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On the distance between $\langle X\rangle$ and L^∞ in the space of continuous BMO-martingales

by

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Abstract. Let $X = (X_t, \mathcal{F}_t)$ be a continuous BMO-martingale, that is,

$$\|X\|_{\mathrm{BMO}} \equiv \sup_T \|E[|X_\infty - X_T| \,|\, \mathcal{F}_T]\|_\infty < \infty,$$

where the supremum is taken over all stopping times T. Define the critical exponent $b(X)$ by

$$b(X) = \{b > 0 : \sup_T \|E[\exp(b^2(\langle X\rangle_\infty - \langle X\rangle_T)) \,|\, \mathcal{F}_T]\|_\infty < \infty\},$$

where the supremum is taken over all stopping times T. Consider the continuous martingale $q(X)$ defined by

$$q(X)_t = E[\langle X\rangle_\infty \,|\, \mathcal{F}_t] - E[\langle X\rangle_\infty \,|\, \mathcal{F}_0].$$

We use $q(X)$ to characterize the distance between $\langle X\rangle$ and the class L^∞ of all bounded martingales in the space of continuous BMO-martingales, and we show that the inequalities

$$\frac{1}{4d_1(q(X), L^\infty)} \le b(X) \le \frac{4}{d_1(q(X), L^\infty)}$$

hold for every continuous BMO-martingale X.

1. Introduction and preliminaries. Throughout this paper, we fix a filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ with the usual conditions, and we assume that every martingale is uniformly integrable and continuous.

Recall that a uniformly integrable martingale $X = (X_t, \mathcal{F}_t)$ is said to be in BMO_p $(p \ge 1)$ if

$$\|X\|_{\mathrm{BMO}_p} \equiv \sup_T \|E[|X_\infty - X_T|^p \,|\, \mathcal{F}_T]^{1/p}\|_\infty < \infty, \tag{1.1}$$

where the supremum is taken over all stopping times T. In particular,

$$\|X\|_{\mathrm{BMO}_2} = \sup_T \|E[\langle X\rangle_\infty - \langle X\rangle_T \,|\, \mathcal{F}_T]^{1/2}\|_\infty.$$

Then, as is well known, $\|\cdot\|_{\mathrm{BMO_p}}$ is a norm for all $p \ge 1$ and

$$\|X\|_{\mathrm{BMO}_1} \le \|X\|_{\mathrm{BMO}_p} \le C_p\|X\|_{\mathrm{BMO}_1},$$

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where $C_p > 0$ is a constant depending only on p. For these, see, for example, [3, p. 28].

Now, let BMO be the class of all uniformly integrable martingales X such that $\|X\|_{\mathrm{BMO}_1} < \infty$. Then BMO is a Banach space with the norm $\|\cdot\|_{\mathrm{BMO}_1}$, and we call the martingale X in BMO a BMO-*martingale*. There exist two important subclasses of BMO, namely, the class L^∞ of all bounded martingales and the class H^∞ of all martingales X such that $\langle X\rangle$ is bounded.

For $X \in \mathrm{BMO}$, let $a(X)$ be the supremum of the set of $a > 0$ for which

$$\sup_T \|E[\exp(a|X_\infty - X_T|)\,|\,\mathcal{F}_T]\|_\infty < \infty,$$

where the supremum is taken over all stopping times T, and for $M, N \in \mathrm{BMO}$ we set

$$d_p(M,N) = \|M - N\|_{\mathrm{BMO}_p} \quad (p \geq 1).$$

Then there is a beautiful relationship between $a(X)$ and $d_1(\cdot,\cdot)$:

$$\frac{1}{4d_1(X, L^\infty)} \leq a(X) \leq \frac{4}{d_1(X, L^\infty)} \tag{1.2}$$

for every $X \in \mathrm{BMO}$. This is the Garnett–Jones theorem. For the proof, see [1], [3], [4].

