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A characterization of F-algebras with all one-sided ideals closed

by

W. Żelazko (Warszawa)

Abstract. We prove that a real or complex F-algebra has all left and right ideals closed if and only if it is noetherian.

A *topological algebra* is a real or complex algebra A which is a topological vector space (t.v.s.) and the multiplication $(x, y) \mapsto xy$ is a jointly continuous map from A^2 to A. In terms of neighbourhoods of zero this means that for each such neighbourhood U there is a neighbourhood V with

$$V^2 \subset U. \tag{1}$$

A unital topological algebra A is called a *Q-algebra* if the set (group) $G(A)$ of all invertible elements of A is open. It is known ([7, Lemma I.6.4, pp. 43–44]) that A is a Q-algebra if and only if its unity element e has a neighbourhood consisting of invertible elements.

An *F-algebra* is a topological algebra which is an F-space, i.e. a complete metrizable t.v.s. The topology of an F-space X can be given by means of an *F-norm*, i.e. a map $x \mapsto \|x\|$ from X to the non-negative real numbers such that

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ iff $x = 0$,
(ii) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$,
(iii) the map $(\lambda, x) \mapsto \|\lambda x\|$ is jointly continuous, $\lambda \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $x \in X$.

For further information on F-spaces and F-norms the reader is referred to [1] and [8].

A topological algebra is said to be *multiplicatively convex* (briefly *m-convex*) if its topology can be given by means of a family of submultiplicative (algebra) seminorms. An m-convex algebra which is also an F-algebra is

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called an *m-convex B_0-algebra*. More information about topological algebras can be found in [7] or [10].

We say that an algebra A is *noetherian* if it satisfies the *ascending chain condition*, i.e. whenever

$$I_1 \subseteq I_2 \subseteq \cdots$$

is a sequence of left (or right) ideals in A, then there is an index n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$. If A is unital, then it is noetherian if and only if every proper one-sided ideal I of A is of the form

$$I = x_1A + \cdots + x_nA, \quad x_i \in I,$$

in the case of a right ideal, and

$$I = Ax_1 + \cdots + Ax_n, \quad x_i \in I,$$

in the case of a left ideal.

The noetherian topological algebras were first considered by Grauert and Remmert in the context of commutative Banach algebras. They showed in [6] that a commutative noetherian Banach algebra is necessarily finite-dimensional. This result was extended to the non-commutative case by Sinclair and Tullo [9]. Ferreira and Tomassini studied in [5] the noetherian m-convex algebras and showed, among other results, that a noetherian commutative complex unital m-convex B_0-algebra has all ideals closed (Theorem 2.6 of [5]). They also observed that there exist such infinite-dimensional algebras (the algebras of all power series in one or a finite number of variables).

Motivated by the work of Ferreira and Tomassini, Carboni and Larotonda ([2], [3]) constructed highly non-trivial examples of commutative semisimple noetherian m-convex B_0-algebras (the examples considered in [5] were radical algebras), which are principal ideal domains. Further, independently, Choukri and El Kinani in [4] and the author in [12] proved that a commutative F-algebra is noetherian if and only if it has all ideals closed. The aim of this paper is to extend this result to the non-commutative case (this solves a problem explicitly posed in [12]).

Our result reads as follows.

THEOREM. *Let A be a real or complex F-algebra. Then A has all one-sided ideals closed if and only if it is noetherian.*

In the proof we shall make use of the following result ([13, Corollary]).

LEMMA 1. *Let A be a real or complex F-algebra with unity e.*

(i) *Let (x_n) be a sequence of elements of A tending to e. Then there is a subsequence $(a_i) \subset (x_n)$ such that for all natural k the products*

$$u_k = \lim_i a_{k+i}a_{k+i-1} \cdots a_k \tag{2}$$

and

$$v_k = \lim_i a_k a_{k+1} \cdots a_{k+i} \tag{3}$$

are convergent, with

$$\lim_k u_k = \lim_k v_k = e. \tag{4}$$

(ii) *If A is not a Q-algebra, then there is a sequence (a_i) of non-invertible elements in A tending to e such that either $I_l = \bigcup_k Av_k$ is a proper dense left ideal, or $I_r = \bigcup_k u_k A$ is a proper dense right ideal.*

The easier part of our theorem follows from the following result, extending Lemma 2 of [12] to the non-commutative situation.

