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Embeddings of finite-dimensional operator spaces into the second dual

by

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Abstract. We show that, if a a finite-dimensional operator space E is such that X contains E C-completely isomorphically whenever X^{**} contains E completely isometrically, then E is $2^{15}C^{11}$-completely isomorphic to $\mathbf{R}_m \oplus \mathbf{C}_n$ for some $n, m \in \mathbb{N} \cup \{0\}$. The converse is also true: if X^{**} contains $\mathbf{R}_m \oplus \mathbf{C}_n$ λ-completely isomorphically, then X contains $\mathbf{R}_m \oplus \mathbf{C}_n$ $(2\lambda + \varepsilon)$-completely isomorphically for any $\varepsilon > 0$.

1. Introduction. Local reflexivity of Banach spaces was first discovered by J. Lindenstrauss and H. Rosenthal in [12]. Later, W. Johnson, H. Rosenthal, and M. Zippin [9] improved on this result, and obtained:

THEOREM 1.1. *Suppose X is a Banach space, E and F are finite-dimensional subspaces of X^{**} and X^*, respectively, and $\varepsilon > 0$. Then there exists an operator $u : E \to X$ such that $\|u\| < 1 + \varepsilon$, $u|_{E\cap X} = I_{E\cap X}$, and $\langle ue, f\rangle = \langle e, f\rangle$ for any $e \in E$ and $f \in F$.*

This immediately implies the result of [12]:

COROLLARY 1.2. *Suppose E and X are Banach spaces, E is a finite-dimensional space, and E is contained in X^{**} C-isomorphically (that is, there exists $E' \hookrightarrow X^{**}$ such that $d(E, E') \le C$). Then E is contained in X $(C + \varepsilon)$-isomorphically for any $\varepsilon > 0$.*

In the non-commutative case, the results quoted above do not hold in general. It is well known that an infinite-dimensional operator space need not be locally reflexive. Moreover, for every $n > 2$ the space ℓ_1^n (equipped with the maximal operator space structure) is contained in $\mathbf{B} = \mathbf{K}^{**}$, while, by Theorem 21.5 of [16], $d_{\mathrm{cb}}(\ell_1^n, E) \ge n/(2\sqrt{n-1})$ for any $E \hookrightarrow \mathbf{K}$ (here and below, $\mathbf{B}$ and $\mathbf{K}$ denote the spaces of bounded and compact operators on ℓ_2, respectively).

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In this paper, we show that a "non-commutative" analogue of Corollary 1.2 holds for a finite-dimensional operator space E if and only if E is completely isomorphic to a direct sum of row and column spaces (Theorem 1.3). We say that X contains E *C-completely isomorphically* (*C-ci*) if there exists $F \hookrightarrow X$ such that $d_{\text{cb}}(E, F) \le C$; and X is said to contain E *C+-ci* if it contains E C_1-ci for every $C_1 > C$. A finite-dimensional operator space E is said to be *C-bidually representable* (*C-BDR*, for short) if an operator space X contains E C-completely isomorphically whenever X^{**} contains a completely isometric copy of E; and E is said to be *C+-BDR* if it is C_1-BDR for any $C_1 > C$.

Below, $\mathbf{R}_n$ and $\mathbf{C}_n$ stand for the spaces spanned by the first row and the first column of $n \times n$ matrices, respectively. $\oplus$ means $\oplus_\infty$ (the ℓ_∞ sum of spaces), unless specified otherwise.

THEOREM 1.3.

(1) *If E is λ-completely isomorphic to $\mathbf{R}_m \oplus \mathbf{C}_n$ for some $m, n \in \mathbb{Z}_+$, then E is $2\lambda^2$+-BDR.*
(2) *If E is λ+-BDR, then, for some $m, n \in \mathbb{Z}_+$, $d_{\text{cb}}(E, \mathbf{R}_m \oplus \mathbf{C}_n) \le 2^{15}\lambda^{11}$.*

THEOREM 1.4. *An operator space is 1-BDR if and only if it is 1-dimensional.*

The rest of the paper is organized as follows: in Section 2, we gather some essential facts about non-dual local reflexivity, and prove item (1) of Theorem 1.3. The proof of Theorem 1.3(2) proceeds in two steps. First, we show that E embeds "nicely" into $\mathbf{R} \oplus \mathbf{C}$ (Section 3). In Section 4 we complete the proof of Theorem 1.3(2). Section 5 is devoted to proving Theorem 1.4. Finally, in Section 6, we consider our problem in the setting of C^*-algebras.

The notation used in this paper is, by and large, either standard, or explained above. The minimal (also called injective, or spatial) tensor product of operator spaces is denoted by $\otimes$. If $T : X \to Y$ is a finite rank operator, $\widetilde{T}$ stands for the corresponding element of $X^* \otimes Y$ (then $\|T\|_{\text{cb}} = \|\widetilde{T}\|$). We often use $\mathbf{M}_n$, $\mathbf{B}$, $\mathbf{K}$, and $\mathbf{K}_0$—the spaces of $n \times n$ matrices, of bounded operators on ℓ_2, of compact operators on ℓ_2, and of compact operators with matrices having finitely many non-zero entries, respectively.

In the proofs, we use the notion of exactness of an operator space, and the notion of a complete M-ideal in an operator space.

We say that an operator space Z is *C-exact* ($C > 0$) if, for any finite-dimensional subspace $E \hookrightarrow Z$, and every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $F \hookrightarrow M_N$ with $d_{\text{cb}}(E, F) < C + \varepsilon$; and Z is said to be *exact* if it is C-exact for some C. The *exactness constant* of Z (denoted by $\text{ex}(Z)$) is the infimum of all the C's with the above property. It is easy to see that the row space $\mathbf{R}$, the column space $\mathbf{C}$, and the space of compact operators $\mathbf{K}$ are 1-exact. On the other hand, it is known that $\mathbf{B}$ is not exact. The reader is referred to [15], Chapter 17 of [16], or Chapter 14 of [6] for more information.

A subspace X of an operator space Y is called a *complete M-summand* if $Y = Y \oplus_\infty Z$ for some $Z \hookrightarrow Y$; and X is a *complete M-ideal* in Y if $X^{\perp\perp}$ is a complete M-summand in Y^{**}. We refer the reader to [5], or to Section 4.8 of [2], for information about complete M-ideals. For the theory of M-ideals in Banach spaces, see [8].

Finally, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ is the set of non-negative integers.

2. Some remarks on bidual representability. To prove Theorem 1.3(1), we begin with

PROPOSITION 2.1. *Suppose E and F are finite-dimensional operator spaces, X is an operator space, $u : E \to X^{**}$ and $v : X^{**} \to F$ are linear maps, and $\varepsilon > 0$. Then there exist linear maps $u_1 : E \to X$ and $v_1 : X \to F$ such that $\|u_1\|_{\mathrm{cb}}\|v_1\|_{\mathrm{cb}} < (1+\varepsilon)\,\mathrm{ex}(E^*)\,\mathrm{ex}(F)\|u\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}$ and $v_1u_1 = vu$.*

Proof. Pick biorthogonal systems $(e_i, e_i^*)_{i=1}^n$ and $(f_j, f_j^*)_{j=1}^m$ in E and F, respectively. Write $\widetilde{u} = \sum_i e_i^* \otimes x_i^{**}$ and $\widetilde{v} = \sum_j f_j \otimes x_j^{***}$ with $x_i^{**} \in X^{**}$ and $x_j^{***} \in X^{***}$. By Proposition 3.2.1 of [6], $(\mathbf{M}_N(X))^{**} = \mathbf{M}_N(X^{**})$ isometrically, hence $\widetilde{v} \in X^{***} \otimes F$ can be "approximated" by $\widetilde{v}_1 = \sum_j f_j \otimes x_j^* \in F \otimes X^*$ in such a way that (1) $\|\widetilde{v}_1\| < \sqrt{1+\varepsilon}\,\mathrm{ex}(F)\|\widetilde{v}\|$, and (2) $\langle x_j^*, x_i^{**}\rangle = \langle x_j^{***}, x_i^{**}\rangle$ for any pair (i,j). Similarly, there exists $\widetilde{u}_1 = \sum_i e_i^* \otimes x_i \in E^* \otimes X$ such that $\|\widetilde{u}_1\| \le \sqrt{1+\varepsilon}\,\mathrm{ex}(E^*)\|\widetilde{u}\|$, and $\langle x_j^*, x_i^{**}\rangle = \langle x_j^*, x_i\rangle$ for any pair (i,j).

