For $d \in \mathcal{D}$ let $K_d = \overline{\operatorname{co} U_d}^*$ denote the weak* closed convex hull of U_d. We note that $K_{d0} \cup K_{d1} \subset K_d$, and hence $\frac{1}{2}(K_{d0} + K_{d1}) \subset K_d$ for every $d \in \mathcal{D}$. We claim that there exists $t = (t_d;\, d \in \mathcal{D}) \in \prod_{d\in\mathcal{D}} K_d$ such that $t_d = \frac{1}{2}(t_{d0} + t_{d1})$ for every $d \in \mathcal{D}$; note that the set $\{t_d;\, d \in \mathcal{D}\}$ is called a *dyadic tree*. Clearly, in order to prove the claim, it is enough to show that $\bigcap_{d\in\mathcal{D}} A_d \neq \emptyset$, where

$$A_d = \Big\{(t_d;\, d \in \mathcal{D}) \in \prod_{d\in\mathcal{D}} K_d;\, t_d = \tfrac{1}{2}(t_{d0} + t_{d1})\Big\}, \qquad d \in \mathcal{D}.$$

Using a compactness argument, it is enough to prove that $\bigcap\{A_d;\, d \in \mathcal{D},\, |d| \le n\}$ is non-empty for every $n \in \mathbb{N}$. So fix $n \in \mathbb{N}$. For $d \in \mathcal{D}$ with $|d| > n$ let t_d be any element of K_d. To give the definition for $d \in \mathcal{D}$ with $|d| \le n$, we use downward induction. Fix any $d \in \mathcal{D}$, with $|d| \le n$, and assume that we have already defined $t_{d0} \in K_{d0}$ and $t_{d1} \in K_{d1}$. Put then $t_d = \frac{1}{2}(t_{d0} + t_{d1})$; hence $t_d \in K_d$. Thus we finally construct t_d for every $d \in \mathcal{D}$. It is clear that every $(t_d;\, d \in \mathcal{D})$ from the non-empty set $\bigcap_{n=1}^{\infty}\{A_d;\, d \in \mathcal{D},\, |d| \le n\}$ satisfies the claim.

Pick some $(t_d;\, d \in \mathcal{D}) \in \prod_{d\in\mathcal{D}} K_d$. By the assumption, there is a weak open halfspace $V \subset X^*$ such that the set $\{t_d;\, d \in \mathcal{D}\} \cap V$ is non-empty and has M-diameter less than ε. Take $d \in \mathcal{D}$ so that $t_d \in V$. Then also $t_{di} \in V$ for a suitable $i \in \{0, 1\}$. Hence $\|t_d - t_{di}\|_M < \varepsilon$. However, from the construction of the sets U_{d0}, U_{d1} we have $2\varepsilon < 2\varepsilon'' < \langle h_d, t_{d0} - t_{d1}\rangle = 2|\langle h_d, t_d - t_{di}\rangle| < 2\|t_d - t_{di}\|_M$, a contradiction. Therefore B_{X^*} is weak* $2\varepsilon'$-M-fragmentable. Proposition 5 then finishes the proof. ∎

PROPOSITION 7. *Let $(X, \|\cdot\|)$ be a Banach space, $M \subset B_X$ a non-empty set, and let $\varepsilon > 0$ be given. Then* (i)⇒(ii)⇒(iii) *where*

(i) *B_{X^*} is weak* ε-M-fragmentable.*
(ii) *M is an ε-Asplund set.*
(iii) *B_{X^*} is weak* 2ε-M-fragmentable.*

Proof. (i)⇒(ii). We follow the argument from [24, p. 742]. Fix any countable set $A \subset M$. By Zorn's lemma we find a set $S \subset B_{X^*}$ such that $\|x_1^* - x_2^*\|_A \ge \varepsilon$ for all distinct $x_1^*, x_2^* \in S$, and for every $x^* \in B_{X^*}$ there is $s \in S$ so that $\|x^* - s\|_A < \varepsilon$. Assume that S is uncountable. On B_{X^*}, consider the topology of pointwise convergence on elements of A—call it τ_A. Clearly, τ_A is semimetrizable. By deleting at most countably many points from S we get an (uncountable) set $S_0 \subset S$ such that each point of S_0 is a τ_A-accumulation point of S_0. This can be done easily by using the concept of condensation points (see [6, p. 85]). Since B_{X^*} is weak* ε-M-fragmentable, we can find a weak* open set $W \subset X^*$ such that $S_0 \cap W$ is non-empty and has M-diameter less than ε. But this leads to a contradiction because $S_0 \cap W$ is not a singleton. Therefore S must be at most countable and hence B_{X^*} is $\|\cdot\|_A$-ε-separable. This holds for every countable $A \subset M$ and hence M is an ε-Asplund set.

(ii)⇒(iii). We guess that this argument goes back to I. Namioka. Let $U \subset B_{X^*}$ be a non-empty set. Let $\{0\} \neq Y_0$ be some fixed separable subspace of X. Let $W_i^0 \subset B_{X^*}$, $i \in \mathbb{N}$, be a basis for the topology on B_{X^*} of pointwise convergence on Y_0. For every $i \in \mathbb{N}$ we find a countable set $A_i^0 \subset M$ such that A_i^0-diam$(U \cap W_i^0) = M$-diam$(U \cap W_i^0)$. Let Y_1 be the closed linear span of $Y_0 \cup \bigcup_{i\in\mathbb{N}} A_i^0$. Generally, consider any fixed $n \in \mathbb{N}$ and assume we have already found separable subspaces $Y_0 \subset Y_1 \subset \cdots \cdots \subset Y_n \subset X$, sets $A_i^0, A_i^1, \ldots, A_i^{n-1} \subset M$, $i \in \mathbb{N}$, and relatively weak* open sets $W_i^0, W_i^1, \ldots, W_i^{n-1} \subset B_{X^*}$, $i \in \mathbb{N}$. Let $W_i^n \subset B_{X^*}$, $i \in \mathbb{N}$, be a basis for the topology on B_{X^*} of pointwise convergence on Y_n. For every $i \in \mathbb{N}$ we find a countable set $A_i^n \subset M$ such that A_i^n-diam$(U \cap W_i^n) = M$-diam$(U \cap W_i^n)$. Let then Y_{n+1} be the closed linear span of the set $Y_n \cup \bigcup_{i\in\mathbb{N}} A_i^n$. Finally, let Y be the closure of $\bigcup_{i\in\mathbb{N}} Y_n$, and put $A = \bigcup_{i,n\in\mathbb{N}} A_i^n$. We observe that Y is separable and A is countable.

Now, we show that M-diam$(U \cap W) < 2\varepsilon$ for a suitable weak* open set $W \subset X^*$. From (ii) we find a countable set $C \subset B_{X^*}$ such that for every $x^* \in B_{X^*}$ there is $c \in C$ satisfying $\|x^* - c\|_A < \varepsilon$. Let τ denote the (possibly non-Hausdorff) topology on B_{X^*} of pointwise convergence on Y, and let $\overline{U}^\tau$ be the closure of U in τ. We note that (B_{X^*}, τ) is a compact space. We can write

$$\overline{U}^\tau = \bigcup_{c\in C} \bigcup_{j\in\mathbb{N}} \{x^* \in \overline{U}^\tau;\ \|x^* - c\|_A \le \varepsilon - 1/j\},$$

and Baire's category theorem yields $c \in C$, $j \in \mathbb{N}$, and a τ-open set $V \subset B_{X^*}$ so that $\emptyset \neq \overline{U}^\tau \cap V \subset \{x^* \in \overline{U}^\tau;\ \|x^* - c\|_A \le \varepsilon - 1/j\}$. Therefore the (non-empty) set $U \cap V$ has A-diameter $\le 2(\varepsilon - 1/j) < 2\varepsilon$. We may and do assume that $V = \{x^* \in B_{X^*};\ \langle y_i, x^*\rangle < \alpha_i,\ i = 1, \ldots, l\}$ for suitable $n, l \in \mathbb{N}$, $y_1, \ldots, y_l \in Y_n$, and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$. Thus V is an open set in the topology of pointwise convergence on Y_n. Hence, there must exist $i \in \mathbb{N}$ such that $\emptyset \neq U \cap W_i^n \subset U \cap V$. Find a weak* open set $W \subset X^*$ such that $W_i^n = W \cap B_{X^*}$. Then $U \cap W = U \cap W_i^n$ and hence

$$\begin{aligned} M\text{-diam}(U \cap W) &= A_i^n\text{-diam}(U \cap W_i^n) \le A\text{-diam}(U \cap W_i^n) \\ &\le A\text{-diam}(U \cap V) < 2\varepsilon. \ \blacksquare \end{aligned}$$

Remark. It is natural to ask what happens if the whole unit ball B_X is an ε-Asplund set. Clearly, this information is vacuous if $\varepsilon > 1$. On the other hand, if $0 < \varepsilon < 1$ and B_X is an ε-Asplund set, then X is already an Asplund space; this easily follows from Riesz' lemma. Thus we get from Proposition 7 a result of M. Muñoz [22]: *If B_{X^*} is weak* ε-B_X-fragmentable, i.e., weak** *ε-fragmentable by the norm, for some* $0 < \varepsilon < 1$, *then* X *is an Asplund space* (*and hence* B_{X^*} *is weak* fragmentable by the norm*). Further, using this fact and Proposition 6 we find that *if* B_{X^*} *is* ε-B_X*-dentable for some* $0 < \varepsilon < 1/2$, *then* X *must be an Asplund space.* Finally, if B_X is 1-Asplund, then X may not be an Asplund space. In other words, the $\|\cdot\|$-1-separability of B_{X^*} may not be enough for the separability of X^*. The following example illustrating this is due to J. Tišer. We thank him for allowing us to include it in our paper.