LEMMA 2. *Let A be a real or complex F-algebra with all ideals closed. Then A is noetherian.*

Proof. Let I be a proper right ideal in A (for left ideals the reasoning is analogous) and choose a non-zero $x_1 \in I$ so that $I_1 = x_1 A \subset I$. If $I = I_1$ we are done. If not, there is an $x_2 \in I \setminus I_1$ and so $I_2 = I_1 + x_2 A$ is a subideal of I. Again either $I_2 = I$ and in this case we are done, or the process can be continued. If it does not finish, we obtain a sequence $I_1 \subset I_2 \subset \cdots \subset I$ of closed right ideals and all imbeddings are proper. Setting $J = \bigcup_{i \geq 1} I_i$ we obtain a proper right ideal in A ($J \subset I$) which is not closed as the union of an increasing sequence of closed subspaces. This gives a contradiction showing that $I = I_n$ for some n, and the conclusion follows.

It remains to show that if A is noetherian, then it has all left and right ideals closed. We obtain this result through Lemmas 3–7; first we need the concept of a topologically invertible element.

DEFINITION. Let A be a unital F-algebra. An element u in A is said to be *topologically right* (resp. *left*) *invertible* if there is a sequence $(z_i) \subset A$ with $\lim_i uz_i = e$ (resp. $\lim_i z_i u = e$). An element u is said to be *topologically invertible* if it is both left and right topologically invertible. This concept can be extended to an arbitrary topological algebra A, but then sequences have to be replaced by nets in case A is non-metrizable.

Let us recall the following simple lemma ([13, Lemma 1]); its proof follows immediately from the joint continuity of multiplication in A.

LEMMA 3. *Let A be a real or complex F-algebra with unity e. Then for any $u, v \in A$ and $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, u, v) > 0$ such that*

$$\|x - e\| < \delta \quad \textit{implies} \quad \|uxv - uv\| < \varepsilon \tag{5}$$

for all x in A.

Taking in (5) a smaller positive ε, we can replace $\|x-e\| < \delta$ by $\|x-e\| \le \delta$, and in this form Lemma 3 will be used in the proof of Lemma 5.

LEMMA 4. *A finite product of topologically left* (*resp. right*) *invertible elements is topologically left* (*resp. right*) *invertible.*

Proof. First we show that the product of two topologically left invertible elements a, b is again topologically invertible (for topologically right invertible elements the proof is the same). We have to show that for any $\varepsilon > 0$ there is a $z \in A$ with

$$\|zab - e\| < \varepsilon. \tag{6}$$

Since b is topologically left invertible, there is a $z_1 \in A$ such that

$$\|z_1 b - e\| < \varepsilon/2. \tag{7}$$

By Lemma 3 (setting $u = z_1$, $v = b$ and replacing ε by $\varepsilon/2$), there is a $\delta > 0$ such that

$$\|z_1 x b - z_1 b\| < \varepsilon/2 \tag{8}$$

whenever $\|x-e\| < \delta$. Finally, by the left topological invertibility of a, there is a $z_2 \in A$ such that $\|z_2 a - e\| < \delta$.

Consequently, setting in (8) $x = z_2 a$, we obtain

$$\|z_1 z_2 ab - z_1 b\| < \varepsilon/2,$$

which together with (7) gives (6). By an easy induction, any finite product of topologically left invertible elements is topologically left invertible. The conclusion follows.

In the proof of the following lemma we shall use a reasoning similar to the proof of Proposition 1 in [12] or Lemma 2 in [13].

LEMMA 5. *Let A be a unital F-algebra, and (x_i) a sequence of topologically left* (*resp. right*) *invertible elements tending to the unity e of A. Then there is a subsequence $(a_i) \subset (x_i)$ such that the products u_k given by* (2) (*resp. v_k given by* (3)), $k = 1, 2, \ldots,$ *are topologically left* (*resp. right*) *invertible.*

Proof. We assume that the x_i are topologically left invertible. For topologically right invertible elements the proof is analogous. The sequence (a_i) will be chosen inductively. First choose x_{j_1} so that $\|e - x_{j_1}\| < 1$, put $a_1 = x_{j_1}$ and choose $z_{0,1}$ in A so that $\|z_{0,1} a_1 - e\| < 1/2$; this is possible because a_1 is topologically left invertible. Using Lemma 3, we choose a $\delta_1 > 0$ such that $\|z_{0,1} x a_1 - z_{0,1} a_1\| < 1/2$ whenever $\|x - e\| \le \delta_1$. Put also $\delta_0 = 1$.