Now go back from tensor products to c.b. maps $u_1 : E \to X$ and $v_1 : X \to F$. By the above, $\|u_1\|_{\mathrm{cb}} < \sqrt{1+\varepsilon}\,\mathrm{ex}(E^*)\|u\|_{\mathrm{cb}}$ and $\|v_1\|_{\mathrm{cb}} < \sqrt{1+\varepsilon}\,\mathrm{ex}(F)\|v\|_{\mathrm{cb}}$. Moreover, for any i,

$$v_1u_1e_i = \sum_j \langle x_j^*, x_i\rangle f_j = \sum_j \langle x_j^{***}, x_i^{**}\rangle f_j = vue_i.$$

Therefore, $vu = v_1u_1$ and $\|u_1\|_{\mathrm{cb}}\|v_1\|_{\mathrm{cb}} < (1+\varepsilon)\,\mathrm{ex}(E^*)\,\mathrm{ex}(F)\|u\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}}$. ∎

Proposition 2.1 implies:

COROLLARY 2.2. *Suppose X is an operator space, and $\delta > 0$. Then*:

(1) *If X^{**} contains a λ-injective finite-dimensional operator space E C-completely isomorphically, then X contains E $(\lambda C\,\mathrm{ex}(E)\,\mathrm{ex}(E^*)+\delta)$-completely isomorphically.*
(2) *If X^{**} contains $\mathbf{R}_m \oplus \mathbf{C}_n$ $(n, m \in \mathbb{Z}_+)$ C-completely isomorphically, then X contains $\mathbf{R}_m \oplus \mathbf{C}_n$ $(2C+\delta)$-completely isomorphically. Consequently, $\mathbf{R}_m \oplus \mathbf{C}_n$ is $2+$-BDR.*

(3) *Suppose* $E = \mathbf{R}_m$, $\mathbf{C}_m$, ℓ^2_∞, $\mathbb{C} \oplus_\infty \mathbf{R}_m$, *or* $\mathbb{C} \oplus_\infty \mathbf{R}_m$ $(m \in \mathbb{N})$. *If* X^{**} *contains* E C*-completely isomorphically*, *then* X *contains* E $(C+\delta)$*-completely isomorphically. Consequently*, E *is* $1+$*-BDR.*

Proof. (1) Consider a subspace $E' \hookrightarrow X^{**}$ for which there exists a completely contractive isomorphism $uE \to E'$ with $\|u^{-1}\|_{\mathrm{cb}} \le C$. As E is λ-injective, there exists $v : X^{**} \to E$ extending u^{-1} (that is, $v|_{E'} = u^{-1}$) of norm not exceeding λC. By Proposition 2.1, for every $\varepsilon > 0$ there exist $u_1 : E \to X$ and $v_1 : X \to E$ such that $\|u_1\|_{\mathrm{cb}}\|v_1\|_{\mathrm{cb}} \le (1+\varepsilon)\operatorname{ex}(E)\operatorname{ex}(E^*)\lambda C$ and $v_1 u_1 = vu = I_E$. Therefore, $d_{\mathrm{cb}}(E, u_1(E)) \le (1+\varepsilon)\operatorname{ex}(E)\operatorname{ex}(E^*)\lambda C$. Since ε can be arbitrarily small, we are done.

(2) The space $\mathbf{R}_m \oplus \mathbf{C}_n$ embeds into $\mathbf{M}_{m+n}$ as a 1-completely complemented subspace, hence $\operatorname{ex}(\mathbf{R}_m \oplus \mathbf{C}_n) = 1$. Moreover, $(\mathbf{R}_m \oplus \mathbf{C}_n)^* = \mathbf{C}_n \oplus_1 \mathbf{R}_m$ is 2-completely isomorphic to $\mathbf{C}_n \oplus \mathbf{R}_m$, hence $\operatorname{ex}((\mathbf{R}_m \oplus \mathbf{C}_n)^*) \le 2$. An application of part (1) yields the desired result.

(3) Reason as in the proof of part (2), and recall that $\operatorname{ex}(E) = \operatorname{ex}(E^*) = 1$ for any E from the list (see Chapter 21 of [16]). ∎

Remark 2.3. We return to the connections between injectivity, exactness, and BDR in Section 5. Meanwhile, note that by Theorem 1.4, $\mathbb{C}$ is the only 1-BDR space. Hence, the condition of being 1-BDR is much stronger than being 1+-BDR.

The next result shows that bidual representability is an isomorphic property.

Proposition 2.4. *Suppose* X *is an operator space*, E *is a finite-dimensional subspace of* X^{**}, $d_{\mathrm{cb}}(E,F) = C_1$, *and* $\operatorname{ex}(F) < C_2$. *Then there exists an operator space* Y *such that* $d_{\mathrm{cb}}(X,Y) < C_1C_2$ *and* Y^{**} *contains a completely isometric copy of* F.

Proof. Suppose the map $u : E \to F$ is such that $\|u\|_{\mathrm{cb}} = C_1$ and $\|u^{-1}\|_{\mathrm{cb}} = 1$. By renorming X, we shall construct a space Y and a map $T : X \to Y$ such that $\|T\|_{\mathrm{cb}} < C_1C_2$, and T^{-1} is completely contractive. To this end, find a subspace $F_1 \hookrightarrow \mathbf{M}_N$ and a map $v : F_1 \to F$ such that $\|v^{-1}\|_{\mathrm{cb}} = 1$ and $\|v\|_{\mathrm{cb}} < C_2$. Embedding F into $\mathbf{B}$ completely isometrically, and applying Stinespring's extension theorem, we obtain an operator $\widetilde{v} : \mathbf{M}_N \to \mathbf{B}$ such that $\widetilde{v}|_{F_1} = v$ and $\|\widetilde{v}\|_{\mathrm{cb}} = \|v\|_{\mathrm{cb}}$. Let $\widetilde{F} = \widetilde{v}(\mathbf{M}_N)$.

Extend $w = v^{-1}u$ to $\widetilde{w} : X^{**} \to \mathbf{M}_N$ such that $\|\widetilde{w}\|_{\mathrm{cb}} = \|w\|_{\mathrm{cb}} \le C_1$ and $\widetilde{w}|_E = w$. Let $C_1' = C_1C_2/\|v\|_{\mathrm{cb}}$. The unit ball of $X^* \otimes \mathbf{M}_N$ is weak* dense in the unit ball of $X^{***} \otimes \mathbf{M}_N$, hence there exists an operator $w_1 : X \to \mathbf{M}_N$ such that $\|w_1\|_{\mathrm{cb}} < C_1'$ and $w_1^{**}|_E = w$.

Define a new operator space structure Y on X by setting, for $x \in X \otimes \mathbf{K}_0$,

$$\|x\|_{Y\otimes\mathbf{K}} = \max\{\|x\|_{X\otimes\mathbf{K}}, \|(\widetilde{v}w_1 \otimes I_{\mathbf{K}})x\|_{\widetilde{F}\otimes\mathbf{K}}\}.$$

Denote by T the formal identity map from X to Y. Clearly, T^{-1} is a complete contraction, and $\|T\|_{\mathrm{cb}} \le \|\widetilde{v}w_1\|_{\mathrm{cb}} < C_1C_2$. Moreover, for $x^{**} \in X^{**} \otimes \mathbf{K}_0$,

$$\|x^{**}\|_{Y^{**}\otimes\mathbf{K}} = \max\{\|x^{**}\|_{X^{**}\otimes\mathbf{K}}, \|(\widetilde{v}w_1^{**} \otimes I_{\mathbf{K}})x^{**}\|_{\widetilde{F}\otimes\mathbf{K}}\},$$

which implies that, for $e \in E \otimes \mathbf{K}_0$, $\|e\|_{Y^{**}\otimes\mathbf{K}} = \|(u \otimes I_{\mathbf{K}})e\|_{F\otimes\mathbf{K}}$. ∎

Proposition 2.5. *Suppose the space E is C+-BDR. Then E is C-exact.*

Proof. Apply the definition of BDR to $X = \mathbf{K}$. ∎

Corollary 2.6. *Suppose E and E' are operator spaces of the same* (*finite*) *dimension*, *E is C-BDR*, $d_{\mathrm{cb}}(E, E') = \lambda$, *and* $\mathrm{ex}(E) < \mu$. *Then an operator space X contains E $C\lambda\mu$-completely isomorphically whenever X^{**} contains E' completely isometrically. Consequently*, *E' is $C\lambda^2\mu$-BDR.*

