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Joint subnormality of n-tuples and C_0-semigroups of composition operators on L^2-spaces, II

by

Piotr Budzyński and Jan Stochel (Kraków)

Abstract. In the previous paper, we have characterized (joint) subnormality of a C_0-semigroup of composition operators on L^2-space by positive definiteness of the Radon–Nikodym derivatives attached to it at each rational point. In the present paper, we show that in the case of C_0-groups of composition operators on L^2-space the positive definiteness requirement can be replaced by a kind of consistency condition which seems to be simpler to work with. It turns out that the consistency condition also characterizes subnormality of C_0-semigroups of composition operators on L^2-space induced by injective and bimeasurable transformations. The consistency condition, when formulated in the language of the Laplace transform, takes a multiplicative form. The paper concludes with some examples.

1. Introduction. The notion of a subnormal operator was introduced into Hilbert space operator theory by Halmos [18], who himself gave a two-condition characterization of it. This characterization was successively simplified by Bram [3], Embry [11] and Lambert [22]. Itô [21] extended the notion (as well as Bram's characterization) of subnormality to the context of commutative families of bounded operators. Embry's and Lambert's criteria for subnormality were adapted to the context of families of operators by Lubin [26]. Itô [21] proved that a C_0-semigroup of subnormal operators is automatically jointly subnormal and as such has an extension to a C_0-semigroup of normal operators. As noted below, this is still true for C_0-groups of subnormal operators (cf. Proposition 2.1). Itô's theorem enabled Nussbaum [29] to prove the subnormality of the infinitesimal generator of a C_0-semigroup of subnormal operators. The interested reader is referred to the monograph [7] for the foundations of the theory of subnormal operators.

Composition operators, which play an essential role in ergodic theory, turn out to be interesting objects of operator theory. The questions of bound-

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edness, normality, quasinormality, subnormality, hyponormality etc. of such operators are the subject of intensive research (cf. [10, 35, 28, 42, 37, 36, 20, 39, 23, 24, 25, 9, 6, 14, 15, 41, 30, 31, 32, 5]; see also [12, 13, 27, 40, 8] for particular classes of composition operators). Our main goal here is to continue the study of (joint) subnormality of C_0-semigroups of composition operators on L^2-space initiated in [4]. The main result of the paper, Theorem 6.5, offers a new criterion for subnormality of C_0-semigroups of composition operators on L^2-space induced by injective and bimeasurable transformations; it enables us to replace positive definiteness requirements appearing in [4, Lemma 4.4] by the consistency condition (b*). As shown in Theorem 8.4, the consistency condition characterizes subnormality of general C_0-groups of composition operators on L^2-space; the injectivity and bimeasurability assumption is then superfluous. Another possibility of employing the consistency condition is elucidated in Propositions 7.3 and 8.5, where the Laplace transform approach is exploited. We conclude the paper with examples of C_0-semigroups and C_0-groups illustrating our considerations.

In a subsequent paper we will continue the study of subnormality of C_0-semigroups of composition operators on L^2-space paying special attention to the case of C_0-groups.

2. Preliminaries. In what follows, $\mathbb{C}$ (respectively, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Q}_+$) stands for the set of all complex (respectively, real, nonnegative real, nonnegative rational) numbers. We also use the notation $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

We say that a sequence $\{t_n\}_{n=0}^\infty$ of real numbers is a *Stieltjes moment sequence* if there exists a positive Borel measure μ on $\mathbb{R}_+$ such that

$$t_n = \int_{\mathbb{R}_+} s^n \, d\mu(s), \qquad n = 0, 1, \ldots;$$

such a μ is called a *representing measure* of $\{t_n\}_{n=0}^\infty$. Note that a Stieltjes moment sequence which has a representing measure with compact support is *determinate*, i.e. the representing measure is uniquely determined (within the class of all Borel measures not necessarily compactly supported, cf. [2, 17]).

A bounded linear operator S on a complex Hilbert space $\mathcal{H}$ is called *subnormal* [18, 7] if there exists a complex Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a bounded normal operator N on $\mathcal{K}$ such that $S \subseteq N$ (i.e. $Sh = Nh$ for all $h \in \mathcal{H}$). A family $\{S_\omega \colon \omega \in \Omega\}$ of bounded linear operators on $\mathcal{H}$ is said to be *jointly subnormal* if there exists a complex Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a family $\{N_\omega \colon \omega \in \Omega\}$ of commuting bounded normal operators on $\mathcal{K}$ such that $S_\omega \subseteq N_\omega$ for all $\omega \in \Omega$.

According to the Itô theorem (cf. [21, Theorem 4]), a C_0-semigroup $\{S_t\}_{t \in \mathbb{R}_+}$ of bounded linear operators on $\mathcal{H}$ is jointly subnormal if and only if each operator S_t is subnormal. Because of this, we shall abbreviate the expression "a C_0-semigroup is jointly subnormal" to "a C_0-semigroup is subnormal".

We now state a C_0-group counterpart of [21, Theorem 4].

PROPOSITION 2.1. *Suppose that* $\{S_t\}_{t\in\mathbb{R}}$ *is a* C_0*-group of bounded linear operators on a complex Hilbert space* $\mathcal{H}$*. Then the following conditions are equivalent*:

(i) *the* C_0*-group* $\{S_t\}_{t\in\mathbb{R}}$ *is jointly subnormal*,
(ii) *each operator* S_t, $t\in\mathbb{R}$, *is subnormal*,
(iii) *the* C_0*-semigroup* $\{S_t\}_{t\in\mathbb{R}_+}$ *is jointly subnormal*,
(iv) *each operator* $S_{1/k}$, $k\in\mathbb{N}$, *is subnormal.*

Moreover, *if* (i) *holds*, *then there exists a* C_0*-group* $\{N_t\}_{t\in\mathbb{R}}$ *of bounded normal operators on a complex Hilbert space* $\mathcal{K}\supseteq\mathcal{H}$ *such that* $S_u\subseteq N_u$ *for every* $u\in\mathbb{R}$.

Proof. The implications (i)⇒(ii), (i)⇒(iii), (ii)⇒(iv) and (iii)⇒(iv) are obvious.

(iv)⇒(i). Arguing as in the proof of [21, Lemma 5], we see that

$$\sum_{i,j=1}^{n}\langle S_{t_i}h_j, S_{t_j}h_i\rangle\geq 0,\qquad t_1,\ldots,t_n\in\mathbb{R}_+,\ h_1,\ldots,h_n\in\mathcal{H},\ n\in\mathbb{N}.\tag{2.1}$$

Take finite sequences $r_1,\ldots,r_n\in\mathbb{R}$ and $g_1,\ldots,g_n\in\mathcal{H}$. Then there exists $u\in\mathbb{R}$ such that $t_j:=r_j+u\geq 0$ for all $j=1,\ldots,n$. Set $h_j=S_{-u}g_j$ for $j=1,\ldots,n$. Then $g_j=S_uh_j$ for $j=1,\ldots,n$. Hence, by (2.1) we have

$$\sum_{i,j=1}^{n}\langle S_{r_i}g_j, S_{r_j}g_i\rangle=\sum_{i,j=1}^{n}\langle S_{t_i}h_j, S_{t_j}h_i\rangle\geq 0.$$

An application of [21, Theorem 1] gives (i).

The "moreover" part can be proved just as [21, Theorem 4]. ∎

Because of Proposition 2.1, we shall abbreviate the expression "a C_0-group is jointly subnormal" to "a C_0-group is subnormal".

REMARK 2.2. It is easily seen that if $\{S_t\}_{t\in\mathbb{R}}$ is a group of bounded linear operators on a Banach space $\mathcal{X}$ (i.e. $S_{u+v}=S_uS_v$ for all $u,v\in\mathbb{R}$ and S_0 is the identity operator on $\mathcal{X}$) and its restriction $\{S_t\}_{t\in\mathbb{R}_+}$ is a C_0-semigroup, then $\{S_t\}_{t\in\mathbb{R}}$ is a C_0-group.