EXAMPLE. Consider $X = \ell_1$. Then $X^* = \ell_\infty$. Endow X^* with the (equivalent dual) norm

$$|||x^*||| = \sup_{n\in\mathbb{N}} |x_n| + \sum_{n=1}^{\infty} 2^{-n}|x_n|, \qquad x^* = (x_1, x_2, \ldots) \in \ell_\infty.$$

Put $e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,\ldots)$, … We shall show that *the closed unit ball in* $(\ell_\infty, |||\cdot|||)$ *can be covered by a countable family of open balls of* $|||\cdot|||$*-radius* 1. For this we shall show that the open balls $\{x^* \in \ell_\infty;\ |||x^* - \frac{1}{2}e_m||| < 1\}$ and $\{x^* \in \ell_\infty;\ |||x^* + \frac{1}{2}e_m||| < 1\}$, $m \in \mathbb{N}$, cover the unit sphere $\{x^* \in \ell_\infty;\ |||x^*||| = 1\}$. Hence, all these balls, together with the open ball $\{x^* \in \ell_\infty;\ |||x^* - 0||| < 1\}$, will cover the whole closed unit ball $\{x^* \in \ell_\infty;\ |||x^*||| \le 1\}$. So fix any $x^* = (x_1, x_2, \ldots) \in \ell_\infty$ with $|||x^*||| = 1$. Define $\|x^*\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. Then $1 \le \|x^*\|_\infty + \sum_{n=1}^{\infty} 2^{-n}\|x^*\|_\infty = 2\|x^*\|_\infty$, and so $\|x^*\|_\infty \ge 1/2$. Assume first that $\|x^*\|_\infty > 1/2$. Find $m \in \mathbb{N}$ so that $|x_m| > 1/2$. Assume, say, $x_m > 0$; then $x_m > 1/2$. We shall show that $|||x^* - \frac{1}{2}e_m||| < 1$. Indeed,

$$\begin{aligned}
|||x^* - \tfrac{1}{2}e_m||| &= |x_m - 1/2| \vee \sup_{n\neq m} |x_n| + 2^{-m}|x_m - 1/2| + \sum_{n\neq m} 2^{-n}|x_n| \\
&= (x_m - 1/2) \vee \sup_{n\neq m} |x_n| + 2^{-m}(x_m - 1/2) + (1 - 2^{-m}|x_m| - \|x^*\|_\infty) \\
&\le \|x^*\|_\infty + 2^{-m}x_m - 2^{-m-1} + 1 - 2^{-m}x_m - \|x^*\|_\infty < 1.
\end{aligned}$$

If $x_m < 0$, then we find similarly that $|||x^* + \frac{1}{2}e_m||| < 1$. Second, assume that $\|x^*\|_\infty = 1/2$. Then, necessarily, $|x_n| = 1/2$ for every $n \in \mathbb{N}$ and so

$$\min |||x^* \pm \tfrac{1}{2}e_1||| = \tfrac{1}{2} + \sum_{n=2}^{\infty} 2^{-n}\tfrac{1}{2} = \tfrac{3}{4} < 1. \ \blacksquare$$

QUESTION. *If* X *is a general separable Banach space with non-separable dual, does it admit an equivalent norm* $|||\cdot|||$ *such that the closed unit ball* $B_{(X,|||\cdot|||)}$ *is a* 1*-Asplund set?*

Next, we shall focus on ε-variants of some concepts of smoothness and rotundity of the norm. An example of a space with a Gateaux smooth norm that has no equivalent M-smooth norm, with M linearly dense, is any non-WCG subspace of a WCG space (see, e.g., [9, Theorem 1]).

PROPOSITION 8. *Let* $(X, \|\cdot\|)$ *be a Banach space*, $M \subset B_X$ *a non-empty set*, *and let* $\varepsilon > 0$ *be given. Then* (i)⇒(ii)⇒(iii) *where*

(i) *The dual norm* $\|\cdot\|$ *on* X^* *is* ε-M-*LUR.*
(ii) *The dual norm* $\|\cdot\|$ *on* X^* *has the weak** ε-M-*Kadec property.*
(iii) *The set* M *is* ε-*Asplund.*

Proof. (i)⇒(ii). Let x^* and a net $(x^*_\tau)_{\tau\in T}$ lie in S_{X^*} and $x^*_\tau \xrightarrow{w^*} x^*$. Then for every $x \in B_X$ we have

$$2 \geq \limsup_\tau \|x^* + x^*_\tau\| \geq \liminf_\tau \|x^* + x^*_\tau\| \geq \lim_\tau \langle x, x^* + x^*_\tau\rangle = 2\langle x, x^*\rangle.$$

Hence $\lim_\tau \|x^* + x^*_\tau\| = 2$. Assume that $\limsup_\tau \|x^*_\tau - x^*\|_M \geq \varepsilon$. Find then $\tau_1 < \tau_2 < \cdots$ in T such that

$$\|x^*_{\tau_n} - x^*\|_M > \varepsilon - 1/n \quad \text{and} \quad \|x^*_{\tau_n} + x^*\| > 2 - 1/n$$

for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|x^*_{\tau_n} + x^*\| = 2$, and therefore, by (i), $\limsup_{n\to\infty} \|x^*_{\tau_n} - x^*\|_M < \varepsilon$, a contradiction.

(ii)⇒(iii). Let $\emptyset \neq A \subset M$ be a countable subset. Denote by Y the closed linear span of A and let $Q : X^* \to Y^*$ be the canonical quotient mapping. Let τ denote the topology, on the dual unit sphere S_{X^*}, of pointwise convergence on elements of A. Then (S_{X^*}, τ) is a separable space (maybe not Hausdorff). Find a countable set $D \subset S_{X^*}$ which is τ-dense in S_{X^*}. Fix any $0 \neq y^* \in B_{Y^*}$. Find a sequence (d_n) in D such that $\langle x, Q(d_n)\rangle \to \langle x, y^*/\|y^*\|\rangle$ as $n \to \infty$ for every $x \in A$. Let $x^* \in X^*$ be a weak* cluster point of (d_n). It is easy to see that $\|x^*\| = 1$. From (ii) we can then find $n \in \mathbb{N}$ so that $\|x^* - d_n\|_M < \varepsilon$, and hence

$$\|y^*/\|y^*\| - Q(d_n)\|_A = \|x^* - d_n\|_A \leq \|x^* - d_n\|_M < \varepsilon.$$

Therefore $\|y^* - \|y^*\|Q(d_n)\|_A < \|y^*\|\varepsilon \leq \varepsilon$. Thus, the (countable) set $\bigcup\{rQ(D);\ r \in \mathbb{Q},\ 0 \leq r \leq 1\}$, where $\mathbb{Q}$ stands for the set of rational numbers, witnesses that B_{Y^*} is $\|\cdot\|_A$-ε-separable. Finally, the (countable) set $\bigcup\{rD;\ r \in \mathbb{Q},\ 0 \leq r \leq 1\}$, witnesses that B_{X^*} is $\|\cdot\|_A$-ε-separable. We have thus proved (iii). ■

We also have the following variant of Proposition 8.

PROPOSITION 9. *Let* $(X, \|\cdot\|)$ *be a Banach space*, $M \subset B_X$ *a non-empty set*, *and let* $\varepsilon > 0$ *be given. Then* (i)⇒(ii)⇒(iii) *where*

(i) *The dual norm* $\|\cdot\|$ *on* X^* *is* $\frac{1}{2}\varepsilon$-M-*LUR.*
(ii) *The norm* $\|\cdot\|$ *on* X *is* ε-M-*smooth.*
(iii) *The set* M *is* ε'-*Asplund for every* $\varepsilon' > \varepsilon$.

Proof. (i)⇒(ii). Consider any $x_0 \in S_X$. Let (x_n^*) and (y_n^*) be two sequences in B_{X^*} such that $\langle x_0, x_n^*\rangle \to 1$ and $\langle x_0, y_n^*\rangle \to 1$. Find $x_0^* \in S_{X^*}$ such that $\langle x_0, x_0^*\rangle = 1$. Then

$$2 \geq \limsup_{n\to\infty} \|x_0^* + x_n^*\| \geq \liminf_{n\to\infty} \|x_0^* + x_n^*\| \geq \lim_{n\to\infty} \langle x_0, x_0^* + x_n^*\rangle = 2,$$

and similarly, $\lim_{n\to\infty} \|x_0^* + y_n^*\| = 2$. Thus (i) implies that

$$\limsup_{n\to\infty} \|x_0^* - x_n^*\|_M < \varepsilon/2 \quad \text{and} \quad \limsup_{n\to\infty} \|x_0^* - y_n^*\|_M < \varepsilon/2.$$

Then $\limsup_{n\to\infty} \|x_n^* - y_n^*\|_M < \varepsilon$. Now, Proposition 4 yields (ii).