Proof. Suppose E' is contained in X^{**}. Consider a map $u : E' \to E$ such that $\|u\|_{\mathrm{cb}} = \lambda$ and $\|u^{-1}\|_{\mathrm{cb}} = 1$. By the proof of Proposition 2.4, there exists an operator space Y and a map $T : X \to Y$ such that $\|T\|_{\mathrm{cb}} < \lambda\mu$, $\|T^{-1}\|_{\mathrm{cb}} \le 1$, and $T^{**}(E') = E$. There exists a subspace $F \hookrightarrow Y$ with $d_{\mathrm{cb}}(E, F) \le C$. Let $F' = T^{-1}(F) \hookrightarrow X$. Then

$$d_{\mathrm{cb}}(E, F') \le d_{\mathrm{cb}}(E, F)d_{\mathrm{cb}}(F, F') \le d_{\mathrm{cb}}(E, F)\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} < C\lambda\mu.$$

Therefore, $d_{\mathrm{cb}}(E', F') \le d_{\mathrm{cb}}(E', E)d_{\mathrm{cb}}(E, F') < C\lambda^2\mu$. ∎

3. Proof of Theorem 1.3(2): E is a subspace of $\mathbf{R} \oplus \mathbf{C}$. In this section, we make the first step toward proving Theorem 1.3(2) by showing that every C-BDR space embed "neatly" into $\mathbf{R} \oplus \mathbf{C}$. More precisely, we prove:

Theorem 3.1. *For every $N \in \mathbb{N}$, there exists a separable operator space X such that*:

1. *X^{**} contains $\mathbf{B}$ as a complete M-summand.*
2. *Suppose E is a finite-dimensional operator space such that $d_{\mathrm{cb}}(E, E')$ $< C$ for some $E' \hookrightarrow \ell_\infty(\mathbf{M}_N)$, and X contains E c-completely isomorphically for some $c < C$. Then E is $4\sqrt{2}C^3$-completely isomorphic to a subspace of $\mathbf{R} \oplus \mathbf{C}$.*

Consequently, *any C+-BDR space is $4\sqrt{2}C^3$-completely isomorphic to a subspace of $\mathbf{R} \oplus \mathbf{C}$.*

We start the proof by constructing the space X. At the Banach space level, let $X = (\bigoplus_{n>N} \mathbf{M}_n)_c$ be the space of all sequences whose elements are $n \times n$ matrices, and which have a limit (in $\mathbf{K}$). Denote by P_n (once again, $n > N$) the canonical "truncation" from X to $(\sum_{k=N+1}^n \mathbf{M}_k)_\infty$. Let $X_n = \mathrm{MAX}_n((\sum_{k=N+1}^n \mathbf{M}_k)_\infty)$ (see [13] or [11] for the definition and properties of the functor MAX_n), and set, for $x \in X \otimes \mathbf{K}_0$,

$$\|x\| = \sup_n \|(P_n \otimes I_{\mathbf{K}})x\|_{X_n \otimes \mathbf{K}}. \tag{3.1}$$

It is easy to notice that the Banach space structure of X is as described above. Denote by Y the space $(\bigoplus_{n>N} \mathbf{M}_n)_{c_0}$, with the operator space structure inherited from X.

For further use, we state the following easy consequence of (3.1).

Lemma 3.2. *Suppose x is an element of $X \otimes \mathbf{M}_m$ ($m \in \mathbb{N}$). Write $x = (x_i)_{i>N}$ with $x_i \in \mathbf{M}_i \otimes \mathbf{M}_m$. Then*

$$\|x\|_{X\otimes \mathbf{M}_m} = \max\{\|x\|_{(\sum_{n>N} \mathbf{M}_n)_c \otimes \mathbf{M}_m}, \max_{N<n\le m} \|(P_n \otimes I_{\mathbf{M}_m})x\|_{X_n \otimes \mathbf{M}_m}\}.$$

Lemma 3.3. *Y is a complete M-ideal in X.*

Proof. For $n > N$, define the map $T_n : X \to Y$ by setting

$$T_n((x_i)_{i>N}) = (x_{N+1}, \ldots, x_n, 0, 0, \ldots).$$

We need to show that the sequence (T_n) is an M-complete approximate identity (see Definition 1.1 of [1]). Clearly, $T_n y \to y$ for any $y \in Y$. Moreover, suppose $x = (x_i)$ and $z = (z_i)$ are elements of $X \otimes \mathbf{M}_m$. By Lemma 3.2, for $n \ge m$,

$$\begin{aligned}
\|(T_n \otimes I_{\mathbf{M}_m})(x) + ((I - T_n) \otimes I_{\mathbf{M}_m})(z)\| &= \|(x_{N+1}, \ldots, x_n, z_{n+1}, z_{n+2}, \ldots)\| \\
&= \max\{\|(x_{N+1}, \ldots, x_n, z_{n+1}, z_{n+2}, \ldots)\|_{(\sum_{n>N} \mathbf{M}_n)_c \otimes \mathbf{M}_m}, \\
&\qquad \max_{N<k\le m} \|(x_{N+1}, \ldots, x_k)\|_{X_k \otimes \mathbf{M}_m}\} \\
&= \max\{\max_{N<k\le n} \|x_k\|, \sup_{k>n} \|z_k\|, \max_{N<k\le m} \|(x_{N+1}, \ldots, x_k)\|_{X_k \otimes \mathbf{M}_m}\} \\
&\le \max\{\|x\|, \|z\|\}.
\end{aligned}$$

Thus, (T_n) is indeed an M-completely approximate identity, and therefore, by Theorem 1.1 of [1], Y is a complete M-ideal in X. ■

Lemma 3.4. *The quotient X/Y is completely isometric to $\mathbf{K}$.*

Proof. Define the map $U : X/Y \to \mathbf{K}$ by setting $U([(x_i)_{i>N}]) = \lim_i x_i$. To show that it is a complete isometry, fix $m \in \mathbb{N}$, and consider $x = (x_i)_{i>N} \in X \otimes \mathbf{M}_m$. By Lemma 3.2, $\|[x]\|_{X/Y \otimes \mathbf{M}_m} = \lim_i \|x_i\| = \|U[x]\|_{\mathbf{K}\otimes \mathbf{M}_m}$. ■

Conclusion of the proof of Theorem 3.1. Part (1) of the theorem follows from Lemma 3.4. Suppose E is a finite-dimensional operator space as in part (2), $u : E \to X$ is a complete contraction, $F = u(E)$, and $\|u^{-1}\|_{\mathrm{cb}} < C$. Let $F_n = P_n(F)$ be a subspace of X_n (here, P_n and X_n are as in (3.1)), and let $u_n = P_n u$. By the definition of MAX_n,

$$\|(x_{N+1}, \ldots, x_n)\|_{X_n \otimes \mathbf{M}_N} = \max_{N<k\le n} \|x_k\|,$$

hence there exists $n > N$ such that $\|(u_n \otimes I_{\mathbf{M}_N})e\| > C^{-1}\|e\|$ for any $e \in E \otimes \mathbf{M}_N$. By Smith's lemma, $\|u_n^{-1}\|_{\mathrm{cb}} < C\|u_n^{-1} \otimes I_{\mathbf{M}_N}\| < C^2$.

Since X_n^* is 1-exact, by [17] there exist operators $v : E \to \mathbf{R} \oplus \mathbf{C}$ and $w : \mathbf{R} \oplus \mathbf{C} \to X_n$ so that $\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \le 4\sqrt{2}C$ and $u_n = wv$. Let $G = v(E)$. Then $d_{\mathrm{cb}}(E, G) \le \|v\|_{\mathrm{cb}}\|u_n^{-1}w\|_{\mathrm{cb}} < 4\sqrt{2}C^3$.

To prove the last assertion, note that, by the reasoning above and Proposition 2.5, any N-dimensional $C+$-BDR space E embeds into $\mathbf{R}_N \oplus \mathbf{C}_N$ $4\sqrt{2}(C+\varepsilon)^3$-completely isomorphically for any $\varepsilon > 0$. Letting ε approach 0, and applying a classical compactness argument, we complete the proof. ■

4. Proof of Theorem 1.3(2): E is $\mathbf{R}_m \oplus \mathbf{C}_n$. In this section, we complete the proof of Theorem 1.3(2). For the convenience of working with Hilbert spaces as much as possible, we use the sum $\oplus_2$: if X and Y are operator spaces, and x and y are elements of $X \otimes \mathbf{K}_0$ and $Y \otimes \mathbf{K}_0$, respectively, define

$$\|x \oplus y\|_{(X\oplus_2 Y)\otimes\mathbf{K}} = \max\{\|x\|, \|y\|, \|x \oplus y\|_{\mathrm{MIN}(X\oplus_2 Y)\otimes\mathbf{K}}\}. \tag{4.1}$$

Clearly, Ruan's axioms are satisfied, so $X \oplus_2 Y$ is indeed an operator space.