3. Composition operators modulo their symbols. Suppose that (X,Σ,μ) is a σ-finite measure space. Put

$$\Sigma_\mu=\{\sigma\in\Sigma\colon \mu(\sigma)<\infty\}.$$

Denote by $L^2(\mu)$ the complex Hilbert space of all square integrable complex functions on X. Let $\phi\colon X \to X$ be a $\varSigma$-measurable transformation of X. Denote by $\mu \circ \phi^{-1}$ the measure on $\varSigma$ defined by $\mu \circ \phi^{-1}(\sigma) = \mu(\phi^{-1}(\sigma))$ for $\sigma \in \varSigma$. If $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ (briefly, $\mu \circ \phi^{-1} \ll \mu$), then the operator $C_\phi\colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\}, \quad C_\phi f = f \circ \phi \quad \text{for } f \in \mathcal{D}(C_\phi),$$

is well-defined and linear. Such an operator is called a *composition operator* induced by ϕ; we also say that ϕ is the *symbol* of C_ϕ. Set

$$h_n^\phi = \frac{d\mu \circ (\phi^n)^{-1}}{d\mu}, \quad n \in \mathbb{Z}_+, \tag{3.1}$$

where ϕ^n denotes the n-fold composition of ϕ with itself for $n \in \mathbb{N}$ and $\phi^0 = I_X =$ the identity transformation of X (as usual, $d\nu/\,d\mu$ stands for the Radon–Nikodym derivative of a measure ν with respect to a measure μ). Note that $h_0^\phi = 1$ a.e. $[\mu]$. Since $\mathcal{D}(C_\phi)$ equipped with the graph norm of C_ϕ coincides with the Hilbert space $L^2((1 + h_1^\phi)\,d\mu)$, we see that the operator C_ϕ is closed. Recall that C_ϕ is bounded on $L^2(\mu)$ if and only if $h_1^\phi \in L^\infty(\mu)$. If ψ is a $\varSigma$-measurable transformation of X such that the mapping $L^2(\mu) \ni f \mapsto f \circ \psi \in L^2(\mu)$ is well-defined, then $\mu \circ \psi^{-1} \ll \mu$ and

$$\|C_\psi\| = \|h_1^\psi\|_\infty^{1/2}, \tag{3.2}$$

where $\|\cdot\|_\infty$ stands for the $L^\infty(\mu)$-norm. The interested reader is referred to [10], [28] and [38] for further information on composition operators.

If Y is a nonempty set and $f, g\colon X \to Y$ are arbitrary functions, then "$f = g$ a.e. $[\mu]$" (or "$f(x) = g(x)$ for μ-a.e. $x \in X$") means that there exists a set $\sigma \in \varSigma$ such that $\mu(X \setminus \sigma) = 0$ and $f(x) = g(x)$ for all $x \in \sigma$.

Henceforth by a *$\varSigma$-bimeasurable* transformation of X we mean a $\varSigma$-measurable transformation $\phi\colon X \to X$ such that $\phi(\sigma) \in \varSigma$ for every $\sigma \in \varSigma$. Note that if ϕ is an injective $\varSigma$-bimeasurable transformation of X, then the mapping $\varSigma \ni \sigma \mapsto \mu(\phi(\sigma)) \in [0, \infty]$, denoted by $\mu \circ \phi$, is a measure. The following simple lemma is stated without proof.

LEMMA 3.1. *Suppose that $\phi\colon X \to X$ is a $\varSigma$-measurable transformation, Y is a nonempty set and $f, g\colon X \to Y$ are arbitrary functions.*

(i) *If $\mu \circ \phi^{-1} \ll \mu$ and $f = g$ a.e. $[\mu]$, then $f \circ \phi = g \circ \phi$ a.e. $[\mu]$.*
(ii) *If ϕ is injective and $\varSigma$-bimeasurable, $\mu \circ \phi \ll \mu$, $\mu(X \setminus \phi(X)) = 0$ and $f \circ \phi = g \circ \phi$ a.e. $[\mu]$, then $f = g$ a.e. $[\mu]$.*

We now investigate under what conditions on ϕ and ψ the equality $C_\phi = C_\psi$ holds. Recall that it is not true in general that $C_\phi = C_\psi$ implies $\phi = \psi$ a.e. $[\mu]$; note also that the set $\{x \in X\colon \phi(x) \neq \psi(x)\}$ may not belong to $\varSigma$ though ϕ and ψ are $\varSigma$-measurable (cf. [4, Example 3.2]).

LEMMA 3.2. *Suppose that $\phi, \phi', \psi, \psi' \colon X \to X$ are Σ-measurable transformations of X.*

(i) *If $\phi = \psi$ a.e. $[\mu]$, then $\mu \circ \phi^{-1} = \mu \circ \psi^{-1}$.*
(ii) *If $\phi = \psi$ a.e. $[\mu]$, $\phi' = \psi'$ a.e. $[\mu]$ and $\mu \circ \phi^{-1} \ll \mu$ (equivalently, $\mu \circ \psi^{-1} \ll \mu$), then $\phi' \circ \phi = \psi' \circ \psi$ a.e. $[\mu]$.*
(iii) *If $\phi = \psi$ a.e. $[\mu]$ and $\mu \circ \phi^{-1} \ll \mu$, then $\mu \circ \psi^{-1} \ll \mu$ and $C_\phi = C_\psi$; moreover, $\phi^n = \psi^n$ a.e. $[\mu]$, $\mu \circ (\phi^n)^{-1} = \mu \circ (\psi^n)^{-1}$ and $h_n^\phi = h_n^\psi$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$.*

Proof. (i) Suppose that $\phi = \psi$ on Y, where $Y \in \Sigma$ and $\mu(X \setminus Y) = 0$. Then

$$\mu(\phi^{-1}(\sigma)) = \mu(Y \cap \phi^{-1}(\sigma)) = \mu(Y \cap \psi^{-1}(\sigma)) = \mu(\psi^{-1}(\sigma)), \quad \sigma \in \Sigma.$$

(ii) If $\phi' = \psi'$ on Z, where $Z \in \Sigma$ and $\mu(X \setminus Z) = 0$, then $\phi' \circ \phi = \psi' \circ \psi$ on $Y \cap \phi^{-1}(Z)$. Since $\mu \circ \phi^{-1} \ll \mu$, we get $\phi' \circ \phi = \psi' \circ \psi$ a.e. $[\mu]$.

(iii) By (i), $\mu \circ \psi^{-1} \ll \mu$ and $C_\phi = C_\psi$ (because $f \circ \phi = f \circ \psi$ on Y for all $f \in \mathbb{C}^X$). It follows from (ii) that $\phi^n = \psi^n$ a.e. $[\mu]$ for all $n \in \mathbb{Z}_+$. By (i), this completes the proof. ■

The following useful fact extends [4, Lemma 3.1]. Below, $\triangle$ stands for symmetric difference of sets.

LEMMA 3.3. *Suppose that ϕ and ψ are Σ-measurable transformations of X inducing bounded composition operators on $L^2(\mu)$.*

(i) *If $C_\phi = C_\psi$, then $\mu \circ (\phi^n)^{-1} = \mu \circ (\psi^n)^{-1}$ and $h_n^\phi = h_n^\psi$ a.e. $[\mu]$ for every $n \in \mathbb{Z}_+$.*
(ii) *$C_\phi \neq C_\psi$ if and only if there exist $Y, Z \in \Sigma$ such that $Y \cap Z = \emptyset$ and $\mu(\phi^{-1}(Y) \cap \psi^{-1}(Z)) > 0$.*
(iii) *$C_\phi = C_\psi$ if and only if $\mu(\phi^{-1}(\sigma) \triangle \psi^{-1}(\sigma)) = 0$ for every $\sigma \in \Sigma_\mu$, or equivalently if $\mu(\phi^{-1}(\sigma) \cap \tau) = \mu(\psi^{-1}(\sigma) \cap \tau)$ for all $\sigma, \tau \in \Sigma_\mu$.*
(iv) *$C_\phi = C_\psi$ if and only if for every Σ-measurable function $g \colon X \to \mathbb{C}$,*

$$g \circ \phi = g \circ \psi \qquad \text{a.e. } [\mu]. \tag{3.3}$$

Proof. (i) and (ii) follow from [4, Lemma 3.1].