(ii)⇒(iii). Take any countable subset $\emptyset \neq A \subset M$. Denote by Y the closed linear span of A; this subspace is separable. Find a countable dense subset C in S_Y. For every $c \in C$ we find $c^* \in S_{Y^*}$ such that $\langle c, c^*\rangle = 1$, and let C^* denote the set of all such c^*'s. Now, fix any $y^* \in S_{Y^*}$ such that there is $y \in S_Y$ satisfying $\langle y, y^*\rangle = 1$. Find a sequence (c_n) in C norm converging to y. For every $n \in \mathbb{N}$ find $c_n^* \in C^*$ such that $\langle c_n, c_n^*\rangle = 1$. We have

$$\langle y, c_n^*\rangle = \langle c_n, c_n^*\rangle + \langle y - c_n, c_n^*\rangle = 1 + \langle y - c_n, c_n^*\rangle \to 1 \quad \text{as } n \to \infty.$$

Hence, by Proposition 4, there is $n \in \mathbb{N}$ so that $\|c_n^* - y^*\|_A \leq \|c_n^* - y^*\|_{M\cap Y} < \varepsilon$. We have proved that the set of all norm attaining elements of S_{Y^*} is $\|\cdot\|_A$-ε-separable. Hence, by the Bishop–Phelps theorem, the whole S_{Y^*} is $\|\cdot\|_A$-ε'-separable for every $\varepsilon' > \varepsilon$. Then the (countable) set $D := \bigcup\{rC^*;\ r \in \mathbb{Q},\ 0 \leq r \leq 1\}$ shows that the whole ball B_{Y^*} is $\|\cdot\|_A$-ε'-separable for every $\varepsilon' > \varepsilon$. Finally, extending every element of $d \in D$ to $\tilde{d} \in B_{X^*}$, the (countable) set $\{\tilde{d};\ d \in D\}$ witnesses that B_{X^*} is $\|\cdot\|_A$-ε'-separable for every $\varepsilon' > \varepsilon$. We have thus proved (iii). ■

If X^* is separable, then it admits an equivalent dual LUR norm [4, Corollary II.4.3]. A quantitative version of this, which can be thought of as a kind of converse to Propositions 8 and 9, reads:

PROPOSITION 10. *Let $(X, \|\cdot\|)$ be a Banach space, let $\Delta > 0$, $\varepsilon > 0$ be given, and let $\emptyset \neq M \subset B_X$ be a separable ε-Asplund set. Then X admits an equivalent norm $|\cdot|$, with $|\cdot| \leq \|\cdot\| \leq (1+\Delta)|\cdot|$, whose dual norm is 2ε-M-LUR.*

Proof. We use Godefroy's method of transfer ([4, Ch. II]). Replace M by $M \cup (-M)$ and call it again M; this set will also be ε-Asplund. Find $d_i^* \in B_{X^*}$, $i \in \mathbb{N}$, such that for every $x^* \in B_{X^*}$ there is $j \in \mathbb{N}$ so that $\|x^* - d_j^*\|_M < \varepsilon$. For $k \in \mathbb{N}$ define

$$|x^*|_k^2 := \inf\left\{\left\|x^* - \sum_{i=1}^{\infty} \frac{1}{i}\, y_i d_i^*\right\|_M^2 + \frac{1}{k}\sum_{i=1}^{\infty} {y_i}^2;\ (y_i) \in \ell_2\right\}, \qquad x^* \in X^*,$$

and then

$$|x^*|^2 := \|x^*\|^2 + \Delta \sum_{k=1}^{\infty} 2^{-k} |x^*|_k^2, \qquad x^* \in X^*.$$

Clearly $\|x^*\|^2 \le |x^*|^2 \le (1+\Delta)\|x^*\|^2$ for every $x^* \in X^*$. A use of the triangle inequality in ℓ_2 reveals that $|\cdot|$ is subadditive. That $|\cdot|$ is weak* lower semicontinuous can be proved similarly to [4, Lemma VII.2.5(i)]. Therefore $|\cdot|$ is an equivalent dual norm on X^*. We denote by $|\cdot|$ also the norm on X predual to $|\cdot|$; then $|x|^2 \le \|x\|^2 \le (1+\Delta)|x|^2$ for every $x \in X$.

Now, consider $x_0^*, x_1^*, x_2^*, \ldots$ in the unit sphere of $(X^*, |\cdot|)$ such that $|x_0^* + x_n^*| \to 2$ as $n \to \infty$. We have to show that $\limsup_{n\to\infty} \|x_0^* - x_n^*\|_M < 2\varepsilon$. Find $j \in \mathbb{N}$ so that $\|x_0^* - d_j^*\|_M < \varepsilon$. Then find $k \in \mathbb{N}$ so large that $\|x_0^* - d_j^*\|_M^2 + j^2/k < \varepsilon^2$. Further, for $n = 0, 1, 2, \ldots$ find $y_n = (y_1^n, y_2^n, \ldots) \in \ell_2$ such that

$$|x_n^*|_k^2 = \Big\| x_n^* - \sum_{i=1}^{\infty} \frac{1}{i}\, y_i^n d_i^* \Big\|_M^2 + \frac{1}{k} \sum_{i=1}^{\infty} (y_i^n)^2;$$

this is possible because the mapping Φ from ℓ_2 with weak topology into $\mathbb{R}$ defined for $y = (y_i) \in \ell_2$ by

$$\Phi(y) := \Big\| x^* - \sum_{i=1}^{\infty} \frac{1}{i} y_i d_i^* \Big\|_M^2 + \frac{1}{k} \sum_{i=1}^{\infty} y_i^2$$

is lower semicontinuous and $\Phi(z_m) \to \infty$ whenever $z_1, z_2, \ldots \in \ell_2$ and $\|z_m\|_2 \to \infty$. We have the estimate

$$\begin{aligned} (1)\qquad \|x_0^* - x_n^*\|_M &\le \Big\| x_0^* - \sum_{i=1}^{\infty} \frac{1}{i}\, y_i^0 d_i^* \Big\|_M + \Big\| \sum_{i=1}^{\infty} \frac{1}{i}\, (y_i^0 - y_i^n) d_i^* \Big\|_M \\ &\quad + \Big\| \sum_{i=1}^{\infty} \frac{1}{i}\, y_i^n d_i^* - x_n^* \Big\|_M \\ &\le |x_0^*|_k + \Big(\sum_{i=1}^{\infty} \frac{1}{i^2} \Big)^{1/2} \Big(\sum_{i=1}^{\infty} (y_i^0 - y_i^n)^2 \Big)^{1/2} + |x_n^*|_k \end{aligned}$$

for every $n = 1, 2, \ldots$. Now, a convexity argument and the parallelogram identity in ℓ_2 yield, for every $n \in \mathbb{N}$,

$$4 - |x_0^* + x_n^*|^2 \ge \Delta 2^{-k} (|x_0^*|_k - |x_n^*|_k)^2$$

and

$$4 - |x_0^* + x_n^*|^2 \ge \Delta 2^{-k}\, \frac{1}{k} \sum_{i=1}^{\infty} (y_i^0 - y_i^n)^2.$$

Hence, because $\lim_{n\to\infty}|x_0^*+x_n^*|=2$, we have

$$|x_n^*|_k\to|x_0^*|_k \quad\text{and}\quad \sum_{i=1}^{\infty}(y_i^0-y_i^n)^2\to 0 \quad\text{as } n\to\infty.$$

Therefore, (1) shows that $\limsup_{n\to\infty}\|x_0^*-x_n^*\|_M\le 2|x_0^*|_k$. As

$$|x_0^*|_k^2\le\Big\|x_0^*-\frac{1}{j}\,jd_j^*\Big\|_M^2+\frac{1}{k}\,j^2=\|x_0^*-d_j^*\|_M^2+\frac{j^2}{k}<\varepsilon^2,$$

we conclude that $\limsup_{n\to\infty}\|x_0^*-x_n^*\|_M<2\varepsilon$. ∎

Next, we focus on ε-weak compactness. Given $\varepsilon>0$, we say that a subset M of a Banach space $(X,\|\cdot\|)$ is *ε-weakly compact* if it is bounded and $\overline{M}^*\subset X+\varepsilon' B_{X^{**}}$ for every $\varepsilon'>\varepsilon$, or equivalently, $\operatorname{dist}(x^{**},X)\le\varepsilon$ for every $x^{**}\in\overline{M}^*$. A simple example of such a set is of course $K+\varepsilon B_X$, where K is a weakly compact subset of X. However, not all ε-weakly compact sets are of this nature (see [15, Remark 4]). For another example, coming naturally from uniform Gateaux smoothness, see [10]. Some results related to this concept are presented in [12] and [17]. Clearly, if a set is ε-weakly compact for every $\varepsilon>0$, then it is relatively weakly compact.

PROPOSITION 11. *Let $(X,\|\cdot\|)$ be a Banach space, $M\subset B_X$ a non-empty set, and let $\varepsilon>0$ be given. Then* (i)⇒(ii)⇒(iii) *where*

(i) *The double dual norm $\|\cdot\|$ on X^{**} is ε-M-smooth.*
(ii) *M is ε-weakly compact.*
(iii) *M is $4\varepsilon'$-Asplund for every $\varepsilon'>\varepsilon$.*

Proof. (i)⇒(ii). The argument is from [10]. Take an arbitrary $x^{**}\in\overline{M}^*$. Put $d=\operatorname{dist}(x^{**},X)$. Assume that $d>0$. By the Hahn–Banach theorem find $F\in X^{***}$ with $\|F\|=1$ such that F vanishes on X and $\langle x^{**},F\rangle=d$. Fix any $\varepsilon'>\varepsilon$. From the Bishop–Phelps theorem find $G\in X^{***}$ and $x_0^{**}\in X^{**}$ such that $\|G-F\|<\frac12(\varepsilon'-\varepsilon)$ and $\langle x_0^{**},G\rangle=1=\|x_0^{**}\|=\|G\|$. Using Goldstine's theorem we find a sequence (x_k^*) in B_{X^*} so that

$$\langle x_0^{**},x_k^*\rangle\to\langle x_0^{**},G\rangle \quad\text{and}\quad \langle x^{**},x_k^*\rangle\to\langle x^{**},G\rangle \quad\text{as } k\to\infty.$$

Since the double dual norm $\|\cdot\|$ on X^{**} is ε-M-differentiable at x_0^{**}, Proposition 4 shows that $\limsup_{k\to\infty}\|x_k^*-G\|_M<\varepsilon$. But F vanishes on X; so

$$\begin{aligned}\limsup_{k\to\infty}\|x_k^*\|_M&=\limsup_{k\to\infty}\|x_k^*-F\|_M\\&\le\limsup_{k\to\infty}\|x_k^*-G\|_M+\|G-F\|_M<\varepsilon+\tfrac12(\varepsilon'-\varepsilon)=\tfrac12(\varepsilon'+\varepsilon).\end{aligned}$$

Hence $\limsup_{k\to\infty}\langle x^{**},x_k^*\rangle<\frac12(\varepsilon'+\varepsilon)$, and so $\langle x^{**},G\rangle<\frac12(\varepsilon'+\varepsilon)$. Now

$$\operatorname{dist}(x^{**},X)=\langle x^{**},F\rangle=\langle x^{**},G\rangle+\langle x^{**},F-G\rangle<\tfrac12(\varepsilon'+\varepsilon)+\tfrac12(\varepsilon'-\varepsilon)=\varepsilon'.$$

Since $\varepsilon'>\varepsilon$ was arbitrary, we get (ii).