Let now $b(X)$ denote the supremum of the set of $b > 0$ for which

$$\sup_T \|E[\exp(b^2(\langle X\rangle_\infty - \langle X\rangle_T))\,|\,\mathcal{F}_T]\|_\infty < \infty$$

for $X \in \mathrm{BMO}$, where T runs through all stopping times. Then we have (see [3])

$$\frac{1}{\sqrt{2}\, d_2(X, H^\infty)} \leq b(X) \quad (X \in \mathrm{BMO}). \tag{1.3}$$

Furthermore, we shall see in Section 2 that $\sqrt{2}\, a(X) \geq b(X)$ for every $X \in \mathrm{BMO}$.

In this paper, we consider the continuous martingale $q(X)$ defined by

$$q(X)_t = E[\langle X\rangle_\infty | \mathcal{F}_t] - E[\langle X\rangle_\infty \,|\, \mathcal{F}_0],$$

where X is a continuous martingale. We use $q(X)$ to characterize the distance between $\langle X\rangle$ and L^∞ in the space of continuous BMO-martingales.

2. Results and proofs. In this section, we give the characterization of the distance between $\langle X\rangle$ and L^∞ in the space of BMO-martingales.

LEMMA 1. *Let* $X, Y \in \mathrm{BMO}$. *Assume that* $q(X)$ *and* $q(Y)$ *are defined as in Section* 1. *Then*

$$\|q(X) - q(Y)\|_{\mathrm{BMO}_1} \leq 2(\|X\|_{\mathrm{BMO}_2} + \|Y\|_{\mathrm{BMO}_2})\|X - Y\|_{\mathrm{BMO}_2}.$$

Proof. Observing that

$$\langle X\rangle - \langle Y\rangle = \langle X - Y, X\rangle + \langle X - Y, Y\rangle,$$

we find

$$q(X)_\infty - q(Y)_\infty - E[q(X)_\infty - q(Y)_\infty \,|\, \mathcal{F}_T]$$
$$= \langle X\rangle_\infty - \langle Y\rangle_\infty - E[\langle X\rangle_\infty - \langle Y\rangle_\infty \,|\, \mathcal{F}_T]$$
$$= \big(\langle X-Y, X\rangle_\infty - \langle X-Y, X\rangle_T\big) - E[\langle X-Y, X\rangle_\infty - \langle X-Y, X\rangle_T \,|\, \mathcal{F}_T]$$
$$+ l(\langle X-Y, Y\rangle_\infty - \langle X-Y, Y\rangle_T) - E[\langle X-Y, Y\rangle_\infty - \langle X-Y, Y\rangle_T \,|\, \mathcal{F}_T].$$

It follows from the Schwarz inequality that

$$E\big[|q(X)_\infty - q(Y)_\infty - E[q(X)_\infty - q(Y)_\infty \,|\, \mathcal{F}_T]| \,\big|\, \mathcal{F}_T\big]$$
$$\le 2E[|\langle X-Y, X\rangle_\infty - \langle X-Y, X\rangle_T| \,|\, \mathcal{F}_T]$$
$$+ 2E[|\langle X-Y, Y\rangle_\infty - \langle X-Y, Y\rangle_T| \,|\, \mathcal{F}_T]$$
$$\le 2E[|\langle X-Y\rangle_\infty - \langle X-Y\rangle_T| \,|\, \mathcal{F}_T]^{1/2} E[\langle X\rangle_\infty - \langle X\rangle_T \,|\, \mathcal{F}_T]^{1/2}$$
$$+ 2E[|\langle X-Y\rangle_\infty - \langle X-Y\rangle_T| \,|\, \mathcal{F}_T]^{1/2} E[\langle Y\rangle_\infty - \langle Y\rangle_T \,|\, \mathcal{F}_T]^{1/2}$$
$$\le 2(\|X\|_{\mathrm{BMO}_2} + \|Y\|_{\mathrm{BMO}_2})\|X-Y\|_{\mathrm{BMO}_2}.$$

This completes the proof. ∎

As a consequence of the lemma, we see that $X \in \mathrm{BMO}$ implies $q(X) \in \mathrm{BMO}$. Furthermore, we have

THEOREM 1. *Let X be a uniformly integrable continuous martingale and let $q(X)$ be defined as in Section* 1. *If $X \in \mathrm{BMO}$, then*

$$\frac{1}{4d_1(q(X), L^\infty)} \le b(X) \le \frac{4}{d_1(q(X), L^\infty)}, \tag{2.1}$$

and furthermore, we have $\sqrt{2}\,a(X) \ge b(X)$ for all $X \in \mathrm{BMO}$.