(iii) If $C_\phi = C_\psi$, then the characteristic function χ_σ of every $\sigma \in \Sigma_\mu$ is in $L^2(\mu)$ and $\chi_\sigma \circ \phi = \chi_\sigma \circ \psi$ a.e. $[\mu]$. Hence

$$\mu(\phi^{-1}(\sigma) \triangle \psi^{-1}(\sigma)) = \int_X |\chi_\sigma \circ \phi - \chi_\sigma \circ \psi|^2 \, d\mu = 0, \quad \sigma \in \Sigma_\mu. \tag{3.4}$$

Conversely, if (3.4) holds, then $C_\phi = C_\psi$ on a dense subset of $L^2(\mu)$ consisting of simple functions and consequently $C_\phi = C_\psi$. Since $C_\phi = C_\psi$ if and only if $\langle C_\phi \chi_\sigma, \chi_\tau \rangle = \langle C_\psi \chi_\sigma, \chi_\tau \rangle$ for all $\sigma, \tau \in \Sigma_\mu$, we get (iii).

(iv) It is enough to show that $C_\phi = C_\psi$ implies (3.3). Since μ is σ-finite, for each Σ-measurable function $g\colon X \to \mathbb{C}$, there exists a sequence $\{g_n\}_{n=1}^\infty \subseteq L^2(\mu)$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ for every $x \in X$ (e.g. $g_n := g \cdot \chi_{Y_n}$, where $Y_n = \sigma_n \cap \{x \in X\colon |g(x)| \le n\}$ and $\{\sigma_k\}_{k=1}^\infty \subseteq \Sigma_\mu$ is an ascending sequence such that $X = \bigcup_{k=1}^\infty \sigma_k$). As $C_\phi = C_\psi$, for each $n \in \mathbb{N}$ there exists $Z_n \in \Sigma$ such that

$$\mu(X \setminus Z_n) = 0 \quad \text{and} \quad g_n(\phi(x)) = g_n(\psi(x)) \quad \text{for all } x \in Z_n.$$

Letting n tend to ∞, we get $g(\phi(x)) = g(\psi(x))$ for every $x \in Z := \bigcap_{n=1}^\infty Z_n$. Since $Z \in \Sigma$ and $\mu(X \setminus Z) = 0$, the proof is finished. ■

4. Injectivity. Proposition 4.1 below is folklore. For the reader's convenience we sketch its proof.

PROPOSITION 4.1. *Assume that ϕ is a Σ-measurable transformation of X inducing a bounded composition operator C_ϕ on $L^2(\mu)$. Set $Z_\phi = \{x \in X\colon h_1^\phi(x) = 0\}$ and $\Sigma(Z_\phi) = \{\sigma \in \Sigma\colon \sigma \subseteq X \setminus Z_\phi\}$. Then:*

(i) *$\mu \circ \phi^{-1} \ll \mu$ and $\mu|_{\Sigma(Z_\phi)} \ll \mu \circ \phi^{-1}|_{\Sigma(Z_\phi)}$,*
(ii) *$\mathcal{N}(C_\phi) = \chi_{Z_\phi} \cdot L^2(\mu)$, where $\mathcal{N}(C_\phi)$ stands for the kernel of C_ϕ,*
(iii) *$\mathcal{N}(C_\phi) = \{0\}$ if and only if the measures $\mu\circ\phi^{-1}$ and μ are mutually absolutely continuous, or equivalently if $h_1^\phi > 0$ a.e. $[\mu]$,*
(iv) *if $\phi(X) \in \Sigma$, then $h_1^\phi = 0$ on $X \setminus \phi(X)$ a.e. $[\mu]$,*
(v) *if ϕ is injective and Σ-bimeasurable, then $C_\phi(L^2(\mu))$ is dense in $L^2(\mu)$,*
(vi) *if ϕ is injective and Σ-bimeasurable, and $\mu \circ \phi \ll \mu$, then*

$$h_{-1}^\phi \circ \phi^{-1} \cdot h_1^\phi = 1 \qquad \text{on } \phi(X) \text{ a.e. } [\mu], \tag{4.1}$$

where $h_{-1}^\phi = d\mu \circ \phi / d\mu$,
(vii) *if ϕ is injective and Σ-bimeasurable, $\mu\circ\phi \ll \mu$ and $\mu(X\setminus\phi(X)) = 0$, then $\mathcal{N}(C_\phi) = \{0\}$, the operator C_ϕ^{-1} is closed and densely defined, and $C_\phi^{-1} = C_{\phi^{-1}}$.*

Note that the set Z_ϕ (and hence $\Sigma(Z_\phi)$) is determined up to a.e. $[\mu]$ equivalence.

Proof of Proposition 4.1. The conditions [1] (i)–(iii) can be deduced from the definition of h_1^ϕ and the well-known equality (which in turn is a consequence of the measure transport theorem [19, Theorem C, p. 163])

$$\|C_\phi f\|^2 = \int_X |f|^2 h_1^\phi \, d\mu, \qquad f \in L^2(\mu).$$

[1] The condition (iii) also follows from [38, Theorem 2.2.2].

(iv) is a consequence of the equalities

$$0 = \mu(\phi^{-1}(X \setminus \phi(X))) = \int_{X\setminus\phi(X)} h_1^{\phi}\, d\mu.$$

(v) According to [20, p. 126] (see also [38, Theorem 2.2.6]), the closure of the range of C_ϕ consists of all $\phi^{-1}(\Sigma)$-measurable members of $L^2(\mu)$. This fact combined with the equality $f = f \circ \phi^{-1} \circ \phi$, which is valid for every $f \in L^2(\mu)$, gives the conclusion of (v).

(vi) It follows from the measure transport theorem that

$$\begin{aligned}\mu(\sigma) = \mu(\phi(\phi^{-1}(\sigma))) &= \int_X \chi_\sigma \circ \phi \cdot h_{-1}^{\phi} \circ \phi^{-1} \circ \phi\, d\mu \\ &= \int_{\phi(X)} \chi_\sigma \cdot h_{-1}^{\phi} \circ \phi^{-1}\, d\mu \circ \phi^{-1} \\ &= \int_\sigma h_{-1}^{\phi} \circ \phi^{-1} \cdot h_1^{\phi}\, d\mu, \qquad \sigma \in \Sigma,\ \sigma \subseteq \phi(X).\end{aligned}$$

By σ-finiteness of μ this implies (4.1).

(vii) Injectivity of C_ϕ is a direct consequence of Lemma 3.1(ii). Since C_ϕ is closed so is its inverse. The operator C_ϕ^{-1} is densely defined due to (v). In turn, because $\mu(X \setminus \phi(X)) = 0$ and $\mu \circ \phi \ll \mu$, the composition operator $C_{\phi^{-1}}$ is well-defined. Finally, the equality $C_\phi^{-1} = C_{\phi^{-1}}$ can be proved in a standard way. ∎

It follows from Proposition 4.1(iii) that a bounded composition operator C_ϕ on $L^2(\mu)$, which is induced by a bijective and Σ-bimeasurable transformation ϕ, is injective if and only if $\mu \circ \phi \ll \mu$; moreover, if this is the case, then, by Lemma 3.1(i) and Proposition 4.1(vi), (vii), we see that $h_{-1}^{\phi} = 1/h_1^{\phi} \circ \phi$ a.e. $[\mu]$, the operator C_ϕ^{-1} is closed and densely defined, and $C_\phi^{-1} = C_{\phi^{-1}}$.