Suppose now that we have already chosen elements $a_1, \ldots, a_n$, $a_i = x_{j_i}$, $j_i < j_{i+1}$, elements $z_{i,k}$ in A, $1 \le k + i \le n$, and positive numbers $\delta_1, \ldots, \delta_n$

in such a way that

$$\|a_{k+i+1}a_{k+i}\cdots a_k - a_{k+i}\cdots a_k\| < 2^{-(k+i+1)} \quad \text{for } 1 \le k \le k+i < n \ (i \ge 0), \tag{9}$$

$$\|a_{k+i}\cdots a_k - e\| < \min\{\delta_{k-1}, k^{-1}\} \quad \text{for } 1 \le k+i \le n, \tag{10}$$

$$\|z_{i,k}a_{k+i}\cdots a_k - e\| < \frac{1}{2(k+i)} \quad \text{for } 1 \le k+i \le n, \tag{11}$$

$$\|z_{i,k}xa_{k+i}\cdots a_k - z_{i,k}a_{k+i}\cdots a_k\| < \frac{1}{2(k+i)} \quad \text{for } 1 \le k+i \le n, \tag{12}$$

provided

$$\|x - e\| \le \delta_{k+i}.$$

We shall show that it is possible to find elements $a_{n+1} \in (x_i), z_{i,n+1-i}$ in A, $0 \le i \le n$, and a positive number δ_{n+1}, so that (9)–(12) are satisfied with n replaced by $n+1$. Let $\varepsilon > 0$. Since $x_i \to e$, we can find an index $i(\varepsilon)$ such that

$$\|a_n\cdots a_k - x_i a_n\cdots a_k\| < \varepsilon \quad \text{for } 1 \le k \le n, \tag{13}$$

whenever $i \ge i(\varepsilon)$. There is also an index j such that

$$\|x_i - e\| < \min\{\delta_n, (n+1)^{-1}\} \quad \text{whenever } i \ge j. \tag{14}$$

Put

$$s_k = \min\{\min\{\delta_{k-1}, k^{-1}\} - \|a_{k+i}\cdots a_k - e\| : 1 \le k \le k+i \le n\}.$$

By (10) all s_k are positive, and hence so is $r = \min\{s_k : 1 \le k \le n\}$. Take now any ε satisfying $0 < \varepsilon < \min\{r, 2^{-(n+1)}\}$, put $i_{n+1} = \max\{i_n + 1, j, i(\varepsilon)\}$, and $a_{n+1} = x_{i_{n+1}}$. Relation (13) implies (9) with n replaced by $n+1$, and also

$$\|a_{n+1}a_n\cdots a_k - a_n\cdots a_k\| < r \le s_k \le \min\{\delta_{k-1}, k^{-1}\} - \|a_n\cdots a_k - e\|$$

for $1 \le k \le n$, and so

$$\begin{aligned}\|a_{n+1}\cdots a_k - e\| &\le \|a_{n+1}\cdots a_k - a_n\cdots a_k\| + \|a_n\cdots a_k - e\| \\ &< \min\{\delta_{k-1}, k^{-1}\}.\end{aligned}$$

Thus (10) holds true for $1 \le k+1 \le n+1$, provided $k \le n$. The case $k = n+1$ is covered by (14), so that (10) is satisfied with n replaced by $n+1$.

By the previous lemma the elements $a_{n+1}\cdots a_k$ are topologically left invertible, so there are $z_{i,k}$, $i+k = n+1$, such that relations (11) hold true with n replaced by $n+1$. Finally, for the newly obtained elements $z_{i,n+1-i}$ we can find, by Lemma 3, the desired $\delta_{n+1} > 0$ so that (12) is satisfied. The induction step is complete.