Proof. Let $X \in \mathrm{BMO}$. Then for any $\lambda > 0$ we have

$$E[\exp(\lambda(\langle X\rangle_\infty - \langle X\rangle_T)) \,|\, \mathcal{F}_T]$$
$$= E[\exp(\lambda E[\langle X\rangle_\infty - \langle X\rangle_T \,|\, \mathcal{F}_T]) \exp(\lambda(\langle X\rangle_\infty - E[\langle X\rangle_\infty \,|\, \mathcal{F}_T])) \,|\, \mathcal{F}_T]$$
$$\le e^{\lambda\|X\|^2_{\mathrm{BMO}_2}} E[\exp(\lambda|\langle X\rangle_\infty - E[\langle X\rangle_\infty \,|\, \mathcal{F}_T]|) \,|\, \mathcal{F}_T]$$
$$\le e^{\lambda\|X\|^2_{\mathrm{BMO}_2}} E[\exp(\lambda|q(X)_\infty - q(X)_T|) \,|\, \mathcal{F}_T]$$

and

$$E[\exp(\lambda|q(X)_\infty - q(X)_T|) \,|\, \mathcal{F}_T]$$
$$= E[\exp(\lambda|\langle X\rangle_\infty - E[\langle X\rangle_\infty \,|\, \mathcal{F}_T]|) \,|\, \mathcal{F}_T]$$
$$\le E[\exp(\lambda(\langle X\rangle_\infty - \langle X\rangle_T)) \exp(\lambda E[\langle X\rangle_\infty - \langle X\rangle_T \,|\, \mathcal{F}_T]) \,|\, \mathcal{F}_T]$$
$$\le e^{\lambda\|X\|^2_{\mathrm{BMO}_2}} E[\exp(\lambda(\langle X\rangle_\infty - \langle X\rangle_T)) \,|\, \mathcal{F}_T],$$

which shows that $b(X) = a(q(X))$. Thus, inequalities (2.1) follow from inequalities (1.2).

On the other hand, it is not difficult to show that the inequality

$$E[\exp(\lambda|X_\infty - X_T|)\,|\,\mathcal{F}_T] \le 2E[\exp(2\lambda^2(\langle X\rangle_\infty - \langle X\rangle_T))\,|\,\mathcal{F}_T]^{1/2} \tag{2.2}$$

holds for all $\lambda > 0$. Indeed, by using the Schwarz inequality and noting that for $X \in \mathrm{BMO}$ the continuous exponential martingale $\mathcal{E}(X)$ defined by

$$\mathcal{E}(X) = \exp\bigl(X - \tfrac{1}{2}\langle X\rangle\bigr)$$

is uniformly integrable (see Theorem 2.3 in [3, p. 31]), for every real $\lambda > 0$ we have

$$\begin{aligned} E[\exp(\lambda(X_\infty - X_T))\,|\,\mathcal{F}_T] &\\ &= E\biggl[\frac{\mathcal{E}(\lambda X)_\infty}{\mathcal{E}(\lambda X)_T}\exp(\lambda^2(\langle X\rangle_\infty - \langle X\rangle_T))\biggm|\mathcal{F}_T\biggr] \\ &\le E\biggl[\frac{\mathcal{E}(2\lambda X)_\infty}{\mathcal{E}(2\lambda X)_T}\biggm|\mathcal{F}_T\biggr]^{1/2} E\bigl[\exp\bigl(2\lambda^2(\langle X\rangle_\infty - \langle X\rangle_T)\bigr)\,|\,\mathcal{F}_T\bigr]^{1/2} \\ &\le E[\exp(2\lambda^2(\langle X\rangle_\infty - \langle X\rangle_T))\,|\,\mathcal{F}_T]^{1/2}. \end{aligned}$$

The same argument works if X is replaced by $-X$. Thus, we obtain (2.2). This shows that $\sqrt{2}\,a(X) \ge b(X)$. ∎

Corollary 1. *If* $X \in \mathrm{BMO}$, *then* $q(X) \in \overline{L^\infty}$ *is equivalent to* $b(X) = \infty$, *where* $\overline{L^\infty}$ *stands for the* BMO-*closure of* L^∞.