Note that any N-dimensional subspace E of $\mathbf{R} \oplus_2 \mathbf{C}$ is contained in $\mathbf{R}_N \oplus_2 \mathbf{C}_N$. In view of Theorem 3.1, the proof of Theorem 1.3(2) follows from:

THEOREM 4.1. *Suppose E is an N-dimensional subspace of $\mathbf{R}_N \oplus_2 \mathbf{C}_N$ which is λ-BDR. Then $d_{\mathrm{cb}}(E, \mathbf{R}_m \oplus_2 \mathbf{C}_n) \le 32\sqrt{2}\lambda$ for suitable $m, n \in \mathbb{Z}_+$.*

Denote by $A_{\mathbf{R}}$ and $A_{\mathbf{C}}$ the orthogonal projections from E on $\mathbf{R}_N$ and $\mathbf{C}_N$, respectively. By polar decomposition, there exists an orthonormal basis $(e_i)_{i=1}^N$ in E such that $A_{\mathbf{R}}e_i = a_i^{(\mathbf{R})}e_i^{(\mathbf{R})}$, with $0 \le a_N^{(\mathbf{R})} \le \cdots \le a_1^{(\mathbf{R})} \le 1$ and $e_i^{(\mathbf{R})}$ being an orthonormal basis for $\mathbf{R}_N$. Then, for $1 \le i \le N$, $A_{\mathbf{C}}e_i = a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}$, where $e_i^{(\mathbf{C})}$ is a unit vector and $a_i^{(\mathbf{C})} = \sqrt{1-(a_i^{(\mathbf{R})})^2}$. Note that, for $i \ne j$,

$$\langle e_i, e_j^{(\mathbf{R})}\rangle = \langle a_i^{(\mathbf{R})}e_i^{(\mathbf{R})} + a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}, e_j^{(\mathbf{R})}\rangle = 0.$$

Moreover,

$$\begin{aligned} 0 = \langle e_i, e_j\rangle &= \langle a_i^{(\mathbf{R})}e_i^{(\mathbf{R})} + a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}, a_j^{(\mathbf{R})}e_j^{(\mathbf{R})} + a_j^{(\mathbf{C})}e_j^{(\mathbf{C})}\rangle \\ &= a_i^{(\mathbf{C})}a_j^{(\mathbf{C})}\langle e_i^{(\mathbf{C})}, e_j^{(\mathbf{C})}\rangle, \end{aligned}$$

and therefore, the vectors $(e_i^{(\mathbf{C})})_{i=1}^N$ form an orthonormal basis in $\mathbf{C}_N$ (certain minor changes to this construction need to be made if $a_i^{(\mathbf{C})} = 0$ for some i's). One can also show that $\langle e_i, e_j^{(\mathbf{C})}\rangle = 0$ for $i \ne j$.

LEMMA 4.2. *$(e_i)_{i=1}^N$ is a 1-completely unconditional basis in E.*

Proof. Suppose $\lambda_1, \ldots, \lambda_N$ are complex numbers of absolute value not exceeding 1. We have to show that the operator $\Lambda = \operatorname{diag}((\lambda_i)_{i=1}^N)$ is completely contractive. To this end, consider an operator $\widetilde{\Lambda}$ on $\mathbf{R}_N \oplus_2 \mathbf{C}_N$ mapping $e_i^{(\mathbf{R})}$ (or $e_i^{(\mathbf{C})}$) into $\lambda_i e_i^{(\mathbf{R})}$ (resp. $\lambda_i e_i^{(\mathbf{C})}$) for $1 \le i \le N$. By the discussion preceding the statement of this lemma, the restrictions of $\widetilde{\Lambda}$ to $\mathbf{R}_N$ and $\mathbf{C}_N$ are contractive, hence completely contractive (row and column spaces are 1-homogeneous). Thus, by the homogeneity of minimal spaces, and by (4.1), $\widetilde{\Lambda}$ is completely contractive. To complete the proof, observe that the restriction of $\widetilde{\Lambda}$ to E coincides with Λ. ■

This lemma, together with (4.1), yields:

COROLLARY 4.3. *Suppose $\mathcal{I}$ is a subset of $\{1, \ldots, N\}$. Let $E_{\mathcal{I}} = \operatorname{span}[e_i \mid i \in \mathcal{I}]$, and $E_{\mathcal{I}}^{\perp} = \operatorname{span}[e_i \mid i \notin \mathcal{I}]$. Then the formal identity map $\mathrm{id} : E \to E_{\mathcal{I}} \oplus_2 E_{\mathcal{I}}^{\perp}$ is completely contractive*, *and* $\|\mathrm{id}^{-1}\|_{\mathrm{cb}} \le \sqrt{2}$.

Proof. For simplicity, denote the space $E_{\mathcal{I}} \oplus_2 E_{\mathcal{I}}^{\perp}$ by F. Consider $x = \sum_i e_i \otimes x_i \in E \otimes \mathbf{K}$. Then

$$\|x\|_{E\otimes\mathbf{K}} = \max\Big\{\Big\|\sum_{i=1}^N (a_i^{(\mathbf{R})})^2 x_i^* x_i\Big\|^{1/2}, \Big\|\sum_{i=1}^N (a_i^{(\mathbf{C})})^2 x_i x_i^*\Big\|^{1/2}, \Big\|\sum_{i=1}^N e_i \otimes x_i\Big\|_{\mathrm{MIN}(\ell_2^N)\otimes\mathbf{K}}\Big\},$$

while

$$\|x\|_{F\otimes\mathbf{K}} = \max\Big\{\Big\|\sum_{i\in\mathcal{I}} (a_i^{(\mathbf{R})})^2 x_i^* x_i\Big\|^{1/2}, \Big\|\sum_{i\in\mathcal{I}^{\mathrm{c}}} (a_i^{(\mathbf{R})})^2 x_i^* x_i\Big\|^{1/2}, \Big\|\sum_{i\in\mathcal{I}} (a_i^{(\mathbf{C})})^2 x_i x_i^*\Big\|^{1/2}, \Big\|\sum_{i\in\mathcal{I}^c} (a_i^{(\mathbf{C})})^2 x_i x_i^*\Big\|^{1/2}, \Big\|\sum_{i=1}^N e_i \otimes x_i\Big\|_{\mathrm{MIN}(\ell_2^N)\otimes\mathbf{K}}\Big\}.$$

Comparing the two displayed expressions yields the result. ■

Turning back to the proof of Theorem 4.1, denote by m the largest number i for which $a_i^{(\mathbf{R})} \ge 1/\sqrt{2}$ (if $a_1^{(\mathbf{R})} < 1/\sqrt{2}$, set $m = 0$), and let $n = N - m$. Let $E_{\mathbf{R}} = \operatorname{span}[e_i \mid 1 \le i \le m]$, $E_{\mathbf{C}} = \operatorname{span}[e_i \mid m < i \le N]$. For a compact operator $T \in B(H, K)$ (H and K are Hilbert spaces), we denote by $\|T\|_2$ its Hilbert–Schmidt norm. That is, $\|T\|_2 = (\sum_n t_n^2)^{1/2}$, where $t_1 \ge t_2 \ge \cdots \ge 0$ are the singular numbers of T. Equivalently, $\|T\|_2^2 = \sum_{i,j} |\langle Te_i, f_j\rangle|^2$, where (e_i) and (f_j) are orthonormal bases in H and K, respectively.

To complete the proof, it suffices to show that

$$\max\{\|A_{\mathbf{C}}|_{E_{\mathbf{R}}}\|_2, \|A_{\mathbf{R}}|_{E_{\mathbf{C}}}\|_2\} \le 16\sqrt{2}\lambda.$$

Indeed, this would imply that $E_{\mathbf{R}}$ and $E_{\mathbf{C}}$ are 32λ-completely isomorphic to $\mathbf{R}_m$ and $\mathbf{C}_n$, respectively. An application of Corollary 4.3 would then yield the result. Thus, it remains to prove:

PROPOSITION 4.4. *In the above notation,* $\|A_{\mathbf{C}}|_{E_{\mathbf{R}}}\|_2 \le 16\sqrt{2}\lambda$.