(ii)⇒(iii). Let $\emptyset \neq A \subset M$ be a countable set. Find a countable set $D \subset B_{X^*}$ which is dense in B_{X^*} in the topology of pointwise convergence on elements of A. We shall show that B_{X^*} lies in the norm closure of the set $\operatorname{co} D + 4\varepsilon A^\circ$, where $\operatorname{co} D$ is the convex hull of D and $A^\circ = \{x^* \in X^*;\ \sup|\langle A, x^*\rangle| \le 1\}$. Assume that this is false. Find then $x^* \in B_{X^*}$ and $F \in X^{**}$ so that

$$\langle F, x^*\rangle > \sup\langle F, \operatorname{co} D + 4\varepsilon A^\circ\rangle \quad (= \sup\langle F, D\rangle + 4\varepsilon \sup\langle F, A^\circ\rangle).$$

As $A \subset B_X$, we may and do assume that $\sup\langle F, A^\circ\rangle = 1$; thus $\langle F, x^*\rangle > \sup\langle F, D\rangle + 4\varepsilon$. Now, the bipolar theorem implies that $F \in \overline{\operatorname{co}(A \cup -A)}^*$ $(\subset \overline{\operatorname{co}(M \cup -M)}^* \subset X^{**})$. But, by [12], $\overline{\operatorname{co}(M \cup -M)}^*$ is 2ε-weakly compact. Hence $\overline{\operatorname{co}(A \cup -A)}^* \subset X + 2\varepsilon' B_{X^{**}}$ for every $\varepsilon' > \varepsilon$. Take

$$\varepsilon < \delta < \tfrac{1}{4}(\langle F, x^*\rangle - \sup\langle F, D\rangle).$$

We can write $F = x + x^{**}$ where $x \in X$, $x^{**} \in X^{**}$, and $\|x^{**}\| < 2\delta$. Thus

$$\begin{aligned}\langle x, x^*\rangle + 2\delta > \langle x + x^{**}, x^*\rangle = \langle F, x^*\rangle > \sup\langle F, D\rangle + 4\delta \\ \ge \sup\langle x, D\rangle - 2\delta + 4\delta = \|x\| + 2\delta \ge \langle x, x^*\rangle + 2\delta,\end{aligned}$$

a contradiction. Thus we have shown that B_{X^*} is a subset of the norm closure of $\operatorname{co} D + 4\varepsilon A^\circ$. Now, let C be a countable dense subset of $\operatorname{co} D$. Then $B_{X^*} \subset C + 4\varepsilon' A^\circ$ for every $\varepsilon' > \varepsilon$. This proves that A is $4\varepsilon'$-Asplund for every $\varepsilon' > \varepsilon$. ■

REMARK. The implication (i)⇒(ii) in Proposition 11 applied for $M := B_X$ and every $\varepsilon > 0$ yields the well-known fact that $(X, \|\cdot\|)$ *is reflexive if its norm is uniformly Fréchet smooth.* Indeed, the corresponding double dual norm on X^{**} is then also uniformly Fréchet smooth. Hence B_X is ε-weakly compact for every $\varepsilon > 0$.

Let us mention two more criteria for ε-weak compactness, which may be of use (see [10]). Given a Banach space $(X, \|\cdot\|)$, and a linear set $Y \subset X^*$, the seminorm $x \mapsto \sup\langle x, Y \cap B_{X^*}\rangle$, $x \in X$, is called the Y-*envelope* of $\|\cdot\|$; clearly, it is lower semicontinuous with respect to the topology of pointwise convergence on Y.

PROPOSITION 12. *Let $(X, \|\cdot\|)$ be a Banach space, $M \subset B_X$ be a nonempty set, and let $\varepsilon > 0$ be given. Assume that for every norming hyperplane $Y \subset X^*$ the Y-envelope of $\|\cdot\|$ is ε-M-smooth. Then M is 2ε-weakly compact.*

Proof. Take $x^{**} \in \overline{M}^*$ and assume that $\operatorname{dist}(x^{**}, X) > 2\varepsilon$. It is well known that the kernel of x^{**}—call it Y—is a norming hyperplane (see, e.g., [10]). Then the Y-envelope of $\|\cdot\|$—call it $|\cdot|$—is an equivalent norm on X. Let $|\cdot|$ also denote the corresponding dual norm on X^* and the bidual norm on X^{**}. Observe that the closed unit ball in $(X^*, |\cdot|)$ is just $\overline{Y \cap B_{X^*}}^*$. By [15, Proposition 8], $|x^{**}| > \varepsilon$. The Bishop–Phelps theorem yields $x \in X$

and $x^* \in \overline{Y \cap B_{X^*}}^*$ so that $\langle x, x^* \rangle = 1 = |x| = |x^*|$ and $\langle x^{**}, x^* \rangle > \varepsilon$. Find a sequence (y_n^*) in $Y \cap B_{X^*}$ such that $\langle x, y_n^* \rangle \to \langle x, x^* \rangle$ as $n \to \infty$. Since $|\cdot|$ is ε-M-smooth, Proposition 4 yields $n \in \mathbb{N}$ so large that $\|x^* - y_n^*\|_M < \varepsilon$. It then follows that $\langle x^{**}, x^* \rangle = \langle x^{**}, x^* - y_n^* \rangle < \varepsilon$, a contradiction. ∎

For the case of a dual Banach space we can dispense with the condition on norming hyperplanes. Actually, we have the following quantitative version of the well-known fact that X *is reflexive if the dual norm on* X^* *is Fréchet smooth.*

PROPOSITION 13. *Let* $(X, \|\cdot\|)$ *be a Banach space*, *let* $\varepsilon > 0$, *let* M *be a non-empty subset of* B_{X^*}, *and assume that the dual norm* $\|\cdot\|$ *on* X^* *is* ε-M*-smooth. Then* M *is* 2ε*-weakly compact.*

Proof. Let $x^{***} = x^* + x^\perp \in \overline{M}^*$, where $x^* \in X^*$ and $x^\perp \in X^\perp$, and assume $\operatorname{dist}(x^\perp, X^*) > 2\varepsilon$. By [15, Proposition 8] applied to $x^\perp$, we get $\sup \langle \overline{B}_Y^*, x^\perp \rangle > \varepsilon$, where $Y \subset X^{**}$ is the kernel of $x^\perp$. Note that Y is then a 1-norming hyperplane in X^{**} (as it contains X). It follows that $\overline{B}_Y^* = B_{X^{**}}$ and we get $\|x^\perp\| = \sup \langle x^\perp, \overline{B}_Y^* \rangle > \varepsilon$. The Bishop–Phelps theorem yields $x_0^{**} \in S_{X^{**}}$ and $x_0^* \in S_{X^*}$ such that $\langle x_0^{**}, x_0^* \rangle = 1$ and $\langle x_0^{**}, x^\perp \rangle > \varepsilon$. Let (x_n) be a sequence in B_X such that $\langle x_0^{**} - x_n, x_0^* \rangle \to 0$ and $\langle x_0^{**} - x_n, x^* \rangle \to 0$ as $n \to \infty$. Then for all large $n \in \mathbb{N}$ we have, by Proposition 4, $\|x_0^{**} - x_n\|_M < \varepsilon$, and hence $\langle x_0^{**} - x_n, x^{***} \rangle < \varepsilon$. Fix any $\delta > 0$. Then $\langle x_n - x_0^{**}, x^* \rangle < \delta$ for all large $n \in \mathbb{N}$, and hence $(\langle x_0^{**}, x^\perp \rangle =)\ \langle x_0^{**} - x_n, x^\perp \rangle < \varepsilon + \delta$ for all large $n \in \mathbb{N}$. As $\delta > 0$ was arbitrary, we get $\langle x_0^{**}, x^\perp \rangle \le \varepsilon$, a contradiction. This proves that $\operatorname{dist}(x^\perp, X^*) \le 2\varepsilon$, that is, $\operatorname{dist}(x^{***}, X^*) \le 2\varepsilon$. ∎

4. Proofs of Theorems 1 and 3

Proof of Theorem 1. The implication (o)⇒(i) is proved in [9, p. 438]. The chain (i)⇒(ii)⇒(iv) follows from Proposition 8, while (i)⇒(iii)⇒(iv) follows from Proposition 9.