Example 4.2. Let X be an infinite countable set. Decompose it into a disjoint countable union $X = \bigsqcup_{n=-1}^{\infty} X_n$ of infinite sets. Consider a bijection $\phi\colon X \to X$ such that $\phi(X_{-1}) = X_{-1} \cup X_0$ and $\phi(X_n) = X_{n+1}$ for all $n \geq 0$. Put $\Sigma = 2^X$. Let $\mu\colon \Sigma \to \mathbb{R}_+$ be a measure such that $\mu(X_{-1}) = 0$ and $\mu(\{x\}) > 0$ for all $x \in X \setminus X_{-1}$. Then ϕ is Σ-bimeasurable, $\mu \circ \phi^{-1} \ll \mu$, $\mu \circ \phi \not\ll \mu$, $h_1^{\phi}(x) = 0$ for $x \in X_0$ and $h_1^{\phi}(x) = \mu(\phi^{-1}(\{x\}))/\mu(\{x\}) > 0$ for $x \in X \setminus (X_{-1} \cup X_0)$. Assume that ϕ induces a bounded composition operator on $L^2(\mu)$ (by (3.2) this assumption has no influence on whether μ is finite or not). Clearly $\mu(Z_\phi) > 0$. In view of Proposition 4.1(ii), $\dim \mathcal{N}(C_\phi) = \infty$. By Proposition 4.1(v), the range of C_ϕ is dense in $L^2(\mu)$.

5. Subnormality and almost surjectivity. Suppose that

(5.1) (X, Σ, μ) is a σ-finite measure space with $\mu \neq 0$ and $\boldsymbol{\phi} = \{\phi_t\}_{t\in\mathbb{R}_+}$ is a family of Σ-measurable transformations of X such that every ϕ_t induces a bounded composition operator C_{ϕ_t} on $L^2(\mu)$, and $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is a C_0-semigroup.

Set

$$h_t^{\boldsymbol{\phi}} = \frac{d\mu\circ\phi_t^{-1}}{d\mu}, \qquad t\in\mathbb{R}_+. \tag{5.2}$$

Note that (cf. [4, Section 4])

$$h_0^{\boldsymbol{\phi}} = 1 \text{ a.e. } [\mu], \qquad h_n^{\phi_t} = h_{nt}^{\boldsymbol{\phi}} \text{ a.e. } [\mu] \text{ for all } t\in\mathbb{R}_+ \text{ and } n\in\mathbb{Z}_+. \tag{5.3}$$

It is clear that for each $t\in\mathbb{R}_+$ the function $h_t^{\boldsymbol{\phi}}$ can be redefined on a set of measure zero (depending on t) without affecting the validity of (5.2), which may improve the properties of the function $t\mapsto h_t^{\boldsymbol{\phi}}(x)$ (cf. [4, Theorem 4.5]).

By (5.3) and Lambert's criterion (cf. [24]), the operator C_{ϕ_t} (with t fixed) is subnormal if and only if for μ-a.e. $x\in X$, there exists a unique probability Borel measure ϑ_x^t on $\mathbb{R}_+$ such that

$$h_{nt}^{\boldsymbol{\phi}}(x) = \int_0^\infty s^n\, d\vartheta_x^t(s), \qquad n\in\mathbb{Z}_+. \tag{5.4}$$

In fact, for μ-a.e. $x\in X$, the closed support of ϑ_x^t is contained in $[0, \|C_{\phi_t}\|^2]$. Moreover, according to (5.3) and (5.4), for μ-a.e. $x\in X$ the closed support of ϑ_x^0 equals $\{1\}$. If the C_0-semigroup $\{C_{\phi_s}\}_{s\in\mathbb{R}_+}$ is subnormal, then by [4, Lemma 4.4] we have

$$\vartheta_x^s(\{0\}) = 0 \quad \text{for } \mu\text{-a.e. } x\in X \text{ and for all } s\in\mathbb{R}_+. \tag{5.5}$$

We now show that under certain circumstances the family $\{\phi_t(X)\}_{t\in\mathbb{R}_+}$ may have a kind of monotonicity property. Below, μ_* stands for the inner measure induced by μ, i.e.

$$\mu_*(\tau) = \sup\{\mu(\sigma)\colon \sigma\in\Sigma,\ \sigma\subseteq\tau\}, \qquad \tau\subseteq X.$$

PROPOSITION 5.1. *If* (5.1) *holds,* $t\in\mathbb{R}_+$ *and* $\mu(\phi_t(\sigma)) = 0$ *for every* $\sigma\in\Sigma$ *such that* $\phi_t(\sigma)\in\Sigma$ *and* $\mu(\sigma)=0$, *then*

$$\mu_*(\phi_t(X)\setminus\phi_s(X)) = 0, \qquad 0\le s\le t. \tag{5.6}$$

Proof. Take $\sigma\in\Sigma_\mu$ such that $\sigma\subseteq\phi_t(X)\setminus\phi_s(X)$. Then $\chi_\sigma\in L^2(\mu)$ and

$$\|C_{\phi_s}(\chi_\sigma)\|^2 = \mu(\phi_s^{-1}(\sigma)) = 0,$$

which implies that

$$\mu(\phi_t^{-1}(\sigma)) = \|C_{\phi_t}(\chi_\sigma)\|^2 = \|C_{\phi_{t-s}}(C_{\phi_s}(\chi_\sigma))\|^2 = 0. \tag{5.7}$$

Since $\sigma \subseteq \phi_t(X)$, we get $\sigma = \phi_t(\phi_t^{-1}(\sigma))$. Hence (5.7) and our assumption on μ imply that $\mu(\sigma) = 0$. Thus, by σ-finiteness of μ, the proof of (5.6) is finished. ∎

In the case when the operator C_{ϕ_t} is subnormal for some $t > 0$, all the transformations ϕ_u, $u \in \mathbb{R}_+$, have to be "almost surjective".

PROPOSITION 5.2. *Suppose that* (5.1) *holds. Then*

$$\mu_*(X \setminus \phi_0(X)) = 0.$$

If C_{ϕ_t} is subnormal for some real $t > 0$ and $\vartheta_x^t(\{0\}) = 0$ for μ-a.e. $x \in X$, then

$$\mu_*(X \setminus \phi_u(X)) = 0, \quad u \in \mathbb{R}_+. \tag{5.8}$$

In particular, (5.8) *holds if the C_0-semigroup $\{C_{\phi_s}\}_{s\in\mathbb{R}_+}$ is subnormal.*

Proof. Clearly, C_{ϕ_0}, being the identity on $L^2(\mu)$, is subnormal and $\vartheta_x^0(\{0\}) = 0$ for μ-a.e. $x \in X$. Thus, the proof reduces to showing that (5.8) holds for each $u \in \mathbb{R}_+$ of the form $u = st$, where $s,t \in \mathbb{R}_+$, C_{ϕ_t} is subnormal and $\vartheta_x^t(\{0\}) = 0$ for μ-a.e. $x \in X$.