Proof. If $m = 0$, there is nothing to prove. If $m \ge 1$, denote $A_{\mathbf{C}}|_{E_{\mathbf{R}}}$ by A for simplicity of notation. Let

$$X_1 = \Big(\sum_{n>N} \mathrm{MAX}_n(E_{\mathbf{R}})\Big)_c, \quad Y = \Big(\sum_{n>N} \mathrm{MAX}_n(E_{\mathbf{R}})\Big)_{c_0}, \quad X = X_1 \oplus_2 E_{\mathbf{C}}.$$

By Proposition 3.2 of [1], Y is a complete M-ideal in X_1. Imitating the proof of Lemma 3.4, one can show that $X_1/Y = E_{\mathbf{R}}$ completely isometrically. Therefore, X_1^{**} contains $E_{\mathbf{R}}$ as a complete M-summand. Finally, $\mathbf{M}_s(X^{**}) = (\mathbf{M}_s(X))^{**}$ for any $s \in \mathbb{N}$, hence $X^{**} = X_1^{**} \oplus_2 E_{\mathbf{C}}$, and therefore, X^{**} contains E $\sqrt{2}$-completely isomorphically. By Corollary 2.6, X contains E $\sqrt{2}\lambda$+-ci.

Pick $C > \lambda$, and consider a complete contraction $u : E \to X$ satisfying $\|u^{-1}\|_{\mathrm{cb}} \le \sqrt{2}C$. Denote the "natural truncation" of u to $E_{\mathbf{C}}$ (or the nth summand of X_1, $n > N$) by u_0 (respectively, u_n). More precisely, we view u_0 (resp. u_n) as a map from E to $E_{\mathbf{C}}$ (resp. $\mathrm{MAX}_n(E_{\mathbf{R}})$). In this notation, $F = \ker u_0$ is an M-dimensional subspace of E ($M \ge m$). Let v be an isometry from $\mathbf{R}_M$ onto F. To complete the proof, it suffices to show that

$$\text{(4.2)} \qquad \begin{aligned} &(1) \quad \|v\|_{\mathrm{cb}} \ge \max\{1, \|A\|_2/2\}, \\ &(2) \quad \|u_n v\|_{\mathrm{cb}} \le 8 \quad \text{for any } n > N. \end{aligned}$$

Indeed, then $\|uv\|_{\mathrm{cb}} = \sup_n \|u_n v\|_{\mathrm{cb}} \le 8$. On the other hand, $v = u^{-1} \circ (uv)$, hence the above inequalities imply

$$\|A\|_2/2 \le \|v\|_{\mathrm{cb}} < \sqrt{2}C\|uv\|_{\mathrm{cb}} \le 8\sqrt{2}C.$$

Since $C > \lambda$ is arbitrary, we conclude that $\|A_2\|_2 \le 16\sqrt{2}\lambda$.

We start by proving (4.2(1)). Denote by Q and $Q^{\perp}$ the orthogonal projections from F onto $E_{\mathbf{R}}$ and $E_{\mathbf{C}}$, respectively. Reasoning as in the proof of Lemma 4.2 (see also the discussion preceding it), we can find an orthonormal basis $(f_i)_{i=1}^M$ in F such that $\langle Qf_i, Qf_j\rangle = \langle Q^{\perp}f_i, Q^{\perp}f_j\rangle = 0$ if $i \ne j$. By changing the numbering if necessary, assume that $\|Q^{\perp}f_i\| \ge 1/\sqrt{2}$ for $1 \le i \le l$, and $\|Qf_i\| > 1/\sqrt{2}$ for $l < i \le M$. Let $F_1 = \mathrm{span}[f_i \,|\, 1 \le i \le l]$, $F_2 = \mathrm{span}[f_i \,|\, l < i \le M]$, and $G_s = v^{-1}(F_s)$ for $s = 1, 2$. We can identify G_1 and G_2 with $\mathbf{R}_l$ and $\mathbf{R}_{M-l}$, respectively. Note that

$$\|v\|_{\mathrm{cb}} \ge \max\{\|v|_{G_1}\|_{\mathrm{cb}}, \|v|_{G_2}\|_{\mathrm{cb}}\} \ge (\|v|_{G_1}\|_{\mathrm{cb}}^2 + \|v|_{G_2}\|_{\mathrm{cb}}^2)^{1/2}/\sqrt{2}.$$

However,

$$\|v|_{G_1}\|_{\mathrm{cb}} \ge \|Q^{\perp}v|_{G_1}\|_{CB(\mathbf{R}_l, \mathbf{C}_M)} = \|Q^{\perp}v|_{G_1}\|_2 \ge \sqrt{l/2}.$$

Similarly,

$$\|v|_{G_2}\|_{\mathrm{cb}} \geq \|AQv|_{G_2}\|_{CB(\mathbf{R}_{M-l},\mathbf{C}_M)} = \|AQv|_{G_2}\|_2 = \Big(\sum_{i=l+1}^{M} \|AQf_i\|^2\Big)^{1/2}.$$

To evaluate $\|A\|_2$, introduce the vectors $f_i' \in E_{\mathbf{R}}$ $(1 \leq i \leq M)$ in such a way that $E_{\mathbf{R}} = \mathrm{span}[f_i' \mid 1 \leq i \leq M]$, and, for each i, $\|f_i'\|$ equals 0 or 1, and $f_i' = Qf_i/\|Qf_i\|$ provided $Qf_i \neq 0$ (this is possible, since $M \geq m = \dim E_{\mathbf{R}}$). Since A is a contraction, we have

$$\|A\|_2^2 = \sum_{i=1}^{M} \|Af_i'\|^2 \leq l + 2\sum_{i=l+1}^{M} \|AQf_i\|^2 \leq 2(\|v|_{G_1}\|_{\mathrm{cb}}^2 + \|v|_{G_2}\|_{\mathrm{cb}}^2) \leq 4\|v\|_{\mathrm{cb}}^2.$$

Moreover, v is an isometry, thus we obtain (4.2(1)).

Next we tackle (4.2(2)). Fix n. By [17], there exist operators $T_{\mathbf{R}} : F \to \mathbf{R}_M$, $T_{\mathbf{C}} : F \to \mathbf{C}_M$, $S_{\mathbf{R}} : \mathbf{R}_M \to \mathrm{MAX}_n(E_{\mathbf{R}})$, and $S_{\mathbf{C}} : \mathbf{C}_M \to \mathrm{MAX}_n(E_{\mathbf{R}})$ so that $u_n|_F = S_{\mathbf{R}}T_{\mathbf{R}} + S_{\mathbf{C}}T_{\mathbf{C}}$ and $\max\{\|T_{\mathbf{R}}\|_{\mathrm{cb}}\|S_{\mathbf{R}}\|_{\mathrm{cb}}, \|T_{\mathbf{C}}\|_{\mathrm{cb}}\|S_{\mathbf{C}}\|_{\mathrm{cb}}\} \leq 2\sqrt{2}$. Then

$$\|S_{\mathbf{R}}T_{\mathbf{R}}v\|_{\mathrm{cb}} \leq \|S_{\mathbf{R}}\|_{\mathrm{cb}}\|T_{\mathbf{R}}v\|_{\mathrm{cb}} = \|S_{\mathbf{R}}\|_{\mathrm{cb}}\|T_{\mathbf{R}}v\| \leq 2\sqrt{2}.$$

Moreover,

$$\|S_{\mathbf{C}}\|_{\mathrm{cb}} \geq \|A_{\mathbf{R}}S_{\mathbf{C}}\|_{CB(\mathbf{C}_M,\mathbf{R}_N)} = \|A_{\mathbf{R}}S_{\mathbf{C}}\|_2 \geq \|S_{\mathbf{C}}\|_2/\sqrt{2},$$

hence

$$\|S_{\mathbf{C}}T_{\mathbf{C}}v\|_{\mathrm{cb}} \leq \|S_{\mathbf{C}}T_{\mathbf{C}}v\|_2 \leq \|S_{\mathbf{C}}\|_2\|T_{\mathbf{C}}v\| \leq 4.$$

Thus,

$$\|u_nv\|_{\mathrm{cb}} \leq \|S_{\mathbf{R}}T_{\mathbf{R}}v\|_{\mathrm{cb}} + \|S_{\mathbf{C}}T_{\mathbf{C}}v\|_{\mathrm{cb}} \leq 8.$$

This establishes (4.2(2)). ■

Proof of Theorem 1.3(2). Suppose an N-dimensional space E is $\lambda+$-BDR. Pick $C > \lambda$. By Theorem 3.1, there exists a subspace F of $\mathbf{C}_N \oplus_2 \mathbf{R}_N$ such that $d_{\mathrm{cb}}(E,F) < 8C^3$. By Corollary 2.6 and Proposition 2.5, F is 2^6C^8-BDR. By Theorem 4.1, F is $2^{23/2}C^8$-ci to $G_C = \mathbf{R}_{m(C)} \oplus_2 \mathbf{C}_{n(C)}$. Thus, $d_{\mathrm{cb}}(E,G_C) \leq 2^{29/2}C^{11}$. Find a sequence (C_j), decreasing to λ, such that $m = m(C_j)$ for any j (then $n = N - m(C_j) = n(C_j)$). Clearly, we have $d_{\mathrm{cb}}(E, \mathbf{R}_m \oplus_\infty \mathbf{C}_n) \leq 2^{15}\lambda^{10}$. ■

5. Proof of Theorem 1.4, and similar lower estimates. This section is devoted to the proof of Theorem 1.4. Clearly, $\mathbb{C}$ is 1-BDR. A series of lemmas helps us rule out other spaces. The first lemma seems to be partly folklore.