It remains to prove (iv)⇒(o) provided that X is WLD. In the course of the proof, we shall use an ε-variant of the Jayne–Rogers selection theorem due to Stegall, a technique of projectional resolutions of the identity, Simons' lemma, transfinite induction, and a separable reduction.

We start with a selection statement, which is a slight variant of [7, Lemma 8.1.1]. Hence, we omit its proof. We recall that $\partial\|\cdot\|$ is the Moreau–Rockafellar subdifferential of the norm $\|\cdot\|$ (see [25, p. 6]).

PROPOSITION 14. *Let* $(Z, \|\cdot\|)$ *be a Banach space*, *let* $\varepsilon > 0$, *and let* $\emptyset \ne M \subset Z$ *be a bounded set. Assume that for every non-empty closed set* $C \subset Z$ *there exist an open set* $U \subset Z$ *and* $\zeta \in B_{Z^*}$ *such that* $C \cap U \ne \emptyset$ *and* $\|\cdot\|_M$-$\operatorname{dist}(\partial\|\cdot\|(z), \zeta) < \varepsilon$ *for every* $z \in C \cap U$. *Then there exists a Baire* *class one mapping* $f : Z \to (Z^*, \|\cdot\|_M)$, *with* $f(Z) \subset B_{Z^*}$, *such that*

$$\|\cdot\|_M\text{-}\operatorname{dist}(\partial\|\cdot\|(z), f(z)) < \varepsilon \quad \textit{for every } z \in Z.$$

A *projectional resolution of the identity* (for short *PRI*) on a non-separable Banach space Z is a family $(P_\alpha;\, \omega \le \alpha \le \mu)$ of norm one projections on Z, where ω is the first infinite ordinal and μ is the first ordinal with cardinality $\operatorname{dens} Z$ such that $P_\omega \equiv 0$, P_μ is the identity mapping on Z, $P_\alpha \circ P_\beta = P_{\min\{\alpha,\beta\}}$ for all α and β in $[\omega, \mu]$, $\operatorname{dens} P_\alpha(Z) \le \#\alpha$ for every $\alpha \in [\omega, \mu]$, and for every $z \in Z$ the mapping $\alpha \mapsto P_\alpha(z)$ is continuous from $[\omega, \mu]$ into Z. For more information about the PRI, see, e.g., [7, Section 6.1].

PROPOSITION 15. *Let* $(Z, \|\cdot\|)$ *be a non-separable Banach space admitting a linearly dense set* $\Gamma \subset B_Z$ *such that* $\#\{\gamma \in \Gamma;\, \langle \gamma, z^* \rangle \ne 0\}$ *is at most countable for every* $z^* \in Z^*$ (*hence* Z *is WLD*). *Assume that for every* $n \in \mathbb{N}$ *we have* $\varepsilon_n > 0$ *and a closed convex symmetric* ε_n*-Asplund set* $M_n \subset B_Z$. *Then there exists a PRI* $(P_\alpha;\, \omega \le \alpha \le \mu)$ *on* Z *such that* $P_\alpha(M_n) \subset M_n$, $P_\alpha(\gamma) \in \{\gamma, 0\}$ *for every* $\alpha \in [\omega, \mu]$, *every* $n \in \mathbb{N}$, *and every* $\gamma \in \Gamma$, *and moreover*, *for every limit ordinal* $\omega < \lambda \le \mu$, *every* $n \in \mathbb{N}$, *and every* $z^* \in B_{Z^*}$ *we have*

$$\limsup_{\beta \uparrow \lambda} \|P_\lambda^* z^* - P_\beta^* z^*\|_{M_n} < 9\varepsilon_n.$$

Proof. We elaborate the argument from [16], which goes back to [19]. Fix any $n \in \mathbb{N}$. We verify the assumptions of Proposition 14. So take any closed set $\emptyset \ne C \subset Z$. Let $F : C \to 2^{(B_{Z^*}, w^*)}$ be a minimal usco mapping such that $F(z) \subset \partial\|\cdot\|(z)$ for every $z \in C$. By Proposition 7, we find a weak* open set (halfspace if one wishes) $W \subset Z^*$ so that $F(C) \cap W \ne \emptyset$ and $\|\cdot\|_{M_n}\text{-}\operatorname{diam} F(C) \cap W < 2\varepsilon_n$. By [7, Lemma 3.1.2], there is an open set $U \subset Z$ so that $\emptyset \ne C \cap U$ and $F(C \cap U) \subset W$. Fix some $\zeta \in F(C \cap U)$. Then

$$\|\cdot\|_{M_n}\text{-}\operatorname{dist}(\partial\|\cdot\|(z), \zeta) \le \|\cdot\|_{M_n}\text{-}\operatorname{dist}(F(z), \zeta) < 2\varepsilon_n$$

for every $z \in C \cap U$. Thus Proposition 14 yields a Baire one mapping $f_n : Z \to (Z^*, \|\cdot\|_{M_n})$ such that

$$\|\cdot\|_{M_n}\text{-}\operatorname{dist}(\partial\|\cdot\|(z), f_n(z)) < 2\varepsilon_n \quad \text{for every } z \in Z. \tag{2}$$

Further, we find continuous mappings $D_n^m : Z \to (B_{Z^*}, \|\cdot\|_{M_n})$, $m \in \mathbb{N}$, such that $\|D_n^m(z) - f_n(z)\|_{M_n} \to 0$ as $m \to \infty$ for every $z \in Z$.

Define $M_0 = B_Z$. Put

$$\Phi(z^*) := \{\gamma \in \Gamma;\, \langle \gamma, z^* \rangle \ne 0\}, \qquad z^* \in Z^*;$$

thus $\Phi : Z^* \to 2^Z$. For $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$ let $\|\cdot\|_{n,m}$ be the Minkowski functional of the set $M_n + (1/m)B_Z$; this will be an equivalent norm on Z. For every $z \in Z$ we find a countable set $\Psi(z) \subset Z^*$ such that $\Psi(z) \supset$

$\{D_n^m(z);\, n, m \in \mathbb{N}\}$ and

$$\|z\|_{n,m} = \sup\{\langle z, z^*\rangle;\ z^* \in \Psi(z) \text{ and } \|z^*\|_{n,m} \le 1\}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Thus we have defined $\Psi : Z \to 2^{Z^*}$.

For the construction of the projections $P_\alpha : Z \to Z$ we shall use a now standard back-and-forth argument (see, e.g., [7, Section 6.1]). We need

CLAIM 1. *Let $\aleph <$ dens Z be any infinite cardinal and consider two non-empty sets $A_0 \subset Z$, $B_0 \subset Z^*$ with $\#A_0 \le \aleph$, $\#B_0 \le \aleph$. Then there exist sets $A_0 \subset A \subset Z$, $B_0 \subset B \subset Z^*$, closed under taking linear combinations with rational coefficients and such that $\#A \le \aleph$, $\#B \le \aleph$, and $\Phi(B) \subset A$, $\Psi(A) \subset B$.*

In order to prove this, let $\mathrm{sp}_{\mathbb{Q}}$ mean the $\mathbb{Q}$-linear hull. Put $A = \bigcup_{n=1}^\infty A_n$, $B = \bigcup_{n=1}^\infty B_n$, where the sets

$$A_n := \mathrm{sp}_{\mathbb{Q}}\,(A_{n-1} \cup \Phi(B_{n-1})), \quad B_n := \mathrm{sp}_{\mathbb{Q}}\,(B_{n-1} \cup \Psi(A_n)), \quad n = 1, 2, \ldots,$$

are defined inductively. Then it is easy to verify all the proclaimed properties of the sets A and B.

Having constructed the sets A, B, we observe that $\Phi(B)^\perp \cap \overline{B}^* = \{0\}$. Indeed, assume there is $0 \ne z^* \in \Phi(B)^\perp \cap \overline{B}^*$. Find $\gamma \in \Gamma$ so that $\langle \gamma, z^*\rangle \ne 0$. Find $b \in B$ so that $\langle \gamma, b\rangle \ne 0$. Then $\gamma \in \Phi(b)$. But $z^* \in \Phi(B)^\perp$ and so $\langle \gamma, z^*\rangle = 0$, a contradiction (we have just proved that Φ is a so called *projectional generator* on Z, see [7, Section 6.1]). Therefore $A^\perp \cap \overline{B}^* = \{0\}$. [7, Lemma 6.1.1] and its proof then yield a linear projection $P : Z \to Z$ with $PZ = \overline{A}$, $P^{-1}(0) = B_\perp$, and $P^*Z^* = \overline{B}^*$, such that $\|P\|_{n,m} = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Thus

$$PM_n \subset \bigcap_{m=1}^\infty P\big(M_n + \tfrac{1}{m} B_Z\big) \subset \bigcap_{m=1}^\infty \overline{M_n + \tfrac{1}{m} B_Z} \subset \bigcap_{m=1}^\infty \big(M_n + \tfrac{2}{m} B_Z\big) = M_n$$

for every $n \in \mathbb{N} \cup \{0\}$, and in particular, $\|P\| = 1$.

Fix any $\gamma \in \Gamma$. We now prove that $P\gamma \in \{\gamma, 0\}$. If $\gamma \in PZ$, then, trivially, $P\gamma = \gamma$. Second, assume that $\gamma \notin PZ$ $(= \overline{A})$. Then $\gamma \notin \Phi(B)$, which implies that $\langle \gamma, b\rangle = 0$ for every $b \in B$, that is, $\gamma \in B_\perp$. But $B_\perp = P^{-1}(0)$. Hence $P\gamma = 0$.