Take $\sigma \in \Sigma_\mu$ such that $\sigma \subseteq X \setminus \phi_{st}(X)$. Then

$$\|C_{\phi_{st}}(\chi_\sigma)\|^2 = \mu(\phi_{st}^{-1}(\sigma)) = 0. \tag{5.9}$$

Let n be an integer with $n \geq s$. Then (5.9) leads to

$$\int_\sigma h_{nt}^{\boldsymbol{\phi}}\, d\mu = \|C_{\phi_{nt}}(\chi_\sigma)\|^2 = \|C_{\phi_{(n-s)t}}(C_{\phi_{st}}(\chi_\sigma))\|^2 = 0.$$

This in turn implies that

$$h_{nt}^{\boldsymbol{\phi}}(x) = 0 \quad \text{for all integers } n \geq s \text{ and for } \mu\text{-a.e. } x \in \sigma.$$

Hence, by (5.4), $\vartheta_x^t((0,\infty)) = 0$ for μ-a.e. $x \in \sigma$. Since $\vartheta_x^t(\{0\}) = 0$ for μ-a.e. $x \in X$, we see that $\vartheta_x^t(\mathbb{R}_+) = 0$ for μ-a.e. $x \in \sigma$. As $\vartheta_x^t(\mathbb{R}_+) = 1$ for μ-a.e. $x \in X$, we conclude that $\mu(\sigma) = 0$. Hence the σ-finiteness of μ implies (5.8). Because each subnormal C_0-semigroup $\{C_{\phi_s}\}_{s\in\mathbb{R}_+}$ satisfies condition (5.5), the proof is complete. ∎

COROLLARY 5.3. *Suppose that* (5.1) *holds and that the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal. Assume moreover that*

(i) $\phi_{s+t}(X) = \phi_s(\phi_t(X))$ *for all* $s,t \in \mathbb{R}_+$,
(ii) $\phi_n(X) \in \Sigma$ *for all* $n \in \mathbb{N}$.

Then $\bigcap_{t\in\mathbb{R}_+} \phi_t(X) \in \Sigma$ and $\mu(X \setminus \bigcap_{t\in\mathbb{R}_+} \phi_t(X)) = 0$.

Proof. By (i), $\phi_t(X) \subseteq \phi_s(X)$ for all $s,t \in \mathbb{R}_+$ such that $s \leq t$. Hence (ii) implies that

$$\bigcap_{t\in\mathbb{R}_+} \phi_t(X) = \bigcap_{n=1}^{\infty} \phi_n(X) \in \Sigma. \tag{5.10}$$

By Proposition 5.2, $\mu(X \setminus \phi_n(X)) = 0$ for all $n \in \mathbb{N}$. Thus, (5.10) yields

$$\mu\Big(X \setminus \bigcap_{t\in\mathbb{R}_+} \phi_t(X)\Big) = \mu\Big(\bigcup_{n=1}^{\infty}(X \setminus \phi_n(X))\Big) = 0. \blacksquare$$

Example 5.4. Let $X = \mathbb{R}_+$, $\varSigma$ be the σ-algebra of all Borel subsets of $\mathbb{R}_+$, μ be the Lebesgue measure on $\mathbb{R}_+$ and $\phi_t(s) = s + t$ for $s, t \in \mathbb{R}_+$. Then $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is a contractive C_0-semigroup of composition operators. By Corollary 5.3, it is not subnormal. Moreover, for every $t \in \mathbb{R}_+$, the transformation ϕ_t is injective and $\varSigma$-bimeasurable, and $\mu \circ \phi_t \ll \mu$.

6. The bimeasurable case. The reader should be aware that if (5.1) is valid, then for every $u \in \mathbb{R}_+$, the density function $h_u^{\boldsymbol{\phi}}$ is determined up to a.e. $[\mu]$ equivalence. Since $\mu \circ \phi_t^{-1} \ll \mu$ and $\mu \circ \phi_s^{-1} \circ \phi_t^{-1} \ll \mu$, also $h_u^{\boldsymbol{\phi}} \circ \phi_t$ and $h_u^{\boldsymbol{\phi}} \circ \phi_t \circ \phi_s$ are determined up to a.e. $[\mu]$ equivalence for all $s, t, u \in \mathbb{R}_+$. The same holds for $x \mapsto \vartheta_x^u$, $x \mapsto \vartheta_{\phi_t(x)}^u$ and $x \mapsto \vartheta_{\phi_t(\phi_s(x))}^u$. These facts will be used without further comments throughout the paper.

We begin by showing a kind of semigroup property for $\{\phi_t\}_{t\in\mathbb{R}_+}$.

Lemma 6.1. *Suppose that* (5.1) *holds and* $s, t, u \in \mathbb{R}_+$. *Then*

$$h_u^{\boldsymbol{\phi}} \circ \phi_s \circ \phi_t = h_u^{\boldsymbol{\phi}} \circ \phi_{s+t} \quad \text{a.e. } [\mu]. \tag{6.1}$$

Moreover, if the composition operator C_{ϕ_1} *is subnormal, then*

$$\vartheta^1_{\phi_s(\phi_t(x))} = \vartheta^1_{\phi_{s+t}(x)} \quad \text{for } \mu\text{-a.e. } x \in X. \tag{6.2}$$

Proof. Since $C_{\phi_s\circ\phi_t} = C_{\phi_t}C_{\phi_s} = C_{\phi_{s+t}}$, we infer (6.1) from Lemma 3.3(iv). If C_{ϕ_1} is subnormal, then (6.1) yields

$$h_m^{\boldsymbol{\phi}}(\phi_s(\phi_t(x))) = h_m^{\boldsymbol{\phi}}(\phi_{s+t}(x)) \quad \text{for all } m \in \mathbb{Z}_+ \text{ and } \mu\text{-a.e. } x \in X.$$

Together with (5.4), this implies that for μ-a.e. $x \in X$,

$$\int_0^\infty r^m \, d\vartheta^1_{\phi_s(\phi_t(x))}(r) = \int_0^\infty r^m \, d\vartheta^1_{\phi_{s+t}(x)}(r) \quad \text{for all } m \in \mathbb{Z}_+. \tag{6.3}$$

Since for μ-a.e. $x \in X$, the Borel measures $\vartheta^1_{\phi_s(\phi_t(x))}$ and $\vartheta^1_{\phi_{s+t}(x)}$ are compactly supported, we infer (6.2) from (6.3). This completes the proof. $\blacksquare$

Next we describe the relationship between $h_{s+t}^{\boldsymbol{\phi}}$, $h_t^{\boldsymbol{\phi}}$ and $h_s^{\boldsymbol{\phi}}$.

Lemma 6.2. *Suppose that* (5.1) *holds and* $t \in \mathbb{R}_+$. *If the transformation* ϕ_t *is injective and* $\varSigma$*-bimeasurable, then for every* $s \in \mathbb{R}_+$,

$$h_{s+t}^{\boldsymbol{\phi}}(\phi_t(x)) = h_t^{\boldsymbol{\phi}}(\phi_t(x)) \cdot h_s^{\boldsymbol{\phi}}(x) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{6.4}$$

Proof. Fix $s \in \mathbb{R}_+$. Applying Lemma 3.3(i) to the equalities $C_{\phi_t \circ \phi_s} = C_{\phi_s} C_{\phi_t} = C_{\phi_{s+t}}$, we get

$$\mu(\phi_{s+t}^{-1}(\sigma)) = \mu(\phi_s^{-1}(\phi_t^{-1}(\sigma))), \quad \sigma \in \Sigma. \tag{6.5}$$

According to our assumption, $\phi_t(X) \in \Sigma$ and $\phi_t \colon X \to \phi_t(X)$ is a bijective Σ-bimeasurable transformation. Hence the measure transport theorem yields

$$\begin{aligned}\int_\sigma h_{s+t}^{\boldsymbol{\phi}} \circ \phi_t \, d\mu &= \int_{\phi_t(X)} \chi_\sigma \circ \phi_t^{-1} \, h_{s+t}^{\boldsymbol{\phi}} \, d\mu \circ \phi_t^{-1} \\ &= \int_{\phi_t(X)} \chi_\sigma \circ \phi_t^{-1} \, h_{s+t}^{\boldsymbol{\phi}} h_t^{\boldsymbol{\phi}} \, d\mu = \int_{\phi_t(X)} \chi_\sigma \circ \phi_t^{-1} \, h_t^{\boldsymbol{\phi}} \, d\mu \circ \phi_{s+t}^{-1} \\ &\overset{(6.5)}{=} \int_{\phi_t(X)} \chi_\sigma \circ \phi_t^{-1} \, h_t^{\boldsymbol{\phi}} \, d\mu \circ \phi_s^{-1} \circ \phi_t^{-1} = \int_\sigma h_t^{\boldsymbol{\phi}} \circ \phi_t \, d\mu \circ \phi_s^{-1} \\ &= \int_\sigma h_t^{\boldsymbol{\phi}} \circ \phi_t \, h_s^{\boldsymbol{\phi}} \, d\mu, \quad \sigma \in \Sigma,\end{aligned}$$

which, together with σ-finiteness of μ, completes the proof. ■

We now distinguish two necessary conditions for a C_0-semigroup of composition operators to be subnormal. As will be shown in Theorem 6.5, they are also sufficient provided the C_0-semigroup in question is more regular.