LEMMA 5.1.

(1) *Suppose $(E_k)_{k=1}^\infty$ is a sequence of Banach spaces, and E is a finite-dimensional subspace of $(\sum_k E_k)_{c_0}$. Then there exists $N \in \mathbb{N}$ such that E is isometric to a finite-dimensional subspace of $(\sum_{k=1}^N E_k)_\infty$.*
(2) *Suppose $(E_k)_{k=1}^\infty$ is a sequence of operator spaces, and E is a finite-dimensional subspace of $(\sum_k E_k)_{c_0}$. Then there exists $N \in \mathbb{N}$ such that E is completely isometric to a finite-dimensional subspace of $(\sum_{k=1}^N E_k)_\infty$.*

Proof. We prove part (2), since (1) can be dealt with in a similar manner. Let $n = \dim E$. Suppose $(e_i)_{i=1}^n$ is an Auerbach basis in E—that is, $\max_i |\alpha_i| \le \| \sum_i \alpha_i e_i\| \le \sum_i |\alpha_i|$ for each sequence (α_i) of scalars. In particular, the projection $R_j : E \to E : \sum_i \alpha_i e_i \mapsto \alpha_j e_j$ is contractive for every j. Since it is a rank one projection, it must also be completely contractive. Thus, for any $(a_i)_{i=1}^n \subset \mathbf{K}$, $\| \sum_i a_i \otimes e_i\| \ge \max_i \|a_i\|$.

Now write $e_i = (e_{ik})_{k=1}^\infty$ with $e_{ik} \in E_k$. There exists $N \in \mathbb{N}$ such that $\|e_{ik}\| < 1/n$ for any $1 \le i \le n$, and any $k > N$. For such k, and for any $(a_i)_{i=1}^n \subset \mathbf{K}$, $\| \sum_i a_i \otimes e_{ik}\| < \max_i \|a_i\|$. Therefore,

$$\Big\| \sum_i a_i \otimes e_i \Big\| = \sup_k \Big\| \sum_i a_i \otimes e_{ik} \Big\| = \max_{1 \le k \le N} \Big\| \sum_i a_i \otimes e_{ik} \Big\|,$$

which implies that E is completely isometric to a subspace of $(\sum E_k)_{k=1}^N$, spanned by the vectors $\widetilde{e_i} = (e_{ik})_{k=1}^N$ $(1 \le i \le n)$. ∎

We next apply this lemma to our situation.

LEMMA 5.2. *Suppose a finite-dimensional operator space E is 1-BDR. Then, for some $N \in \mathbb{N}$, E embeds into ℓ_∞^N isometrically, and into $\mathbf{M}_N$ completely isometrically.*

Proof. For $n \in \mathbb{N}$, find $e_{1n}^*, \dots, e_{M_n n}^* \in E^*$ of norm 1 so that, for any $e \in E$,

$$\max_{1 \le k \le M_n} |\langle e_{kn}^*, e \rangle| \ge (1 - 2^{-n}) \|e\|.$$

Define the operator space E_n by setting, for $e \in E \otimes \mathbf{K}_0$,

$$\|e\|_{E_n \otimes \mathbf{K}} = \max\{(1 - 2^{-n}) \|e\|_{E \otimes \mathbf{K}}, \max_{1 \le k \le M_n} \|(e_{kn}^* \otimes I_{\mathbf{K}}) e\|_{\mathbf{K}}\}.$$

Note that, for $e \in E$,

$$\|e\|_{E_n} = \max_{1 \le k \le M_n} |\langle e_{kn}^*, e \rangle|. \tag{5.1}$$

Let $X = (\sum_{n=1}^\infty E_n)_{c_0}$. Then the map

$$u : E \to X^{**} = \Big(\sum_{n=1}^\infty E_n \Big)_{\ell_\infty} : e \mapsto (e, e, \dots)$$

is a complete isometry. On the other hand, if E embeds into X isometrically, then, by Lemma 5.1, it must embed into $(\sum_{n=1}^{K} E_n)_\infty$ for some $K \in \mathbb{N}$. By (5.1), E_n embeds isometrically into $\ell_\infty^{M_n}$. Therefore, E embeds isometrically into ℓ_∞^N for $N = \sum_{n=1}^{K} M_n$.

Next test 1-bidual representability with $X = (\sum_{k=1}^{\infty} \mathbf{M}_k)_{c_0}$. The space $X^{**} = (\sum_{k=1}^{\infty} \mathbf{M}_k)_\infty$ contains $\mathbf{B}$, hence it also contains E. Therefore, X contains E. By Lemma 5.1, E embeds into $(\sum_{k=1}^{M} \mathbf{M}_k)_\infty$ for some M. *A fortiori*, E embeds completely isometrically into $\mathbf{M}_N$ with $N = M(M+1)/2$. ∎

Corollary 5.3. *If a* 1*-BDR space E has dimension at least* 2, *then E is not strictly convex.*

Recall that a Banach space E is called *strictly convex* if, for any e_1, e_2 in E, the equality $\|e_1 + e_2\|^2 = 2(\|e_1\|^2 + \|e_2\|^2)$ implies $e_1 = e_2$.

Proof. By Lemma 5.2, there exist linear functionals $f_1, \ldots, f_N \in E^*$ such that $\|e\| = \max_{1 \le i \le N} |\langle f_i, e\rangle|$ for any $e \in E$. Therefore, there exist i and two distinct elements e_1, e_2 of the unit ball of E satisfying

$$\|e_1\| = \|e_2\| = |\langle f_i, e_1\rangle| = |\langle f_i, e_2\rangle| = 1.$$

Then $4 = \|e_1 + e_2\|^2 = 2(\|e_1\|^2 + \|e_2\|^2)$. ∎

Proof of Theorem 1.4. Suppose, for the sake of contradiction, that $\dim E > 1$, and E is 1-BDR. By Lemma 5.2, E embeds into $\mathbf{M}_N$ completely isometrically. We then construct an operator space X, isomorphic to $c_0(\mathbf{M}_N)$, which is strictly convex (hence, by Corollary 5.3, X cannot contain E isometrically), but such that X^{**} contains $\mathbf{M}_N$ completely isometrically.

To this end, find a dense sequence $(f_j)_{j=1}^{\infty}$ in the unit ball of $\mathbf{M}_N^*$. For $x = (x_1, x_2, \ldots) \in c_0(\mathbf{M}_N)$, set

$$\|x\|_X = \sup_n \|x\|_n, \quad \text{where} \tag{5.2}$$

$$\|x\|_n = \Big(\|x_n\|^2 + \sum_{j,k=1}^{\infty} 10^{-(n+j+k)} |\langle f_j, x_k\rangle|^2\Big)^{1/2}.$$

Observe that

$$\sum_{j,k=1}^{\infty} 10^{-(n+j+k)} |\langle f_j, x_k\rangle|^2 \le 10^{-(n+1)} \sum_{j,k=1}^{\infty} 10^{-(j+k-1)} = 10^{-(n+1)} \sum_{\ell=0}^{\infty} l 10^{-l}$$
$$= 10^{-(n+1)} \Big(\frac{1}{1-1/10}\Big)^2 < 1.25 \cdot 10^{-(n+1)},$$

so

$$\|x\|_n < \|x_n\| + 10^{-n} \|x\|_{c_0(\mathbf{M}_N)}, \tag{5.3}$$

hence $\|\cdot\|_X$ is well defined. To show that $\|\cdot\|_n$ is a norm, note that $\|x\|_n = \|J_n x\|_{\mathbf{M}_N \oplus_2 \ell_2(\mathbb{N}\times\mathbb{N})}$, where

$$J_n : c_0(\mathbf{M}_N) \to \mathbf{M}_N \oplus_2 \ell_2(\mathbb{N}\times\mathbb{N}) : x \mapsto (x_n, (10^{-(n+j+k)}\langle f_j, x_k\rangle)_{j,k}).$$

By (5.3), $\lim_n \|J_n x\| = 0$ for any x, hence (5.2) describes X as a subspace of $(\sum(\mathbf{M}_N \oplus_2 \ell_2))_{c_0}$, and $\|\cdot\|_X$ is a norm. Thus, X^{**} is isomorphic to $\ell_\infty(\mathbf{M}_N)$, and the norm is also given by (5.2).