Relations (9) imply

$$\begin{aligned}\|a_{m+n}\cdots a_k - a_n\cdots a_k\| &\le \|a_{m+n}\cdots a_k - a_{m+n-1}\cdots a_k\| + \cdots \\ &\quad + \|a_{n+1}\cdots a_k - a_n\cdots a_k\| \\ &\le 2^{-(m+n+1)} + \cdots + 2^{-(n+1)} < 2^{-n}\end{aligned}$$

for all natural m, n and $k \le n$. This shows the convergence of the products u_k given by (2) for all natural k. Relations (10) imply

$$\|u_{k+1} - e\| \le \delta_k \tag{15}$$

for all natural k. Fix such a k. By (11) and (12) we have

$$\|z_{i,k} x a_{k+i}\cdots a_k - e\| < \frac{1}{k+i}$$

for $\|x - e\| \le \delta_{k+i}$. Setting now $x = u_{k+i+1}$ and taking into account the equality $u_k = u_{k+i+1}a_{k+i}\cdots a_k$, we obtain, by (15),

$$\|z_{i,k}u_k - e\| < \frac{1}{k+i}.$$

Letting $i \to \infty$, we see that u_k is topologically left invertible. The conclusion follows.

We now pass to our crucial lemma.

LEMMA 6. *Let A be a unital real or complex F-algebra which is noetherian. Then A is a Q-algebra.*

Proof. Assume that A is not a Q-algebra. By Lemma 1(ii), there is a sequence (a_i) of non-invertible elements in A tending to the unity e such that either $I_r = \bigcup_{k=1}^{\infty} u_k A$ is a proper dense right ideal, or $I_l = \bigcup_{k=1}^{\infty} A v_k$ is a proper dense left ideal. We can assume that $J = I_r$ is a proper ideal (the case of I_l can be treated in an analogous way, or reduced to the former by considering A with the reverse multiplication $x \circ y = yx$).

Since A is noetherian, there are $y_1, \ldots, y_n \in J$ such that

$$J = y_1 A + \cdots + y_n A.$$

Let $I_k = u_k A$, so that I_k is a right ideal contained in J. Since $u_k = u_{k+1}a_k$ we have $I_k = u_{k+1}a_k A \subset I_{k+1}$ and $J = \bigcup_k I_k$. Thus there is the smallest index k_0 such that all elements $y_1, \ldots, y_n$ are in I_{k_0}. Consequently, $J = I_{k_0} = I_k$ for all $k \ge k_0$. Since J is dense in A, for each $k \ge k_0$ there is a sequence $(z_i^{(k)})_{i=1}^{\infty}$ of elements of A such that

$$\lim_i u_k z_i^{(k)} = e \quad \text{for all } k \ge k_0. \tag{16}$$

This means that all elements u_k, $k \ge k_0$, are topologically right invertible.

We now show that they cannot be left invertible. Otherwise there are $c_k \in A$ such that $c_k u_k = e$, $k \ge k_0$, and (16) implies $c_k = \lim_i c_k u_k z_i^{(k)} =$

$\lim_i z_i^{(k)}$, so that $u_k c_k = \lim_i u_k z_i^{(k)} = e$. Thus all u_k, $k \geq k_0$, are invertible with inverses c_k. Writing $u_k = u_{k+1} a_k$, $k \geq k_0$, we obtain $a_k = u_{k+1}^{-1} u_k$, so that all a_k, $k \geq k_0$, are invertible, contrary to assumption. Consequently, the elements u_k, $k \geq k_0$, are not left invertible.

Setting $x_i = u_{k_0+i-1}$ we obtain a sequence of elements of A which are topologically right invertible and not left invertible, and tend to e. By Lemma 5 there is a subsequence $(b_k) \subset (x_i)$ such that the elements v_k given by (3) with a_i replaced by b_i are topologically right invertible and tend to e. A reasoning similar to the first part of the proof shows that the elements v_k are not left invertible. Consequently, all $I_k' = Av_k$ are proper left ideals, and so is $J' = \bigcup_k Av_k$. Since $v_k \to e$, the ideal J' is dense. As before, we find an index k_1 such that $J' = I_{k_1}' = I_k'$ for $k \geq k_1$. The relation $Av_k = Av_{k+1}$, $k \geq k_1$, implies $v_{k+1} \in Av_k$, and so there is a $d_k \in A$ with $v_{k+1} = d_k v_k$. But $v_k = b_k v_{k+1}$ and so $v_{k+1} = d_k b_k v_{k+1}$, or