Recall that the continuous exponential martingale $\mathcal{E}(X)$ is said to satisfy the (A_p)-condition $(1 < p < \infty)$, in symbols $\mathcal{E}(X) \in (A_p)$, if

$$\sup_T \biggl\| E\biggl[\biggl\{\frac{\mathcal{E}(X)_T}{\mathcal{E}(X)_\infty}\biggr\}^{1/(p-1)}\biggm|\mathcal{F}_T\biggr]\biggr\|_\infty < \infty,$$

where the supremum is taken over all stopping times T. It is known (see Theorem 3.12 of [3, p. 72]) that $q(X) \in \overline{L^\infty}$ is equivalent to $\mathcal{E}(X)$ and $\mathcal{E}(-X)$ satisfying all (A_p) $(1 < p < \infty)$. Thus, the following corollary is clear.

Corollary 2. *If* $X \in \mathrm{BMO}$ *and* $1 < p < \infty$, *then* $b(X) = \infty$ *is equivalent to* $\mathcal{E}(X)$ *and* $\mathcal{E}(-X)$ *satisfying all* (A_p).

Finally, we consider a subspace $\mathcal{H}$ of BMO,

$$\mathcal{H} \equiv \{X \in \mathrm{BMO} : q(X) \in \overline{L^\infty}\}.$$

Corollary 3. *Let* $\overline{H^\infty}$ *denote the* BMO-*closure of* H^∞. *Then*

$$\overline{H^\infty} \subset \mathcal{H} \subset \overline{L^\infty}.$$

Proof. $\overline{H^\infty} \subset \mathcal{H}$ follows from (1.3) and Corollary 1, and $\mathcal{H} \subset \overline{L^\infty}$ follows from Theorem 1. ∎

LEMMA 2. *The mapping $q : X \mapsto q(X)$ is continuous on* BMO.

Proof. Let $\{X^n\}$ be a sequence of BMO-martingales such that $X^n \to X$ in BMO. Then

$$K \equiv \sup_n \|X^n\|_{\mathrm{BMO}_2} < \infty \quad \text{and} \quad \|X\|_{\mathrm{BMO}_2} \le K.$$

It follows from Lemma 1 that

$$\|q(X^n) - q(X)\|_{\mathrm{BMO}_1} \le 4K\|X^n - X\|_{\mathrm{BMO}_2} \to 0$$

as $n \to \infty$. This shows that the mapping q is continuous on BMO. ∎

THEOREM 2. *$\mathcal{H}$ is a closed linear subspace of* BMO.

Proof. The closedness of $\mathcal{H}$ follows from Lemma 2.

On the other hand, for any two real α, β and any two BMO-martingales X, Y, we have

$$\langle \alpha X + \beta Y\rangle_\infty - \langle \alpha X + \beta Y\rangle_T \le 2(\alpha^2(\langle X\rangle_\infty - \langle X\rangle_T) + \beta^2(\langle Y\rangle_\infty - \langle Y\rangle_T))$$

and so, for any $\lambda > 0$,

$$\begin{aligned} E[\exp(\lambda(\langle \alpha X + \beta Y\rangle_\infty - \langle \alpha X + \beta Y\rangle_T)) \,|\, \mathcal{F}_T] \\ \le E[\exp(4\alpha^2\lambda(\langle X\rangle_\infty - \langle X\rangle_T)) \,|\, \mathcal{F}_T]^{1/2} \\ \times E[\exp(4\beta^2\lambda(\langle Y\rangle_\infty - \langle Y\rangle_T)) \,|\, \mathcal{F}_T]^{1/2}, \end{aligned}$$

which shows that $b(\alpha X + \beta Y) = \infty$ for $b(X) = \infty$, $b(Y) = \infty$. Thus, $X, Y \in \mathcal{H}$ implies that $\alpha X + \beta Y \in \mathcal{H}$. This completes the proof. ∎

Now, it is natural to ask if the relationship $\overline{H^\infty} = \mathcal{H}$ holds. But we have not been able to settle this question so far.

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