Finally, note that the supremum in (5.2) is attained. Indeed, if $\|x\|_{c_0(\mathbf{M}_N)} = 1$, find $K \in \mathbb{N}$ such that $\|x_n\|^2 + 10^{-n} < 1/4$ for $n \geq K$. For such n, $\|x\|_n < 1/2$, while $\|x\|_X > 1$. Therefore, $\|x\|_X = \max_{n<K} \|x\|_n$.

To show that X is strictly convex, suppose $\|x^{(1)} + x^{(2)}\|^2 = 2(\|x^{(1)}\|^2 + \|x^{(2)}\|^2)$ for some $x^{(1)}, x^{(2)} \in X$. By the observation above, there exists $n \in \mathbb{N}$ such that $\|x^{(1)} + x^{(2)}\| = \|x^{(1)} + x^{(2)}\|_n$. Therefore (writing $x^{(s)} = (x_n^{(s)})_{n=1}^\infty$),

$$\begin{aligned}0 &\geq 2(\|x^{(1)}\|_n^2 + \|x^{(2)}\|_n^2) - \|x^{(1)} + x^{(2)}\|_n^2 \\ &= 2(\|x_n^{(1)}\|^2 + \|x_n^{(2)}\|^2) - \|x_n^{(1)} + x_n^{(2)}\|^2 \\ &\quad + \sum_{j,k=1}^\infty 10^{-(n+j+k)}(2(|\langle f_j, x_k^{(1)}\rangle|^2 + |\langle f_j, x_k^{(2)}\rangle|^2) - |\langle f_j, x_k^{(1)} + x_k^{(2)}\rangle|^2).\end{aligned}$$

But, by the triangle inequality,

$$2(\|x_n^{(1)}\|^2 + \|x_n^{(2)}\|^2) \geq \|x_n^{(1)} + x_n^{(2)}\|^2,$$

and

$$2(|\langle f_j, x_k^{(1)}\rangle|^2 + |\langle f_j, x_k^{(2)}\rangle|^2) \geq |\langle f_j, x_k^{(1)} + x_k^{(2)}\rangle|^2.$$

In the last display, equality holds if and only if $x_k^{(1)} = x_k^{(2)}$ for every k. Thus, $x^{(1)} = x^{(2)}$.

Define the operator space structure on X by setting, for $x \in X \otimes \mathbf{K}_0$,

$$\|x\| = \max\{\|x\|_{\mathrm{MIN}(X)\otimes\mathbf{K}_0}, \|x\|_{c_0(\mathbf{M}_N)\otimes\mathbf{K}_0}\}.$$

In other words, the operator space structure on X is generated by its "natural" embedding into $\mathrm{MIN}(X) \oplus c_0(\mathbf{M}_N)$. Then X^{**} embeds into $\mathrm{MIN}(X^{**}) \oplus \ell_\infty(\mathbf{M}_N)$, and, for $x \in X^{**} \otimes \mathbf{K}_0$,

$$\|x\| = \max\{\|x\|_{\mathrm{MIN}(X^{**})\otimes\mathbf{K}_0}, \|x\|_{\ell_\infty(\mathbf{M}_N)\otimes\mathbf{K}_0}\}. \tag{5.4}$$

Define $U : \mathbf{M}_N \to X^{**} : a \mapsto ((1-10^{-n})a)_{n=1}^\infty$. By (5.2) and the discussion following it, and by (5.4), U is a complete contraction. Furthermore, for any $a \in \mathbf{M}_N \otimes \mathbf{K}_0$,

$$\|(U \otimes I_{\mathbf{K}})a\|_{X^{**}\otimes\mathbf{K}} \geq \sup_n (1-10^{-n})\|a\|_{\mathbf{M}_N\otimes\mathbf{K}} = \|a\|,$$

hence U is a complete isometry. ∎

Remark 5.4. In a similar way one can prove the commutative version of Theorem 1.4: if a finite-dimensional Banach space E is such that a Banach space X contains E isometrically whenever X^{**} contains E isometrically, then E is 1-dimensional. Indeed, one can imitate the proof of Lemma 5.2 to show that such an E embeds into ℓ_∞^N for some N. Then, as above, one constructs a strictly convex Banach space Z whose dual contains ℓ_∞^N.

Next we investigate the smallest C for which the given space E is C-BDR.

THEOREM 5.5. *Suppose E is a finite-dimensional C-injective operator space which is $\lambda+$-BDR. Then $\lambda \geq \mathrm{ex}(E^*)/(C\,\mathrm{ex}(E))$.*

Proof. Suppose, for the sake of contradiction, that $\lambda < \mathrm{ex}(E^*)/(C\,\mathrm{ex}(E))$. Pick $C_1 > \mathrm{ex}(E)$, $C_2 < \mathrm{ex}(E^*)$, and $C_3 > \lambda$ such that $C_2/(C_1C_3) > C$. Following the proof of Theorem 3.1, find $E_1 \hookrightarrow \mathbf{M}_N$ such that $d_{\mathrm{cb}}(E, E_1) < C_1$. Also, define the spaces X_n ($n > N$) and X as in the proof of that theorem. We know that X^{**} contains $\mathbf{B}$ as a complete M-summand. Suppose X contains E C_3-ci. As in the proof of Theorem 3.1, we conclude that, for some $n > N$, X_n contains E C_1C_3-ci. That is, there exists a subspace $F \hookrightarrow X_n$ and a complete contraction $u : F \to E$ with $\|u^{-1}\|_{\mathrm{cb}} \leq C_1C_3$. Since E is C-injective, there exists a map $\widetilde{u} : X_n \to E$ such that $\widetilde{u}|_F = u$ and $\|\widetilde{u}\|_{\mathrm{cb}} \leq C$.

As $\mathrm{ex}(E^*) > C_2$, there exists an operator $v : E \to \mathbf{B}$ such that $\|v\|_{\mathrm{cb}} > C_2$ and $\|v \otimes I_{\mathbf{M}_n}\| < 1$ (to see this, apply Theorem 18 of [15] to E^*, and dualize). Note that

$$\|v\widetilde{u}\|_{\mathrm{cb}} \geq \|vu\|_{\mathrm{cb}} \geq \frac{\|v\|_{\mathrm{cb}}}{\|u^{-1}\|_{\mathrm{cb}}} \geq \frac{C_2}{C_1C_3}.$$

On the other hand, by definition of X_n,

$$\|v\widetilde{u}\|_{\mathrm{cb}} = \|v\widetilde{u} \otimes I_{\mathbf{M}_n}\| \leq \|v \otimes I_{\mathbf{M}_n}\|\,\|\widetilde{u}\|_{\mathrm{cb}} < C.$$

This contradicts our assumption that $C_2/(C_1C_3) > C$. ∎

COROLLARY 5.6. *If ℓ_∞^n is C-BDR, then $C \geq n/(2\sqrt{n-1})$.*

Proof. We know that $E = \ell_\infty^n$ is 1-injective and 1-exact. Moreover (see Theorem 21.5 of [16]), $\mathrm{ex}(E^*) \geq n/(2\sqrt{n-1})$. We complete the proof by applying Theorem 5.5. ∎

6. C^*-algebras and bidual representability. Passing from operator spaces to C^*-algebras, we obtain:

PROPOSITION 6.1. *Suppose X is a C^*-algebra, and $n \in \mathbb{N}$. Then the following are equivalent*:

(1) *X^{**} contains $\mathbf{M}_n$ completely isometrically.*
(2) *X^{**} contains $\mathbf{M}_n$ as a sub-C^*-algebra.*
(3) *X contains $\mathbf{M}_n$ completely isometrically.*
(4) *X^{**} contains $\mathbf{M}_n$ c-completely isomorphically for some $c < n/(n-1)$.*

(5) *There exists an irreducible representation of X into $B(H)$ with $\dim H \geq n$.*

REMARK 6.2. J. Roydor [18] proved a related result: if a C^*-algebra X embeds into $C(\Omega, \mathbf{M}_n)$ (for some Ω) completely isometrically, then X is a C^*-subalgebra of $C(\Omega', \mathbf{M}_n)$ for some set Ω'.

Below, we denote by $(E_{ij})_{i,j=1}^n$ the canonical matrix units in $\mathbf{M}_n$.