$$(e - d_k b_k) v_{k+1} = 0. \tag{17}$$

Since the elements v_k, $k \geq k_1$, are topologically right invertible, there are sequences $(w_i^{(k)})_{i=1}^{\infty}$, $k \geq k_1$, with $\lim_i v_k w_i^{(k)} = e$. Fixing a $k \geq k_1$, multiplying (17) from the right by $w_i^{(k+1)}$ and passing to the limit with respect to i, we obtain $d_k b_k = e$, i.e. b_k is left invertible. This is the desired contradiction and the conclusion follows.

The main idea of the next lemma is due to Grauert and Remmert ([6, Chapter I, Remark 2 in the Appendix to §5]); they needed it to prove that a commutative noetherian Banach algebra is necessarily finite-dimensional. Here we give a modification of the proof of a similar lemma proved in [12] in the commutative case. Similarly to [12] we shall use neighbourhoods of the origin instead of F-norms.

LEMMA 7. *Let A be a real or complex unital F-algebra which is also a Q-algebra. Let I be a proper left* (*resp. right*) *ideal in A whose closure $\bar{I}$ is a finitely generated ideal. Then I is closed.*

Proof. We shall give the proof for a right ideal. The other case is analogous. Since A is a Q-algebra, $\bar{I}$ is a proper ideal in A and, by assumption,

$$\bar{I} = x_1 A + \cdots + x_n A \quad \text{with } x_1, \ldots, x_n \in \bar{I}.$$

Put

$$\Phi(u_1, \ldots, u_n) = x_1 u_1 + \cdots + x_n u_n;$$

this is a continuous linear map from A^n to A. Since A^n is also an F-space (with the product topology), and Φ is onto, the Banach theorem (see [1] or [8]) says that Φ is open, and so, for every neighbourhood V of zero in A, the set $S(V) = \Phi(V, \ldots, V) = x_1 V + \cdots + x_n V$ is also a neighbourhood of zero in $\bar{I}$. Since I is dense in $\bar{I}$ we have

$$I + S(V) = \bar{I} \tag{18}$$

for each such V. Since $x_i \in \bar{I}$, (18) implies that for any neighbourhood V of zero in A there are $u_{k,i}$ in V and y_i in I, $1 \le i, k \le n$, such that $x_k = y_k + \sum_{i=1}^{n} x_i u_{k,i}$, or

$$y_k = x_k - \sum_{i=1}^{n} x_i u_{k,i}, \qquad k = 1, \ldots, n. \tag{19}$$

We can treat (19) as a system of linear equations with given $y_k \in I$ and $u_{k,i} \in V$, and by solving it for suitably chosen V, we show that the elements x_i must belong to I.

Grauert and Remmert solved this system by the use of the classical Cramer formulas. Since we are in the non-commutative situation, and using determinants of non-commuting elements would be troublesome, we shall solve the system by induction with respect to the number of variables. By solving this system, for a suitable neighbourhood V, we shall show that the elements x_i are in I, which implies $\bar{I} \subset I$, as required. In each step of the induction we shall modify the neighbourhood V. Let $n = 1$ and denote by V_1 a neighbourhood of the origin in A so that $e - V_1 \subset G(A)$. Such a V_1 exists, since A is a Q-algebra. The system (19) is now of the form

$$y_1 = x_1 - x_1 u_{1,1}$$

with $y_1 \in I$, $u_{1,1} \in V_1$ and $x_1 \in \bar{I}$. Thus $y_1 = x_1(e - u_{1,1})$, so that $x_1 = y_1(e - u_{1,1})^{-1}$ and $x_1 \in I$ since I is a right ideal and $e - u_{1,1}$ is invertible.