Now, once we know how to construct one projection $P : Z \to Z$, the construction of the whole PRI $(P_\alpha;\, \omega \le \alpha \le \mu)$ is standard (see, e.g., [7, Section 6.1]). We just recall that the projections P_α "come" from sets $A_\alpha \subset Z$ and $B_\alpha \subset Z^*$ with several properties; in particular, they are *rationally linear* (that is, closed under taking linear combinations with rational coefficients) and satisfy

$$\Phi(B_\alpha) \subset A_\alpha, \quad \Psi(A_\alpha) \subset B_\alpha, \quad \overline{A}_\alpha = P_\alpha Z, \quad \text{and} \quad \overline{B}_\alpha^* = P_\alpha^* Z^*$$

for all $\omega \le \alpha \le \mu$, and $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$ for every limit ordinal $\lambda \in (\omega, \mu]$.

Until the end of the proof, we fix any $n \in \mathbb{N}$ and any limit ordinal $\omega < \lambda \le \mu$. It remains to show the last assertion of our proposition. Fix for a while a subspace $Y \subset P_\lambda Z$. Put

$$\Delta_Y := [f_n(Y) + 2\varepsilon_n(P_\lambda(M_n) \cap Y)^\circ] \cap B_{P_\lambda^* Z^*},$$

where $(P_\lambda(M_n) \cap Y)^\circ = \{z^* \in Z^*; \sup \langle P_\lambda(M_n) \cap Y, z^*\rangle \le 1\}$. This Δ_Y is a so called *boundary* for Y, that is, for every $z \in Y$ there exists $\delta \in \Delta_Y$ so that $\|z\| = \langle z, \delta\rangle$. Indeed, fix any $z \in Y$. By (2) we find $z^* \in \partial\|\cdot\|(z)$ so that $\|z^* - f_n(z)\|_{M_n} < 2\varepsilon_n$. Put $\delta = P_\lambda^* z^*$. Then $\delta \in B_{P_\lambda^* Z^*}$, $\langle z, \delta\rangle = \langle z, z^*\rangle = \|z\|$, and $\delta = f_n(z) + (\delta - f_n(z)) \in f_n(Y) + 2\varepsilon_n(P_\lambda(M_n) \cap Y)^\circ$, and so $\delta \in \Delta_Y$. Here we have used the estimate

$$\|\delta - f_n(z)\|_{P_\lambda(M_n) \cap Y} \le \|z^* - f_n(z)\|_{M_n} < 2\varepsilon_n.$$

From now on fix any $z^* \in B_{Z^*}$. Let $C \subset A_\lambda$ be any countable set which is rationally linear.

CLAIM 2. *There are $k \in \mathbb{N}$, rational numbers $a_1, \ldots, a_k$, vectors $y_1, \ldots, y_k \in C$, and $m \in \mathbb{N}$ such that*

$$\sup \Big\langle P_\lambda(M_n) \cap \overline{C}, P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i) \Big\rangle < \frac{9}{2}\,\varepsilon_n.$$

To prove this, put $Y = \overline{C}$; this is a subspace of $P_\lambda Z$. First, we show that $P_\lambda^* z^*$ *lies in the norm closure of* $\operatorname{co} \Delta_Y + 2\varepsilon_n(P_\lambda(M_n) \cap Y)^\circ$. Assume this is not true. Find then $\varphi \in Z^{**}$ and $\alpha \in \mathbb{R}$ so that

$$\langle \varphi, P_\lambda^* z^*\rangle > \alpha > \sup \langle \varphi, \operatorname{co} \Delta_Y + 2\varepsilon_n(P_\lambda(M_n) \cap Y)^\circ\rangle.$$

As $P_\lambda(M_n) \cap Y \subset B_Z$, we may and do assume that $\sup \langle \varphi, (P_\lambda(M_n) \cap Y)^\circ\rangle = 1$. Thus, by the bipolar theorem, $\varphi \in \overline{P_\lambda(M_n) \cap Y}^*$, and hence, the above inequalities have the form

$$\langle \varphi, P_\lambda^* z^*\rangle > \alpha > \sup \langle \varphi, \Delta_Y\rangle + 2\varepsilon_n. \tag{3}$$

Since $P_\lambda(M_n) \cap Y$ is a separable ε_n-Asplund set, there exists a countable set $S \subset \Delta_Y$ such that for every $\delta \in \Delta_Y$ there is $s \in S$ such that $\sup \langle P_\lambda(M_n) \cap Y, \delta - s\rangle < \varepsilon_n$. As $\varphi \in \overline{P_\lambda(M_n) \cap Y}^*$, there is a sequence $(z_i)_{i \in \mathbb{N}}$ in $P_\lambda(M_n) \cap Y$ such that

$$\langle z_i, P_\lambda^* z^*\rangle \to \langle \varphi, P_\lambda^* z^*\rangle, \quad \text{and} \quad \langle z_i, s\rangle \to \langle \varphi, s\rangle \quad \text{for every } s \in S$$

as $i \to \infty$. We may and do assume that $\langle z_i, P_\lambda^* z^*\rangle > \alpha$ for all $i \in \mathbb{N}$. We now verify the assumptions of Simons' lemma (see, e.g., [7, Lemma 8.1.3]). Put $\Gamma = B_{P_\lambda^* Z^*}$, and for $i \in \mathbb{N}$ define $g_i \in \ell_\infty(\Gamma)$ by $g_i(\gamma) = \langle z_i, \gamma\rangle$, $\gamma \in \Gamma$. As Δ_Y is a boundary for Y, for any positive numbers $\lambda_1, \lambda_2, \ldots$ with $\sum_{i=1}^\infty \lambda_i = 1$ there is $\gamma \in \Delta_Y$ so that $\|\sum_{i=1}^\infty \lambda_i g_i\|_\infty = \sum_{i=1}^\infty \lambda_i g_i(\gamma)$. Thus, by Simons' lemma, we have

$$\sup\{\limsup_{i\to\infty}\langle z_i,\delta\rangle;\ \delta\in\varDelta_Y\}\ge\inf\{\|g\|_\infty;\ g\in\operatorname{co}\{g_i;\ i\in\mathbb{N}\}\}.$$

But

$$\begin{aligned}\inf\{\|g\|;\ g\in\operatorname{co}\{g_i;\ i\in\mathbb{N}\}\}&\ge\inf\{\langle y,P_\lambda^*z^*\rangle;\ y\in\operatorname{co}\{z_i;\ i\in\mathbb{N}\}\}\\&=\inf\{\langle z_i,P_\lambda^*z^*\rangle;\ i\in\mathbb{N}\}\ge\alpha.\end{aligned}$$

Hence

$$\sup\{\limsup_{i\to\infty}\langle z_i,\delta\rangle;\ \delta\in\varDelta_Y\}\ge\alpha. \tag{4}$$

Now fix any $\delta\in\varDelta_Y$. Find $s\in S$ so that $\sup\langle P_\lambda(M_n)\cap Y,\delta-s\rangle<\varepsilon_n$. Thus

$$\begin{aligned}&\limsup_{i\to\infty}\langle z_i,\delta\rangle\\&\quad=\lim_{i\to\infty}\langle z_i,s\rangle+\limsup_{i\to\infty}\langle z_i,\delta-s\rangle<\langle\varphi,s\rangle+\varepsilon_n\qquad(\text{as } z_i\in P_\lambda(M_n)\cap Y)\\&\quad=\langle\varphi,\delta\rangle+\langle\varphi,s-\delta\rangle+\varepsilon_n<\langle\varphi,\delta\rangle+2\varepsilon_n\qquad(\text{as } \varphi\in\overline{P_\lambda(M_n)\cap Y}^{\,*}).\end{aligned}$$

Then (4) gives $\alpha\le\sup\langle\varphi,\varDelta_Y\rangle+2\varepsilon_n$, contrary to (3). We have thus proved that $P_\lambda^*z^*\in\overline{\operatorname{co}\varDelta_Y+2\varepsilon_n(P_\lambda(M_n)\cap Y)^\circ}$.

Having this at hand, we find $k\in\mathbb{N}$, rational numbers $a_1,\ldots,a_k>0$ with $a_1+\cdots+a_k=1$, and $\delta_1,\ldots,\delta_k\in\varDelta_Y$ so that

$$\sup\Big\langle P_\lambda(M_n)\cap Y,P_\lambda^*z^*-\sum_{i=1}^k a_i\delta_i\Big\rangle<\frac{5}{2}\,\varepsilon_n.$$

For $i=1,\ldots,k$ write

$$\delta_i=f_n(z_i)+2\varepsilon_n z_i^*,$$

where $z_i\in Y$ and $z_i^*\in(P_\lambda(M_n)\cap Y)^\circ$; this comes from the definition of $\varDelta_Y$. Then

$$\sup\Big\langle P_\lambda(M_n)\cap Y,P_\lambda^*z^*-\sum_{i=1}^k a_i(f_n(z_i)+2\varepsilon_n z_i^*)\Big\rangle<\frac{5}{2}\,\varepsilon_n,$$

and so,

$$\sup\Big\langle P_\lambda(M_n)\cap Y,P_\lambda^*z^*-\sum_{i=1}^k a_if_n(z_i)\Big\rangle<\frac{9}{2}\,\varepsilon_n.$$

Since $\|D_n^m(z_i)-f_n(z_i)\|_{M_n}\to 0$ as $m\to\infty$ for every $i=1,\ldots,k$, we can find $m\in\mathbb{N}$ so that we still have (note that $P_\lambda(M_n)\subset M_n$)

$$\sup\Big\langle P_\lambda(M_n)\cap Y,P_\lambda^*z^*-\sum_{i=1}^k a_iD_n^m(z_i)\Big\rangle<\frac{9}{2}\,\varepsilon_n.$$

Since D_n^m is $\|\cdot\|_{M_n}$-continuous and $Y = \overline{C}$, there are $y_1, \ldots, y_k \in C$ so that

$$\sup\Big\langle P_\lambda(M_n) \cap Y, P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i)\Big\rangle < \frac{9}{2}\,\varepsilon_n.$$

We have thus proved Claim 2.