For $t \in \mathbb{R}_+$, we define the function $\xi_t \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\xi_t(s) = s^t, \quad s \in \mathbb{R}_+ \quad (\text{with } 0^0 = 1).$$

LEMMA 6.3. *Assume that* (5.1) *holds and the C_0-semigroup $\{C_{\phi_s}\}_{s \in \mathbb{R}_+}$ is subnormal. Then*

(a) $\vartheta_x^1(\{0\}) = 0$ *for μ-a.e. $x \in X$.*

Moreover, if $t \in \mathbb{R}_+$ and the transformation ϕ_t is injective and Σ-bimeasurable, then

(b) $\xi_t \, d\vartheta_{\phi_t(x)}^1 = h_t^{\boldsymbol{\phi}}(\phi_t(x)) \, d\vartheta_x^1$ *for μ-a.e. $x \in X$.*

Proof. (a) Apply [4, Lemma 4.4(v)].

(b) It follows from [4, Lemma 4.4(vii)] that

$$h_r^{\boldsymbol{\phi}}(x) = \int_0^\infty s^r \, d\vartheta_x^1(s) \quad \text{for } \mu\text{-a.e. } x \in X \text{ and every } r \in \mathbb{R}_+. \tag{6.6}$$

Hence Lemma 3.1(i) implies that for μ-a.e. $x \in X$,

$$\begin{aligned}\int_0^\infty s^m h_t^{\boldsymbol{\phi}}(\phi_t(x)) \, d\vartheta_x^1(s) &\overset{(6.6)}{=} h_m^{\boldsymbol{\phi}}(x) h_t^{\boldsymbol{\phi}}(\phi_t(x)) \overset{(6.4)}{=} h_{m+t}^{\boldsymbol{\phi}}(\phi_t(x) \\ &\overset{(6.6)}{=} \int_0^\infty s^m \xi_t(s) \, d\vartheta_{\phi_t(x)}^1(s), \quad m \in \mathbb{Z}_+.\end{aligned} \tag{6.7}$$

Since for μ-a.e. $x \in X$, the measures $d\vartheta^1_x$ and $\xi_t \, d\vartheta^1_{\phi_t(x)}$ are finite and compactly supported, we can infer (b) from (6.7). ∎

REMARK 6.4. Regarding Lemmata 6.2 and 6.3, note that if (5.1) holds and the C_0-semigroup $\{C_{\phi_u}\}_{u\in\mathbb{R}_+}$ is subnormal, then for every $t \in \mathbb{R}_+$, the following two conditions are equivalent (we do not assume that the transformation ϕ_t is injective and $\varSigma$-bimeasurable):

(i) $\xi_t \, d\vartheta^1_{\phi_t(x)} = h^{\boldsymbol{\phi}}_t(\phi_t(x)) \, d\vartheta^1_x$ for μ-a.e. $x \in X$,

(ii) $h^{\boldsymbol{\phi}}_{s+t}(\phi_t(x)) = h^{\boldsymbol{\phi}}_t(\phi_t(x)) \cdot h^{\boldsymbol{\phi}}_s(x)$ for μ-a.e. $x \in X$ and for every $s \in \mathbb{R}_+$.

For the idea of the proof see Lemma 7.1. That (ii) implies (i) is essentially shown in the proof of Lemma 6.3(b).

We are now in a position to state a new criterion for subnormality of C_0-semigroups of composition operators induced by injective $\varSigma$-bimeasurable transformations. It seems to be simpler to use (though less general) than the criterion given in [4, Lemma 4.4]. The reason is that we do not impose any requirement of moment type on the operators $\{C_{\phi_{1/k}}\}_{k=2}^{\infty}$, assuming instead a kind of consistency condition.

THEOREM 6.5. *Assume that* (5.1) *holds and for every $k \in \mathbb{N}$, the transformation $\phi_{1/k}$ is injective and $\varSigma$-bimeasurable*, *and $\mu \circ \phi_{1/k} \ll \mu$. Then the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal if and only if the operator C_{ϕ_1} is subnormal*, *condition* (a) *of Lemma* 6.3 *is satisfied and*

(b*) $\xi_{1/k} \, d\vartheta^1_{\phi_{1/k}(x)} = h^{\boldsymbol{\phi}}_{1/k}(\phi_{1/k}(x)) \, d\vartheta^1_x$ *for μ-a.e. $x \in X$ and every $k \in \mathbb{N}$.*

Proof. In view of Lemma 6.3, it is enough to prove the "if" part. We split the proof into three steps.

STEP 1: There is no loss of generality in assuming that for every $t \in \mathbb{Q}_+$, the transformation ϕ_t is injective and $\varSigma$-bimeasurable, $\mu \circ \phi_t \ll \mu$ and $\phi_0 = I_X$.

Indeed, if this is not the case, then we define a new family $\{\widetilde{\phi}_t\}_{t\in\mathbb{R}_+}$ of $\varSigma$-measurable transformations of X by

$$\widetilde{\phi}_t = \begin{cases} I_X & \text{for } t = 0, \\ (\phi_{1/k})^j & \text{for } t = j/k \text{ where } j, k \in \mathbb{N} \text{ are relatively prime}, \\ \phi_t & \text{for } t \in \mathbb{R}_+ \setminus \mathbb{Q}_+. \end{cases} \tag{6.8}$$

Note that $\widetilde{\phi}_{1/k} = \phi_{1/k}$ for all $k \in \mathbb{N}$. It is easily seen that for every $t \in \mathbb{Q}_+$, the transformation $\widetilde{\phi}_t$ is injective and $\varSigma$-bimeasurable, and $\mu \circ \widetilde{\phi}_t \ll \mu$. According to (6.8) and (5.1), we have

$$C_{\widetilde{\phi}_{j/k}} = C_{(\phi_{1/k})^j} = C^j_{\phi_{1/k}} = C_{\phi_{j/k}} \quad \text{for all } j, k \in \mathbb{N} \text{ relatively prime},$$

which, together with Lemma 3.3(i), implies $h_t^{\boldsymbol{\phi}} = h_t^{\widetilde{\boldsymbol{\phi}}}$ a.e. $[\mu]$ for all $t \in \mathbb{R}_+$. This means that the C_0-semigroup $\{C_{\widetilde{\phi}_t}\}_{t\in\mathbb{R}_+}$ has all the desired properties.

We now strengthen condition (b*).

STEP 2: $\xi_{j/k}\, d\vartheta^1_{\phi_{j/k}(x)} = h^{\boldsymbol{\phi}}_{j/k}(\phi_{j/k}(x))\, d\vartheta^1_x$ for μ-a.e. $x \in X$ and all $j \in \mathbb{Z}_+$ and $k \in \mathbb{N}$.