Suppose now that for a given natural m and for all natural k with $1 \le k < m$ there are neighbourhoods V_k of zero in A, with $V_j \subset V_{j-1}$ for $1 < j \le k$, such that whenever $y_1, \ldots, y_k \in I$ and $u_{k,i} \in V_k$ for $1 \le i \le k$, then there are unique $x_1, \ldots, x_k$ satisfying (19), and these elements are in I. We are now looking for a suitable $V_m \subset V_{m-1}$ so that the above holds with $k = m$. Consider (19) with $n = m$ and take into account the last equation

$$y_m = x_m(e - u_{m,m}) - \sum_{i=1}^{m-1} x_i u_{m,i}. \tag{20}$$

Since we demand $V_m \subset V_{m-1} \subset V_1$, and $u_{m,m} \in V_m$, the element $e - u_{m,m}$ is invertible and relation (20) can be replaced by

$$y_m(e - u_{m,m})^{-1} = x_m - \sum_{i=1}^{m-1} x_i u_{m,i}(e - u_{m,m})^{-1}.$$

Hence

$$x_m = y_m(e - u_{m,m})^{-1} + \sum_{i=1}^{m-1} x_i u_{m,i}(e - u_{m,m})^{-1} \tag{21}$$

and substituting this to the first $m-1$ equations in (19) with $n=m$, we have

$$y_k = x_k - \sum_{i=1}^{m-1} x_i u_{k,i} - \Big[y_m (e-u_{m,m})^{-1} + \sum_{i=1}^{m-1} x_i u_{m,i} (e-u_{m,m})^{-1} \Big] u_{m,m} \quad \text{for } k = 1\ldots, m-1,$$

or

$$y_k + y_m (e-u_{m,m})^{-1} u_{m,m} = x_k - \sum_{i=1}^{m-1} x_i [u_{k,i} + u_{m,i}(e-u_{m,m})^{-1} u_{m,m}] \quad \text{for } k = 1\ldots, m-1. \tag{22}$$

Clearly, $y_k + y_m(e-u_{m,m})^{-1}u_{m,m}$ belongs to I for each choice of $V_m \subset V_1$, so in order to make the induction step, we have to find V_m so that

$$u_{k,i} + u_{m,i}(e-u_{m,m})^{-1}u_{m,m} \in V_{m-1} \tag{23}$$

whenever $u_{k,i} \in V_m$. Note that the elements in (23) are the coefficients of $x_1, \ldots, x_{m-1}$ in (22).

To this end first choose a neighbourhood U_1 of the origin so that $U_1 + U_1 \subset V_{m-1}$, and a neighbourhood U_2 with $U_2 + U_2 \subset U_1$.

We now find a neighbourhood U_3 of zero so that both U_3^2 and U_3^3 are contained in U_2, and a neighbourhood $U_4 \subset V_1$ so that $x \in e + U_4$ implies $(e-x)^{-1} \in e + U_3$. The possibility of choosing U_4 follows from the fact that $x \mapsto x^{-1}$ is continuous on Q-algebras of type F (taking the inverse is continuous in an F-algebra if and only if the set $G(A)$ is a G_δ-set, see [7] or [10]), so in particular it is continuous at $x = e$. We now put

$$V_m = U_1 \cap U_2 \cap U_3 \cap U_4.$$

If all the $u_{m,i}, 1 \le i \le m$, are in V_m, then $(e-u_{m,m})^{-1} \in U_3$, and so

$$u_{m,i}(e-u_{m,m})^{-1}u_{m,m} \in U_3(e+U_3)U_3 \subset U_3^2 + U_3^3 \subset U_2 + U_2 \subset U_1.$$

Consequently, the left hand side of (23) is in $U_1 + U_1 \subset V_{m-1}$ and we are done. By the inductive assumption, $x_1, \ldots, x_{m-1}$ are in I. Consequently, by (21), we also have $x_m \in I$, completing the induction step. Thus the conclusion follows.

Proof of the Theorem. Assume first that A is unital. In view of Lemma 2, we have to show that if A is noetherian, then its ideals are all closed. By Lemma 6, A is a Q-algebra. Let now I be a left ideal in A (for right ideals the proof is analogous). Since A is a Q-algebra, the closure $\overline{I}$ is a proper ideal, finitely generated by assumption. Lemma 7 now implies that I is closed.