It remains to perform a separable reduction in the sense that we want to show that Claim 2 holds also for $C = A_\lambda$, that is,

$$\sup\Big\langle P_\lambda(M_n), P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i)\Big\rangle < \frac{9}{2}\,\varepsilon_n \tag{5}$$

with suitable $k, m \in \mathbb{N}$, rational numbers $a_1, \ldots, a_k$, and vectors $y_1, \ldots, y_k \in A_\lambda$. To do so, let $\mathcal{A}$ denote the set of all sequences $a = (a_1, a_2, \ldots)$ with rational entries such that $a_i = 0$ for all large $i \in \mathbb{N}$. Note that $\mathcal{A}$ is a countable set. Pick a non-empty, at most countable set $S_1 \subset P_\lambda(M_n)$ and find a countable rationally linear set $C_1 \subset A_\lambda$ so that $\overline{C}_1 \supset S_1$. Enumerate C_1 as $\{v_1^1, v_2^1, \ldots\}$. For every $m \in \mathbb{N}$ and every $a = (a_i) \in \mathcal{A}$ we find $z(1, m, a) \in P_\lambda(M_n)$ so that

$$\sup\Big\langle P_\lambda(M_n), P_\lambda^* z^* - \sum_{i=1}^\infty a_i D_n^m(v_i^1)\Big\rangle - \frac{1}{1} < \Big\langle z(1, m, a), P_\lambda^* z^* - \sum_{i=1}^\infty a_i D_n^m(v_i^1)\Big\rangle.$$

Put then $S_2 = S_1 \cup \{z(1, m, a);\ m \in \mathbb{N},\ a \in \mathcal{A}\}$ and find a countable rationally linear set $C_2 \subset A_\lambda$ such that $C_2 \supset C_1$ and $\overline{C}_2 \supset S_2$. Let $l \in \mathbb{N}$, and assume we have already constructed countable sets $S_1 \subset \cdots \subset S_l \subset P_\lambda(M_n)$ and rationally linear countable sets $C_1 \subset \cdots \subset C_l \subset A_\lambda$ with $S_1 \subset \overline{C}_1, \ldots, S_l \subset \overline{C}_l$. Enumerate C_l as $\{v_1^l, v_2^l, \ldots\}$. For every $m \in \mathbb{N}$ and every $a \in \mathcal{A}$ we find a vector $z(l, m, a) \in P_\lambda(M_n)$ such that

$$\sup\Big\langle P_\lambda(M_n), P_\lambda^* z^* - \sum_{i=1}^\infty a_i D_n^m(v_i^l)\Big\rangle - \frac{1}{l} < \Big\langle z(l, m, a), P_\lambda^* z^* - \sum_{i=1}^\infty a_i D_n^m(v_i^l)\Big\rangle.$$

Put then $S_{l+1} = S_l \cup \{z(l, m, a);\ m \in \mathbb{N},\ a \in \mathcal{A}\}$, and find a rationally linear countable set $C_{l+1} \subset A_\lambda$ such that $C_{l+1} \supset C_l$ and $\overline{C}_{l+1} \supset S_{l+1}$. Finally, having performed this for every $l \in \mathbb{N}$, put $S = \bigcup_{l=1}^\infty S_l$ and $C = \bigcup_{l=1}^\infty C_l$. Clearly, C is rationally linear.

By Claim 2, we find $k \in \mathbb{N}$, rational numbers $a_1, \ldots, a_k$, vectors $y_1, \ldots, y_k \in C$, $m \in \mathbb{N}$, and $l \in \mathbb{N}$ such that

$$\sup\Big\langle P_\lambda(M_n) \cap \overline{C}, P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i)\Big\rangle + \frac{1}{l} < \frac{9}{2}\,\varepsilon_n.$$

By enlarging l if necessary, we can achieve that $y_1 = v_{i_1}^l, \ldots, y_k = v_{i_k}^l$ with suitable $i_1, \ldots, i_k \in \mathbb{N}$. Define $b = (b_1, b_2, \ldots)$ by $b_{i_1} = a_1, \ldots, b_{i_k} = a_k$, and

$b_i = 0$ for the remaining $i \in \mathbb{N}$. Then $b \in \mathcal{A}$ and we can estimate

$$\begin{aligned}\sup\Big\langle P_\lambda(M_n), P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i)\Big\rangle & \\ &= \sup\Big\langle P_\lambda(M_n), P_\lambda^* z^* - \sum_{i=1}^\infty b_i D_n^m(v_i^l)\Big\rangle \\ &< \sup\Big\langle z(l,m,b), P_\lambda^* z^* - \sum_{i=1}^\infty b_i D_n^m(v_i^l)\Big\rangle + \frac{1}{l} \\ &\le \sup\Big\langle P_\lambda(M_n)\cap \overline{C}, P_\lambda^* z^* - \sum_{i=1}^k a_i D_n^m(y_i)\Big\rangle + \frac{1}{l} < \frac{9}{2}\,\varepsilon_n.\end{aligned}$$

This proves (5) and completes the separable reduction.

Finally, find $\beta_0 < \lambda$ so that $y_1, \ldots, y_k \in A_{\beta_0}$; it exists as $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$. Put then $\zeta = \sum_{i=1}^k a_i D_n^m(y_i)$. We observe that

$$\zeta \in \sum_{i=1}^k a_i \Psi(y_i) \subset \sum_{i=1}^k a_i \Psi(A_{\beta_0}) \subset B_{\beta_0} \subset \overline{B}_{\beta_0}^{\,*} = P_{\beta_0}^* Z^*.$$

Therefore for $\beta_0 < \beta < \lambda$ we have, from (5),

$$\begin{aligned}\|P_\lambda^* z^* - P_\beta^* z^*\|_{M_n} &\le \|P_\lambda^* z^* - \zeta\|_{M_n} + \|\zeta - P_\beta^* z^*\|_{M_n} \\ &= \sup\langle P_\lambda(M_n), P_\lambda^* z^* - \zeta\rangle + \sup\langle P_\beta(M_n), \zeta - P_\lambda^* z^*\rangle \\ &\le 2\sup\langle P_\lambda(M_n), P_\lambda^* z^* - \zeta\rangle < 9\varepsilon_n\end{aligned}$$

as $P_\beta(M_n) \subset P_\lambda(M_n)$. ∎

We are now ready to prove (iv)⇒(o) in Theorem 1 when X is WLD. Note that if X is WLD, then it contains a linearly dense set $\Gamma \subset B_X$ which *countably supports* all elements of X^*, that is, for every $x^* \in X^*$ the set $\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle \neq 0\}$ is at most countable [9, Theorem 5]. In order to prove (o), by [9, Theorem 2], it suffices to show the following

Claim 1. *There exists a linearly dense set $\Gamma \subset B_X$ such that for every $\varepsilon > 0$ there are subsets $\Gamma_i^\varepsilon \subset \Gamma$, $i \in \mathbb{N}$, satisfying $\Gamma = \bigcup_{n=1}^\infty \Gamma_n^\varepsilon$, and such that for all $n \in \mathbb{N}$ and $x^* \in X^*$ the set $\{\gamma \in \Gamma_n^\varepsilon;\ \langle \gamma, x^*\rangle > \varepsilon\}$ is finite.*

Instead of proving this, we shall prove a subtler statement:

Claim 2. *Let X be a WLD space which is simultaneously σ-Asplund generated, with sets A_n^ε, $\varepsilon > 0$, $n \in \mathbb{N}$, witnessing that. Let $\Gamma \subset B_X$ be any set which is linearly dense in X and countably supports X^*. Then there exist subsets $\Gamma_i \subset \Gamma$, $i \in \mathbb{N}$, satisfying $\Gamma = \bigcup_{i=1}^\infty \Gamma_i$ and such that for every rational $\varepsilon > 0$, every $n, i \in \mathbb{N}$, and every $x^* \in X^*$ the set $\{\gamma \in \Gamma_i \cap A_n^{\varepsilon/9};\ \langle \gamma, x^*\rangle > \varepsilon\}$ is finite.*

Now, if for a rational $\varepsilon > 0$ we put $\Gamma^\varepsilon_{n,i} := \Gamma_i \cap A_n^{\varepsilon/9}$, $n, i \in \mathbb{N}$, and for an irrational $\varepsilon > 0$ we put $\Gamma^\varepsilon_{n,i} := \Gamma_i \cap A_n^{\varepsilon'/9}$, $n, i \in \mathbb{N}$, where ε' is a fixed rational number from the interval $(\varepsilon/2, \varepsilon)$, then, enumerating $\mathbb{N} \times \mathbb{N}$ by elements of $\mathbb{N}$, we get Claim 1.