Indeed, fix $k \in \mathbb{N}$. Since the case $j = 0$ follows from Step 1 and (5.3), we can assume that $j \geq 1$. In view of Lemma 3.1(i), we see that for μ-a.e. $x \in X$,

$$\begin{aligned}\xi_{j/k}\, d\vartheta^1_{\phi_{j/k}(x)} &\overset{(6.2)}{=} \xi_{(j-1)/k}(\xi_{1/k}\, d\vartheta^1_{\phi_{1/k}(\phi_{(j-1)/k}(x))})\\ &\overset{(\mathrm{b}^*)}{=} \xi_{(j-1)/k}(h^{\boldsymbol{\phi}}_{1/k}(\phi_{1/k}(\phi_{(j-1)/k}(x)))\, d\vartheta^1_{\phi_{(j-1)/k}(x)})\\ &\overset{(6.1)}{=} h^{\boldsymbol{\phi}}_{1/k}(\phi_{j/k}(x))(\xi_{(j-1)/k}\, d\vartheta^1_{\phi_{(j-1)/k}(x)}).\end{aligned}$$

Iterating the above procedure, we deduce that

$$\xi_{j/k}\, d\vartheta^1_{\phi_{j/k}(x)} = h^{\boldsymbol{\phi}}_{1/k}(\phi_{j/k}(x))h^{\boldsymbol{\phi}}_{1/k}(\phi_{(j-1)/k}(x))\cdots h^{\boldsymbol{\phi}}_{1/k}(\phi_{1/k}(x))\, d\vartheta^1_x \tag{6.9}$$

for μ-a.e. $x \in X$. We now use an induction argument on j to prove that

$$\overbrace{h^{\boldsymbol{\phi}}_{1/k}(\phi_{j/k}(x))h^{\boldsymbol{\phi}}_{1/k}(\phi_{(j-1)/k}(x))\cdots h^{\boldsymbol{\phi}}_{1/k}(\phi_{1/k}(x))}^{\Delta_j(x)} = h^{\boldsymbol{\phi}}_{j/k}(\phi_{j/k}(x)) \tag{6.10}$$

for μ-a.e. $x \in X$. Clearly (6.10) is true for $j = 1$. If (6.10) holds for a fixed $j \geq 1$, then by Lemma 3.1(i) we see that for μ-a.e. $x \in X$,

$$\begin{aligned}\Delta_{j+1}(x) = h^{\boldsymbol{\phi}}_{1/k}(\phi_{(j+1)/k}(x))\Delta_j(x) &\overset{(6.1)\&(6.10)}{=} h^{\boldsymbol{\phi}}_{1/k}(\phi_{1/k}(\phi_{j/k}(x)))h^{\boldsymbol{\phi}}_{j/k}(\phi_{j/k}(x))\\ \overset{(6.4)}{=} h^{\boldsymbol{\phi}}_{(j+1)/k}(\phi_{1/k}(\phi_{j/k}(x))) &\overset{(6.1)}{=} h^{\boldsymbol{\phi}}_{(j+1)/k}(\phi_{(j+1)/k}(x)),\end{aligned}$$

which completes the induction argument. Therefore Step 2 follows from (6.9) and (6.10).

STEP 3: Fix $k \in \mathbb{N}$. If $n \in \mathbb{Z}_+$, then there are $j, m \in \mathbb{Z}_+$ such that $n = mk + j$ and $0 \leq j < k$. Employing Step 1 and the measure transport theorem, we see that for μ-a.e. $y \in X$,

$$\begin{aligned}h^{\boldsymbol{\phi}}_{n/k}(\phi_{j/k}(y)) = h^{\boldsymbol{\phi}}_{m+j/k}(\phi_{j/k}(y)) &\overset{(6.4)}{=} h^{\boldsymbol{\phi}}_{j/k}(\phi_{j/k}(y))h^{\boldsymbol{\phi}}_m(y)\\ &\overset{(5.4)}{=} \int_0^\infty s^m h^{\boldsymbol{\phi}}_{j/k}(\phi_{j/k}(y))\, d\vartheta^1_y(s) \overset{\text{Step 2}}{=} \int_0^\infty s^{n/k}\, d\vartheta^1_{\phi_{j/k}(y)}(s)\\ &= \int_0^\infty s^n\, d(\vartheta^1_{\phi_{j/k}(y)} \circ \xi_k)(s).\end{aligned} \tag{6.11}$$

It follows from condition (a) of Lemma 6.3, Step 1 and Proposition 5.2 (with $t = 1$) that $\mu(X \setminus \phi_{j/k}(X)) = 0$. Step 1 and (6.11), when combined with Lemma 3.1(ii), now lead to

$$h^{\boldsymbol{\phi}}_{n/k}(x) = \int_0^\infty s^n \, d(\vartheta^1_x \circ \xi_k)(s) \quad \text{for all } n \in \mathbb{Z}_+ \text{ and } \mu\text{-a.e. } x \in X.$$

Summarizing, we have shown that for μ-a.e. $x \in X$ and each $k \in \mathbb{N}$, the sequence $\{h^{\boldsymbol{\phi}}_{n/k}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence. Therefore, by [4, Lemma 4.3], the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal. ∎

Remark 6.6. It is worth noticing that under the assumptions of Theorem 6.5, if the C_0-semigroup $\{C_{\phi_u}\}_{u\in\mathbb{R}_+}$ is subnormal, then each operator C_{ϕ_t} is injective and has dense range. Indeed, in view of Steps 1 and 3 of the proof of Theorem 6.5 we can assume that for every $s \in \mathbb{Q}_+$, the transformation ϕ_s is injective and Σ-bimeasurable, $\mu \circ \phi_s \ll \mu$ and $\mu(X \setminus \phi_s(X)) = 0$. It follows from Proposition 4.1(vii) that

$$C_{\phi_s} \text{ is injective and has dense range for all } s \in \mathbb{Q}_+. \tag{6.12}$$

In turn, the semigroup property of $\{C_{\phi_u}\}_{u\in\mathbb{R}_+}$ implies that

$$\text{if } 0 \le t \le s, \text{ then } \mathcal{N}(C_{\phi_t}) \subseteq \mathcal{N}(C_{\phi_s}) \text{ and } \mathcal{R}(C_{\phi_s}) \subseteq \mathcal{R}(C_{\phi_t}), \tag{6.13}$$

where $\mathcal{N}(C_{\phi_u})$ and $\mathcal{R}(C_{\phi_u})$ stand for the kernel and range of C_{ϕ_u} respectively. Combining (6.12) and (6.13) completes the proof.

7. The Laplace transform approach. Our next goal is to show how condition (b) of Lemma 6.3 can be translated into the language of the Laplace transform. Given a finite positive Borel measure ζ on $\mathbb{R}_+$, we define the function $\mathfrak{L}(\zeta)\colon \mathbb{R}_+ \to \mathbb{R}_+$, called the *Laplace transform* of ζ, by

$$\mathfrak{L}(\zeta)(t) = \int_0^\infty e^{-ts} \, d\zeta(s), \qquad t \in \mathbb{R}_+.$$

Denote by $\mathfrak{B}(\mathbb{R}_+)$ the σ-algebra of all Borel subsets of $\mathbb{R}_+$. If (5.1) holds and the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal, then by [4, Theorem 4.5] there exists a function $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ such that:

1° for every $x \in X$, $P(x,\cdot)$ is a probability Borel measure on $\mathbb{R}_+$,
2° for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, $P(\cdot,\sigma)$ is Σ-measurable,
3° for every $t \in \mathbb{R}_+$, the function $X \ni x \mapsto \mathfrak{L}(P(x,\cdot))(t) \in \mathbb{R}_+$ is Σ-measurable,
4° for μ-a.e. $x \in X$ and every $t \in \mathbb{R}_+$, $h^{\boldsymbol{\phi}}_t(x) = e^{\delta t}\mathfrak{L}(P(x,\cdot))(t)$, where $\delta := 2\log\|C_{\phi_1}\|$.