For a non-unital F-algebra A, denote by A_e its *unitization*, i.e. the direct sum $\mathbb{K} \oplus A$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, with coordinatewise addition and scalar multiplication, and with the multiplication given by $(\lambda e + a)(\mu e + b) = \lambda\mu e + \lambda b + \mu a + ab$, where $\lambda e + b = (\lambda, b)$. It is a unital F-algebra with F-norm $\|\lambda e + a\| = |\lambda| + \|a\|$ and with the unity e. Clearly A_e contains A as a closed two-sided ideal. The proof in the non-unital case now follows from the fact that A is noetherian if and only if A_e is, and from the following proposition.

PROPOSITION. *Let A be a real or complex non-unital algebra of type F. Then A has all one-sided ideals closed if and only if the same holds for its unitization A_e.*

Proof. We give the proof for left ideals. (For right ideals it is analogous.) Observe first that if I is a left ideal in A, then it is also a left ideal in A_e (we have $A_e I = (\mathbb{K}e + A)I \subset I + AI \subset I$). Thus if all left ideals are closed in A_e, they are also closed in A. Conversely, suppose that all left ideals are closed in A and let J be a proper left ideal in A_e. Put $I = J \cap A$. This is a left ideal in A, for if $x \in A$ and $y \in I$, then xy is both in J and in A, so it is in I. We are already done if $J \subset A$, so assume there is an element u_0 in J of the form $u_0 = e + a_0$, $a_0 \in A$. We cannot have $a_0 \in I$, since otherwise $e \in J$ and J is not proper. Let now u be an arbitrary element in J, so $u = \lambda e + a$ with $a \in A$. We claim that $u = \lambda u_0 + b$ with $b \in A$. In fact,

$$u - \lambda u_0 = a - \lambda a_0,$$

so this difference is both in A and in J and hence in I. Denoting it by b, we obtain $u = \lambda u_0 + b$. Thus, $J = \mathbb{K} \oplus I$. This implies that J is closed. For if $\lambda_n u_0 + b_n \to v$, $v \in A_e$, then the sequence of scalars (λ_n) is bounded. Otherwise, passing to a subsequence if necessary, we can assume $\lambda_n \to \infty$, so that $u_0 + b_n/\lambda_n \to 0$ and $u_0 \in I$, since $b_n \in I$ and I is closed. This gives a contradiction, and so, passing again to a subsequence, we can assume that (λ_n) is convergent, say to λ_0. Consequently, $a_n \to v - \lambda_0 u_0$, so that $v - \lambda_0 u_0 \in I$ and hence v is in J. Thus J is closed and the conclusion follows.

It was shown in [12, Example 6] that there exists a complete commutative locally convex non-metrizable algebra which has all ideals closed, but is not noetherian. One can also give an easy example of such a non-commutative algebra. To this end consider a real or complex free algebra A in countably many variables $t_1, t_2, \ldots$, provided with the maximal locally convex topology $\tau_{\max}^{\mathrm{LC}}$ (it is given by all seminorms on A, and its linear subspaces are all closed). It is shown in [11] that A is a topological algebra. Clearly all one-sided ideals in A are closed, but it is non-noetherian. To see this, let I be the linear span of all monomials of degree at least two. It is a two-sided ideal in A and it is neither left nor right finitely generated. For, if $x_1, \ldots, x_n$ were such generators, then they would contain only finitely many variables, say $t_1, \ldots, t_k$, and each element of I would contain at least one of those variables. But t_{k+1}^2 is in I and does not contain any of those variables and so our claim follows.

Thus our result does not extend to the non-metrizable case. However, we do not know whether there is a complete topological algebra which is noetherian but has a non-closed one-sided ideal; the answer is not known even in the commutative case (in [12] we gave an example of an incomplete normed noetherian commutative algebra which has non-closed ideals).

We do not know either whether there exists an F-algebra with all left ideals closed but with some right ideals non-closed ([1]). Nor do we know whether left noetherian F-algebras coincide with such algebras with all left ideals closed (our Lemma 2 says, in fact, that if all left ideals of an F-algebra A are closed then A is left noetherian, but we do not know whether the converse is true).

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Institute of Mathematics
Polish Academy of Sciences
P.O. Box 21, 00-956 Warszawa, Poland
E-mail: zelazko@impan.gov.pl

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([1]) *Added in proof*: The author has constructed such an m-convex B_0-algebra. The construction will be published elsewhere.