On the distance between $\langle X \rangle$ and $L^\infty$ in the space of continuous BMO-martingales

by

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Abstract. Let $X = (X_t, \mathcal{F}_t)$ be a continuous BMO-martingale, that is,
\[ \|X\|_{\text{BMO}} = \sup_T \|E[|X_\infty - X_T| \mid \mathcal{F}_T]\|_\infty < \infty, \]
where the supremum is taken over all stopping times $T$. Define the critical exponent $b(X)$ by
\[ b(X) = \{ b > 0 : \sup_T \|E[\exp(b^2(\langle X \rangle_\infty - \langle X \rangle_T)) \mid \mathcal{F}_T]\|_\infty < \infty \}, \]
where the supremum is taken over all stopping times $T$. Consider the continuous martingale $q(X)$ defined by
\[ q(X)_t = E[\langle X \rangle_\infty \mid \mathcal{F}_t] - E[\langle X \rangle_\infty \mid \mathcal{F}_0]. \]
We use $q(X)$ to characterize the distance between $\langle X \rangle$ and the class $L^\infty$ of all bounded martingales in the space of continuous BMO-martingales, and we show that the inequalities
\[ \frac{1}{4d_1(q(X), L^\infty)} \leq b(X) \leq \frac{4}{d_1(q(X), L^\infty)} \]
hold for every continuous BMO-martingale $X$.

1. Introduction and preliminaries. Throughout this paper, we fix a filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ with the usual conditions, and we assume that every martingale is uniformly integrable and continuous.

Recall that a uniformly integrable martingale $X = (X_t, \mathcal{F}_t)$ is said to be in $\text{BMO}_p$ ($p \geq 1$) if
\[ \|X\|_{\text{BMO}_p} \equiv \sup_T \|E[|X_\infty - X_T|^p \mid \mathcal{F}_T]^{1/p}\|_\infty < \infty, \]
where the supremum is taken over all stopping times $T$. In particular,
\[ \|X\|_{\text{BMO}_2} = \sup_T \|E[\langle X \rangle_\infty - \langle X \rangle_T \mid \mathcal{F}_T]^{1/2}\|_\infty. \]
Then, as is well known, $\| \cdot \|_{\text{BMO}_p}$ is a norm for all $p \geq 1$ and
\[ \|X\|_{\text{BMO}_1} \leq \|X\|_{\text{BMO}_p} \leq C_p \|X\|_{\text{BMO}_1}, \]

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where $C_p > 0$ is a constant depending only on $p$. For these, see, for example, [3, p. 28].

Now, let BMO be the class of all uniformly integrable martingales $X$ such that $\|X\|_{\text{BMO}_1} < \infty$. Then BMO is a Banach space with the norm $\| \cdot \|_{\text{BMO}_1}$, and we call the martingale $X$ in BMO a BMO-martingale. There exist two important subclasses of BMO, namely, the class $L^\infty$ of all bounded martingales and the class $H^\infty$ of all martingales $X$ such that $\langle X \rangle$ is bounded.

For $X \in \text{BMO}$, let $a(X)$ be the supremum of the set of $a > 0$ for which
$$\sup_T \|E[\exp(a|X_\infty - X_T|) | \mathcal{F}_T]\|_\infty < \infty,$$
where the supremum is taken over all stopping times $T$, and for $M, N \in \text{BMO}$ we set
$$d_p(M, N) = \|M - N\|_{\text{BMO}_p} \quad (p \geq 1).$$

Then there is a beautiful relationship between $a(X)$ and $d_1(\cdot, \cdot)$:
$$\frac{1}{4d_1(X, L^\infty)} \leq a(X) \leq \frac{4}{d_1(X, L^\infty)}$$
for every $X \in \text{BMO}$. This is the Garnett–Jones theorem. For the proof, see [1], [3], [4].

Let now $b(X)$ denote the supremum of the set of $b > 0$ for which
$$\sup_T \|E[\exp(b^2(\langle X \rangle_\infty - \langle X \rangle_T)) | \mathcal{F}_T]\|_\infty < \infty$$
for $X \in \text{BMO}$, where $T$ runs through all stopping times. Then we have (see [3])
$$\frac{1}{\sqrt{2} d_2(X, H^\infty)} \leq b(X) \quad (X \in \text{BMO}).$$

Furthermore, we shall see in Section 2 that $\sqrt{2} a(X) \geq b(X)$ for every $X \in \text{BMO}$.

In this paper, we consider the continuous martingale $q(X)$ defined by
$$q(X)_t = E[(X)_\infty | \mathcal{F}_t] - E[(X)_\infty | \mathcal{F}_0],$$
where $X$ is a continuous martingale. We use $q(X)$ to characterize the distance between $\langle X \rangle$ and $L^\infty$ in the space of continuous BMO-martingales.

2. Results and proofs. In this section, we give the characterization of the distance between $\langle X \rangle$ and $L^\infty$ in the space of BMO-martingales.