Claim 2 will be proved by transfinite induction on the density of X. The case when X is separable is simple. Further, let $\aleph$ be an uncountable cardinal, and assume that Claim 2 has already been proved for all spaces X with density less than $\aleph$. Now let X be a Banach space with density $\aleph$ and satisfying the assumptions of Claim 2. Let $\Gamma \subset B_X$ be any set which is linearly dense in X and countably supports X^*. We replace each set A_n^ε by its absolutely convex closed hull; this change will not affect anything. Applying Proposition 15, we get a PRI $(P_\alpha;\, \omega \le \alpha \le \mu)$ on X such that $P_\alpha(A_n^{\varepsilon/9}) \subset A_n^{\varepsilon/9}$, $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu]$, every $n \in \mathbb{N}$, every rational $\varepsilon > 0$, and every $\gamma \in \Gamma$; and moreover, for every limit ordinal $\omega < \lambda \le \mu$, every $n \in \mathbb{N}$, every $\varepsilon > 0$, and every $x^* \in B_{X^*}$ we have

$$\limsup_{\beta \uparrow \lambda} \sup \langle A_n^{\varepsilon/9}, P_\lambda^* x^* - P_\beta^* x^* \rangle < \varepsilon. \tag{6}$$

Fix any $\omega \le \alpha < \mu$ and write $Q_\alpha = P_{\alpha+1} - P_\alpha$. Then the (complemented) subspace $Q_\alpha X$ is also WLD, $B_{Q_\alpha X} = \bigcup_{n=1}^\infty (Q_\alpha X \cap A_n^\varepsilon)$ for every $\varepsilon > 0$, and each set $Q_\alpha X \cap A_n^\varepsilon$ is ε-Asplund in $Q_\alpha X$. Also $Q_\alpha \Gamma$ is linearly dense in $Q_\alpha X$, $Q_\alpha \Gamma \subset \Gamma$, and $Q_\alpha \Gamma$ countably supports the dual $(Q_\alpha X)^*$. Thus the assumptions of Claim 2 are satisfied for the subspace $Q_\alpha X$. Then, by the induction assumption, there are $\Gamma_i^\alpha \subset Q_\alpha \Gamma$, $i \in \mathbb{N}$, satisfying $Q_\alpha \Gamma = \bigcup_{i=1}^\infty \Gamma_i^\alpha$ and such that for every rational $\varepsilon > 0$,

$$\forall n, i \in \mathbb{N}\ \forall y^* \in (Q_\alpha X)^* \quad \#\{\gamma \in \Gamma_i^\alpha \cap (Q_\alpha X \cap A_n^{\varepsilon/9});\ \langle \gamma, y^* \rangle > \varepsilon\} < \omega.$$

Assume that we have done the above for every $\omega \le \alpha < \mu$. Put then $\Gamma_i = \bigcup_{\omega \le \alpha < \mu} \Gamma_i^\alpha$, $i \in \mathbb{N}$. Clearly $\Gamma = \bigcup_{i=1}^\infty \Gamma_i$.

Now fix any rational $\varepsilon > 0$, $n, i \in \mathbb{N}$, and $x^* \in X^*$. Define

$$\Lambda = \{\alpha \in [\omega, \mu);\ \langle \gamma, x^* \rangle > \varepsilon \text{ for some } \gamma \in \Gamma_i^\alpha \cap A_n^{\varepsilon/9}\}.$$

We observe that

$$\{\gamma \in \Gamma_i \cap A_n^{\varepsilon/9};\ \langle \gamma, x^* \rangle > \varepsilon\} = \bigcup_{\alpha \in \Lambda} \{\gamma \in \Gamma_i^\alpha \cap A_n^{\varepsilon/9};\ \langle \gamma, x^*_{|Q_\alpha X} \rangle > \varepsilon\}$$

and that each set from this union is finite, by the induction assumption. Hence it remains to show that Λ is finite.

Assume not. Then there exists an infinite injective sequence $\gamma_1, \gamma_2, \ldots$ in $\Gamma_i \cap A_n^{\varepsilon/9}$ such that $\langle \gamma_j, x^* \rangle > \varepsilon$ for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ we find $\alpha_j < \mu$ so that $\gamma_j \in \Gamma_i^{\alpha_j}$ $(\subset Q_{\alpha_j} X)$. By suppressing some α_j's and reindexing, we can achieve that $\alpha_1 < \alpha_2 < \cdots$. Put $\lambda = \lim_{j\to\infty} \alpha_j$; then $\lambda \le \mu$. From (6) we find $j \in \mathbb{N}$ such that $\sup \langle A_n^{\varepsilon/9}, P_\lambda^* x^* - P_{\alpha_l}^* x^* \rangle < \varepsilon$

whenever $l \in \mathbb{N}$ and $l \geq j$. Then we have

$$\begin{aligned}\varepsilon < \langle \gamma_{j+1}, x^* \rangle &= \langle P_\lambda \gamma_{j+1}, x^* \rangle \\ &= \langle \gamma_{j+1}, P_\lambda^* x^* - P_{\alpha_j}^* x^* \rangle + \langle P_{\alpha_j} \gamma_{j+1}, x^* \rangle < \varepsilon + 0,\end{aligned}$$

a contradiction. Therefore the set Λ must be finite.

Claim 2 is thus proved and the proof of Theorem 1 is finished. ∎

REMARK. From Proposition 15 we can also deduce a well known result that *a Banach space X is WCG if (and only if) it is simultaneously WLD and Asplund generated* (see, e.g., [7, Theorem 8.3.4]). Indeed, let M be a linearly dense convex symmetric closed Asplund set in a WLD space X. Put $\varepsilon_n = 1/n$ and $M_n = M$, $n \in \mathbb{N}$. Then Proposition 15 yields a PRI (P_α) on X so that $P_\alpha M \subset M$ for all α, and for every limit ordinal λ and every $x^* \in X^*$ we have $\|P_\lambda^* x^* - P_\beta^* x^*\|_M \to 0$ as $\beta \uparrow \lambda$. Now a standard argument shows that X is WCG (see, e.g., [14]).

Proof of Theorem 3. (i)⇒(ii). Assume that the compact space K is Eberlein. The space $C(K)$ is then WCG (see [1]), and Theorem 1 implies that $C(K)$ is σ-Asplund generated. That K is a Corson compact follows for instance from [7, Theorem 7.2.7].

The equivalence (ii)⇔(iii) follows from Avilés' result [3, Theorem 20].

(ii)⇒(i). In the proof of [7, Theorem 8.3.5] we can find the following

CLAIM. *Let $\varepsilon > 0$, let μ be a probability measure on K, let $M \subset K$ be a Borel set with $\mu(M) > 0$, and let $A \subset B_{C(K)}$ be an ε-Asplund set. Then there exists a closed set $L \subset M$ such that $\mu(L) > 0$ and A-*diam $L < \varepsilon$.

Let $A_n^\varepsilon \subset B_{C(K)}$, $\varepsilon > 0$, $n \in \mathbb{N}$, be the sets witnessing that $C(K)$ is σ-Asplund generated. Fix any $n, m \in \mathbb{N}$. Let $\mathcal{F}_n^m$ be a maximal family of mutually disjoint closed subsets $L \subset K$ such that $\mu(L) > 0$ and $A_n^{1/m}$-diam $L < 1/m$. Put $H_n^m = \bigcup \mathcal{F}_n^m$; this is an F_σ, hence Borel set. We now show that $\mu(H_n^m) = 1$. Assume not. Then $\mu(K \setminus H_n^m) > 0$. By the above claim, we find a closed set $L \subset K \setminus H_n^m$ such that $\mu(L) > 0$ and $A_n^{1/m}$-diam $L < 1/m$. But this contradicts the maximality of the family $\mathcal{F}_n^m$. Therefore $\mu(H_n^m) = 1$. Clearly, the family $\mathcal{F}_n^m$ is at most countable. Find an at most countable subset $S_n^m \subset H_n^m$ such that for every $k \in H_n^m$ there is $s \in S_n^m$ such that $\sup\{|f(k) - f(s)|;\ f \in A_n^{1/m}\} < 1/m$.

Having done the above for all $n, m \in \mathbb{N}$, put

$$H = \bigcap_{n,m=1}^{\infty} H_n^m \quad \text{and} \quad S = \bigcup_{n,m=1}^{\infty} S_n^m.$$

Note that H is a Borel set, $\mu(H) = 1$, and S is at most countable. We now show that $H \subset \overline{S}$. Let $f \in B_{C(K)}$ be any function. It is enough to show that $f(H) \subset \overline{f(S)}$. So fix any $k \in H$ and any $\varepsilon > 0$. Find $m \in \mathbb{N}$ so that $1/m < \varepsilon$.

Find then $n \in \mathbb{N}$ so that $f \in A_n^{1/m}$. Since $k \in H_n^m$, there is an $s \in S_n^m$ $(\subset S)$ such that $|f(k) - f(s)| < \varepsilon$. We have proved that $f(H) \subset \overline{f(S)}$ for every $f \in B_{C(K)}$, and therefore $H \subset \overline{S}$.

Now, since K is a Corson compact space, we may and do assume that $K \subset (\Sigma(\Gamma), \text{pointwise})$ for a suitable set Γ. Find a countable set $\Gamma_0 \subset \Gamma$ such that $s(\gamma) = 0$ whenever $s \in S$ and $\gamma \in \Gamma \setminus \Gamma_0$. Then also $k(\gamma) = 0$ whenever $k \in H$ and $\gamma \in \Gamma \setminus \Gamma_0$. It follows that H is a separable subset of K. Recalling that $\mu(H) = 1$, we can conclude that μ has a separable support. Then, by, e.g., [7, Lemma 7.3.5], the Banach space $C(K)$ is WLD.

Finally, once we know that $C(K)$ is both WLD and σ-Asplund generated, Theorem 1 guarantees that $C(K)$ is a subspace of a WCG space. Then [9, Theorem 2] shows that the ball $(B_{C(K)^*}, w^*)$ is an Eberlein compact space, and hence, a fortiori, so is K. ∎

Remark. We have recently proved that (ii)⇒(i) in Theorem 1. The proof, mostly based on ideas of M. Raja [26, 27], is longer and completely different from the methods used in this paper. Therefore we have decided to publish it elsewhere.

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