LEMMA 6.3. *Suppose $n > m$, and $T : \mathbf{M}_n \to \mathbf{M}_m$ is a complete contraction. Then*

$$\Big\| T \otimes I_{\mathbf{M}_n}\Big(\sum_{i=1}^n E_{ij} \otimes E_{ij}\Big)\Big\|_{\mathbf{M}_m \otimes \mathbf{M}_n} \leq m.$$

COROLLARY 6.4. *If $\mathcal{I}$ is a set, $m < n$, and E is an n^2-dimensional subspace of $\ell_\infty(\mathcal{I}, \mathbf{M}_m)$, then $d_{\mathrm{cb}}(\mathbf{M}_n, E) \geq n/m$.*

Proof. Consider a complete contraction $T : \mathbf{M}_n \to E$. By Lemma 6.3, $\|T \otimes I_{\mathbf{M}_n}(\sum_{i=1}^n E_{ij} \otimes E_{ij})\| \leq m$. However, $\|\sum_{i,j=1}^n E_{ij} \otimes E_{ij}\|_{\mathbf{M}_n \otimes \mathbf{M}_n} = n$. ∎

Proof of Lemma 6.3. Consider a complete contraction $T : \mathbf{M}_n \to \mathbf{M}_m$. By Stinespring's representation theorem, there exists a Hilbert space $\widetilde{H}$, a unital representation $\pi : \mathbf{M}_n \to B(\widetilde{H})$, and contractions $U \in B(\widetilde{H}, \ell_2^m)$, $V \in B(\ell_2^m, \widetilde{H})$ such that $Ta = U\pi(a)V$ for any $a \in \mathbf{M}_n$. Note that, for each i, $p_i = \pi(E_{ii})$ is a projection. Denote its range by H_i. Then $\pi(E_{ij})$ is a partial isometry, with initial and terminal projections H_i and H_j, respectively. Let $H = H_1$ and $u_i = \pi(E_{i1}) : H \to H_i$ (once again, $1 \leq i \leq n$). Then $\ell_2^n(H)$ can be identified with $\widetilde{H}$ via $(\xi_i)_{i=1}^n \mapsto \sum_{i=1}^n u_i \xi_i$.

For $1 \leq i \leq n$, let $U_i = U u_i \in B(H, \ell_2^m)$ and $V_i = u_i p_i V \in B(\ell_2^m, H)$. Then we can identify U with $\sum_{i=1}^n E_{1i} \otimes U_i$ (viewed as an operator from $\ell_2^n(H)$ to ℓ_2^m). Similarly, we identify V with $\sum_{i=1}^n E_{i1} \otimes V_i \in B(\ell_2^m, \ell_2^n(H))$. Then

$$\|V\| = \Big\|\sum_{i=1}^n E_{i1} \otimes V_i\Big\| = \Big\|\sum_{i=1}^n V_i^* V_i\Big\|^{1/2} \leq 1,$$

$$\|U\| = \Big\|\sum_{i=1}^n E_{1i} \otimes U_i\Big\| = \Big\|\sum_{i=1}^n U_i U_i^*\Big\|^{1/2} \leq 1.$$

Moreover, $TE_{ij} = U_i V_j$ for any i and j. Therefore,

$$\begin{aligned}\Big\| T \otimes I_{\mathbf{M}_n}\Big(\sum_{i,j=1}^n E_{ij} \otimes E_{ij}\Big)\Big\|_{\mathbf{M}_m \otimes \mathbf{M}_n} &= \Big\|\sum_{i,j=1}^n U_i V_j \otimes E_{ij}\Big\| \\ &= \Big\|\Big(\sum_i U_i \otimes E_{i1}\Big)\Big(\sum_j V_j \otimes E_{1j}\Big)\Big\|.\end{aligned}$$

However,

$$\Big\|\sum_i U_i \otimes E_{i1}\Big\| = \Big\|\sum_{i=1}^n U_i^* U_i\Big\|^{1/2} \leq \Big(\mathrm{Tr}\Big(\sum_{i=1}^n U_i^* U_i\Big)\Big)^{1/2}$$
$$= \Big(\mathrm{Tr}\Big(\sum_{i=1}^n U_i U_i^*\Big)\Big)^{1/2} \leq \sqrt{m}\Big\|\sum_{i=1}^n U_i U_i^*\Big\| \leq \sqrt{m},$$

and similarly, $\|\sum_j V_j \otimes E_{1j}\| \leq \sqrt{m}$. Thus,

$$\Big\|T \otimes I_{\mathbf{M}_n}\Big(\sum_{i,j=1}^n E_{ij} \otimes E_{ij}\Big)\Big\| \leq \Big\|\sum_i U_i \otimes E_{i1}\Big\| \cdot \Big\|\sum_j V_j \otimes E_{1j}\Big\| \leq m. \ \blacksquare$$

REMARK 6.5. In a similar fashion, one can show that

$$\Big\|\sum_{i=1}^n E_{i1} \otimes E_{i1}\Big\|_{\mathbf{C}_n \otimes \mathbf{C}_n} = \sqrt{n},$$

and

$$\Big\|T \otimes I_{\mathbf{C}_n}\Big(\sum_{i=1}^n E_{i1} \otimes E_{i1}\Big)\Big\|_{\mathbf{M}_m \otimes \mathbf{C}_n} \leq \sqrt{m}$$

whenever $T : \mathbf{C}_n \to \mathbf{M}_m$ is a complete contraction. From this, one concludes that $d_{\mathrm{cb}}(\mathbf{C}_n, E) \geq \sqrt{n/m}$ for every n-dimensional subspace E of $\ell_\infty(\mathcal{I}, \mathbf{M}_m)$ (provided $n > m$).

LEMMA 6.6. *Suppose $\pi : Y \to B(H)$ is an irreducible representation of a C^*-algebra Y on a Hilbert space H of dimension at least n. Then Y contains $\mathbf{M}_n$ completely isometrically as an operator system.*

Proof. If p is a projection of rank n on H, then, by the transitivity of irreducible representations (Theorem II.4.18 of [20]), $\pi(Y)p = B(H)p$. Denote by A the set of all $y \in Y$ satisfying $\pi(y)p = p\pi(y)$ $(= p\pi(y)p)$. Clearly, A is a C^*-subalgebra of Y, and $\pi(A)$ can be identified with $B(p(H)) \sim \mathbf{M}_n$. Moreover, A has a separable subalgebra (call it A_1) such that $\pi(A_1)$ can again be identified with $\mathbf{M}_n$. In other words, $\mathbf{M}_n = A_1/J$, where $J = \ker \pi \cap A_1$ is a closed two-sided ideal. By Theorem 3.10 of [3], $\mathbf{M}_n$ lifts to A_1 completely positively. Thus, A_1 contains $\mathbf{M}_n$ completely isometrically. ∎

REMARK 6.7. A similar lifting technique was used in [7]. Earlier, lifting methods were used in [19, 21] to prove that a C^*-algebra which is not $(n-1)$-subhomogeneous contains a completely positive copy of $\mathbf{M}_n$.

Proof of Proposition 6.1. The implications (2)⇒(1)⇒(4) and (3)⇒(1) are clear.

(1)⇒(2): By Section 6.4 of [10], we can decompose the von Neumann algebra X^{**} into a direct sum of components of type I_k ($k \in \mathbb{N} \cup \{\infty\}$), II, and III. By Lemma 6.4, at least one of the summands of type I_k ($k \geq n$), II, and III is non-trivial. All such summands contain $\mathbf{M}_n$ as a subalgebra (see e.g. comparison theorem for projections in a von Neumann algebra, Theorem 6.2.7 of [10]).

(1)⇒(5): Suppose, for the sake of contradiction, that (5) fails to hold. Viewing X^{**} as the enveloping algebra of X, we embed it into $\ell_\infty(\mathcal{I}, \mathbf{M}_{n-1})$ for some set $\mathcal{I}$. Therefore, by Lemma 6.4, X^{**} cannot contain $\mathbf{M}_n$ c-completely isomorphically with $c < n/(n-1)$.

(4)⇒(1) and (5)⇒(3): If there are no irreducible representations of X^{**} (or X) on Hilbert spaces with dimension $\geq n$, then X^{**} (respectively, X) embeds into $\ell_\infty(\mathcal{I}, \mathbf{M}_{n-1})$, hence, by Lemma 6.4, X^{**} cannot contain $\mathbf{M}_n$ c-completely isomorphically when $c < n/(n-1)$. Otherwise, X^{**} (or X) contains $\mathbf{M}_n$ completely isometrically, by Lemma 6.6. ∎

Remark 6.8. By Lemma 6.6, items (1) and (4) of Proposition 6.1 guarantee that X (or X^{**}) contains $\mathbf{M}_n$ as an operator system. However, X need not contain $\mathbf{M}_n$ as a subalgebra. Indeed, let X be the left regular algebra of a free group on two generators. By [4], X has no non-trivial projections, hence it cannot contain $\mathbf{M}_n$ as a subalgebra.

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