LEMMA 1. Let $X, Y \in \text{BMO}$. Assume that $q(X)$ and $q(Y)$ are defined as in Section 1. Then
$$\|q(X) - q(Y)\|_{\text{BMO}_1} \leq 2(\|X\|_{\text{BMO}_2} + \|Y\|_{\text{BMO}_2}) \|X - Y\|_{\text{BMO}_2}.$$

Proof. Observing that
$$\langle X \rangle - \langle Y \rangle = \langle X - Y, X \rangle + \langle X - Y, Y \rangle,$$
we find
\[ q(X)_\infty - q(Y)_\infty - E[q(X)_\infty - q(Y)_\infty | \mathcal{F}_T] = \langle X \rangle_\infty - \langle Y \rangle_\infty - E[\langle X \rangle_\infty - \langle Y \rangle_\infty | \mathcal{F}_T] = \langle (X - Y, X)_\infty \rangle - \langle (X - Y, X)_T \rangle - E[\langle X - Y, X \rangle_\infty - \langle X - Y, X \rangle_T | \mathcal{F}_T] + l(\langle X - Y, Y \rangle_\infty - \langle X - Y, Y \rangle_T) - E[\langle X - Y, Y \rangle_\infty - \langle X - Y, Y \rangle_T | \mathcal{F}_T]. \]

It follows from the Schwarz inequality that
\[ \ldots \]
\[ = (\langle X - Y, X \rangle_\infty - \langle X - Y, X \rangle_T) - E[\langle X - Y, X \rangle_\infty - \langle X - Y, X \rangle_T | \mathcal{F}_T] + l(\langle X - Y, Y \rangle_\infty - \langle X - Y, Y \rangle_T) - E[\langle X - Y, Y \rangle_\infty - \langle X - Y, Y \rangle_T | \mathcal{F}_T]. \]

This completes the proof. ■

As a consequence of the lemma, we see that \( X \in \text{BMO} \) implies \( q(X) \in \text{BMO} \). Furthermore, we have

**Theorem 1.** Let \( X \) be a uniformly integrable continuous martingale and let \( q(X) \) be defined as in Section 1. If \( X \in \text{BMO} \), then
\[ \frac{1}{4d_1(q(X), L^\infty)} \leq b(X) \leq \frac{4}{d_1(q(X), L^\infty)}, \]

and furthermore, we have \( \sqrt{2} a(X) \geq b(X) \) for all \( X \in \text{BMO} \).

**Proof.** Let \( X \in \text{BMO} \). Then for any \( \lambda > 0 \) we have
\[ E[\exp(\lambda(\langle X \rangle_\infty - \langle X \rangle_T)) | \mathcal{F}_T] = E[\exp(\lambda E[\langle X \rangle_\infty - \langle X \rangle_T | \mathcal{F}_T]) \exp(\lambda(\langle X \rangle_\infty - E[\langle X \rangle_\infty | \mathcal{F}_T])) | \mathcal{F}_T] \leq e^{\lambda \|X\|_{\text{BMO}}^2} E[\exp(\lambda(\langle X \rangle_\infty - E[\langle X \rangle_\infty | \mathcal{F}_T])) | \mathcal{F}_T] \leq e^{\lambda \|X\|_{\text{BMO}}^2} E[\exp(\lambda q(X)_\infty - q(X)_T) | \mathcal{F}_T] \]

and
\[ E[\exp(\lambda q(X)_\infty - q(X)_T) | \mathcal{F}_T] = E[\exp(\lambda(\langle X \rangle_\infty - E[\langle X \rangle_\infty | \mathcal{F}_T])) | \mathcal{F}_T] \leq e^{\lambda \|X\|_{\text{BMO}}^2} E[\exp(\lambda(\langle X \rangle_\infty - \langle X \rangle_T)) | \mathcal{F}_T], \]
which shows that \( b(X) = a(q(X)) \). Thus, inequalities (2.1) follow from inequalities (1.2).

On the other hand, it is not difficult to show that the inequality
\[
E \exp(\lambda |X_\infty - X_T|) |\mathcal{F}_T| \leq 2E \exp(2\lambda^2((X_\infty - (X)_T)) |\mathcal{F}_T|^{1/2}
\]
holds for all \( \lambda > 0 \). Indeed, by using the Schwarz inequality and noting that for \( X \in \text{BMO} \) the continuous exponential martingale \( \mathcal{E}(X) \) defined by
\[
\mathcal{E}(X) = \exp(X - \frac{1}{2}(X))
\]
is uniformly integrable (see Theorem 2.3 in [3, p. 31]), for every real \( \lambda > 0 \) we have
\[
E \exp(\lambda (X_\infty - X_T)) |\mathcal{F}_T| = E \left[ \frac{\mathcal{E}(\lambda X)_{\infty}}{\mathcal{E}(\lambda X)_T} \exp(\lambda^2((X)_\infty - (X)_T)) \right] |\mathcal{F}_T|^{1/2}
\]
\[
\leq E \left[ \frac{\mathcal{E}(2\lambda X)_{\infty}}{\mathcal{E}(2\lambda X)_T} \right]^{1/2} E \exp(2\lambda^2((X)_\infty - (X)_T)) |\mathcal{F}_T|^{1/2}
\]
\[
\leq E \exp(2\lambda^2((X)_\infty - (X)_T)) |\mathcal{F}_T|^{1/2}.
\]
The same argument works if \( X \) is replaced by \(-X\). Thus, we obtain (2.2). This shows that \( \sqrt{2} a(X) \geq b(X) \).

Corollary 1. If \( X \in \text{BMO} \), then \( q(X) \in \overline{L}\infty \) is equivalent to \( b(X) = \infty \), where \( \overline{L}\infty \) stands for the BMO-closure of \( L\infty \).

Recall that the continuous exponential martingale \( \mathcal{E}(X) \) is said to satisfy the \((A_p)\)-condition \((1 < p < \infty)\), in symbols \( \mathcal{E}(X) \in (A_p) \), if
\[
\sup_T \left\| E \left[ \left\{ \frac{\mathcal{E}(X)_T}{\mathcal{E}(X)_\infty} \right\}^{1/(p-1)} \right] |\mathcal{F}_T| \right\|_\infty < \infty,
\]
where the supremum is taken over all stopping times \( T \). It is known (see Theorem 3.12 of [3, p. 72]) that \( q(X) \in \overline{L}\infty \) is equivalent to \( \mathcal{E}(X) \) and \( \mathcal{E}(-X) \) satisfying all \((A_p)\) \((1 < p < \infty)\). Thus, the following corollary is clear.

Corollary 2. If \( X \in \text{BMO} \) and \( 1 < p < \infty \), then \( b(X) = \infty \) is equivalent to \( \mathcal{E}(X) \) and \( \mathcal{E}(-X) \) satisfying all \((A_p)\).

Finally, we consider a subspace \( \mathcal{H} \) of \( \text{BMO} \),
\[
\mathcal{H} = \{ X \in \text{BMO} : q(X) \in \overline{L}\infty \}.
\]

Corollary 3. Let \( \overline{H}\infty \) denote the BMO-closure of \( H\infty \). Then
\[
\overline{H}\infty \subset \mathcal{H} \subset \overline{L}\infty.
\]

Proof. \( \overline{H}\infty \subset \mathcal{H} \) follows from (1.3) and Corollary 1, and \( \mathcal{H} \subset \overline{L}\infty \) follows from Theorem 1.
**Lemma 2.** The mapping $q : X \mapsto q(X)$ is continuous on BMO.

**Proof.** Let $\{X^n\}$ be a sequence of BMO-martingales such that $X^n \to X$ in BMO. Then

$$K \equiv \sup_n \|X^n\|_{\text{BMO}} < \infty \quad \text{and} \quad \|X\|_{\text{BMO}} \leq K.$$ 

It follows from Lemma 1 that

$$\|q(X^n) - q(X)\|_{\text{BMO}} \leq 4K\|X^n - X\|_{\text{BMO}} \to 0$$

as $n \to \infty$. This shows that the mapping $q$ is continuous on BMO.

**Theorem 2.** $\mathcal{H}$ is a closed linear subspace of BMO.

**Proof.** The closedness of $\mathcal{H}$ follows from Lemma 2.

On the other hand, for any two real $\alpha, \beta$ and any two BMO-martingales $X, Y$, we have

$$\langle \alpha X + \beta Y \rangle_\infty - \langle \alpha X + \beta Y \rangle_T \leq 2(\alpha^2(\langle X \rangle_\infty - \langle X \rangle_T) + \beta^2(\langle Y \rangle_\infty - \langle Y \rangle_T))$$

and so, for any $\lambda > 0$,

$$E[\exp(\lambda(\langle \alpha X + \beta Y \rangle_\infty - \langle \alpha X + \beta Y \rangle_T))] | \mathcal{F}_T]$$

$$\leq E[\exp(4\alpha^2\lambda(\langle X \rangle_\infty - \langle X \rangle_T)) | \mathcal{F}_T]^{1/2}$$

$$\times E[\exp(4\beta^2\lambda(\langle Y \rangle_\infty - \langle Y \rangle_T)) | \mathcal{F}_T]^{1/2},$$

which shows that $b(\alpha X + \beta Y) = \infty$ for $b(X) = \infty$, $b(Y) = \infty$. Thus, $X, Y \in \mathcal{H}$ implies that $\alpha X + \beta Y \in \mathcal{H}$. This completes the proof.

Now, it is natural to ask if the relationship $\overline{\mathcal{H}}^\infty = \mathcal{H}$ holds. But we have not been able to settle this question so far.

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**References**