Since $L^2(\mu) \neq \{0\}$, Proposition 1 of [29] implies that $\delta \in \mathbb{R}$ and $e^{\delta t} = \|C_{\phi_t}\|^2$ for all $t \in \mathbb{R}_+$.

LEMMA 7.1. *Assume that* (5.1) *holds and the* C_0*-semigroup* $\{C_{\phi_u}\}_{u\in\mathbb{R}_+}$ *is subnormal. Let* $P\colon X\times\mathfrak{B}(\mathbb{R}_+)\to[0,1]$ *satisfy conditions* 1° *to* 4°. *Then for every* $t\in\mathbb{R}_+$, *the following conditions are equivalent*:

(i) $\mathcal{L}(\chi_\sigma P(\phi_t(x),\cdot))(t)=\mathcal{L}(P(\phi_t(x),\cdot))(t)\,P(x,\sigma)$ *for all* $\sigma\in\mathfrak{B}(\mathbb{R}_+)$ *and* μ*-a.e.* $x\in X$,

(ii) $h^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=h^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot h^{\boldsymbol{\phi}}_s(x)$ *for all* $s\in\mathbb{R}_+$ *and* μ*-a.e.* $x\in X$.

Proof. Set $H^{\boldsymbol{\phi}}_u(x)=e^{\delta u}\mathcal{L}(P(x,\cdot))(u)$ for $x\in X$ and $u\in\mathbb{R}_+$. Fix $t\in\mathbb{R}_+$. Denote by $\nu^{\mathrm{L}}_x(\sigma)$ (respectively, $\nu^{\mathrm{R}}_x(\sigma)$) the left-hand (respectively, right-hand) side of the equality in (i). Clearly, ν^{L}_x and ν^{R}_x are finite positive Borel measures on $\mathbb{R}_+$.

(i)⇒(ii). Let $X_0\in\Sigma$ be a set of full μ-measure such that $h^{\boldsymbol{\phi}}_u(x)=H^{\boldsymbol{\phi}}_u(x)$ and $\nu^{\mathrm{L}}_x=\nu^{\mathrm{R}}_x$ for all $u\in\mathbb{R}_+$ and $x\in X_0$. Integrating the functions $\mathbb{R}_+\ni u\mapsto e^{-su}\in\mathbb{R}_+$, $s\in\mathbb{R}_+$, with respect to ν^{L}_x and ν^{R}_x, we see that $H^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=H^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot H^{\boldsymbol{\phi}}_s(x)$ for all $s\in\mathbb{R}_+$ and $x\in X_0$. This implies that $h^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=h^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot h^{\boldsymbol{\phi}}_s(x)$ for all $s\in\mathbb{R}_+$ and $x\in X_0\cap\phi_t^{-1}(X_0)$. Since $\mu\circ\phi_t^{-1}\ll\mu$, we deduce that $X_0\cap\phi_t^{-1}(X_0)$ is a set of full μ-measure. Summarizing, we have shown that (ii) holds.

(ii)⇒(i). Let $X_0\in\Sigma$ be a set of full μ-measure such that $h^{\boldsymbol{\phi}}_s(x)=H^{\boldsymbol{\phi}}_s(x)$ and $h^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=h^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot h^{\boldsymbol{\phi}}_s(x)$ for all $s\in\mathbb{R}_+$ and $x\in X_0$. Then $H^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=H^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot H^{\boldsymbol{\phi}}_s(x)$ for all $s\in\mathbb{R}_+$ and $x\in Y_0:=X_0\cap\phi_t^{-1}(X_0)$. This implies that $e^{\delta t}\mathcal{L}(\nu^{\mathrm{L}}_x)=\mathcal{L}(H^{\boldsymbol{\phi}}_t(\phi_t(x))P(x,\cdot))$ for every $x\in Y_0$. Consequently, $e^{\delta t}\nu^{\mathrm{L}}_x=H^{\boldsymbol{\phi}}_t(\phi_t(x))P(x,\cdot)$ for every $x\in Y_0$ (cf. [43]). Since Y_0 is a set of full μ-measure, the proof is finished. ■

Now we rewrite condition (b) of Lemma 6.3 in terms of the Laplace transform. The reader should be aware of the difference between conditions (6.4) and (7.2), which lies in the order of the quantifiers "for every $s\in\mathbb{R}_+$" and "for μ-a.e. $x\in X$".

PROPOSITION 7.2. *Assume that* (5.1) *holds*, *the* C_0*-semigroup* $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ *is subnormal and each transformation* ϕ_t *is injective and* Σ*-bimeasurable. Let* $P\colon X\times\mathfrak{B}(\mathbb{R}_+)\to[0,1]$ *satisfy conditions* 1° *to* 4° *preceding Lemma* 7.1. *Then for every* $t\in\mathbb{R}_+$ *and* μ*-a.e.* $x\in X$,

$$\mathcal{L}(\chi_\sigma P(\phi_t(x),\cdot))(t)=\mathcal{L}(P(\phi_t(x),\cdot))(t)\,P(x,\sigma),\qquad \sigma\in\mathfrak{B}(\mathbb{R}_+),\tag{7.1}$$

$$h^{\boldsymbol{\phi}}_{s+t}(\phi_t(x))=h^{\boldsymbol{\phi}}_t(\phi_t(x))\cdot h^{\boldsymbol{\phi}}_s(x),\qquad s\in\mathbb{R}_+.\tag{7.2}$$

Proof. First we justify (7.1). By [4, Theorem 4.5], we have for μ-a.e. $x\in X$,

$$P(x,\sigma)=\vartheta^1_x(\omega^{-1}(\sigma)),\qquad \sigma\in\mathfrak{B}(\mathbb{R}_+),\tag{7.3}$$

where $\omega\colon (0,e^{\delta}] \to [0,\infty)$ is defined by $\omega(s) = \delta - \log s$. Fix $t \in \mathbb{R}_+$. Lemma 3.1(i), the measure transport theorem and Lemma 6.3(b) imply that for μ-a.e. $x \in X$,

$$\begin{aligned}\mathcal{L}(\chi_\sigma P(\phi_t(x),\cdot))(t) &\overset{(7.3)}{=} \int_0^\infty \chi_\sigma(s)e^{-ts}\, d\,(\vartheta^1_{\phi_t(x)} \circ \omega^{-1})(s)\\ &= e^{-\delta t} \int_{(0,e^\delta]} \chi_\sigma \circ \omega \cdot \xi_t\, d\vartheta^1_{\phi_t(x)}\\ &\overset{\text{(b)}}{=} e^{-\delta t} h^{\boldsymbol{\phi}}_t(\phi_t(x)) \int_{(0,e^\delta]} \chi_\sigma \circ \omega\, d\vartheta^1_x\\ &\overset{(7.3)}{=} \mathcal{L}(P(\phi_t(x),\cdot))(t)\, P(x,\sigma), \qquad \sigma \in \mathfrak{B}(\mathbb{R}_+).\end{aligned}$$

Applying Lemma 7.1 completes the proof. ∎

Motivated by Lemma 7.1 and Proposition 7.2, we propose yet another criterion for subnormality of C_0-semigroups of composition operators written in terms of the Laplace transform.

PROPOSITION 7.3. *Assume that* (5.1) *holds and for every* $k \in \mathbb{N}$, *the transformation* $\phi_{1/k}$ *is injective and* Σ*-bimeasurable, and* $\mu \circ \phi_{1/k} \ll \mu$. *Suppose also that* $\mu(X \setminus \phi_1(X)) = 0$ *and there exists* $\delta \in \mathbb{R}$ *and a family* $\{\zeta_x\}_{x\in X}$ *of probability Borel measures on* $\mathbb{R}_+$ *satisfying the following two conditions for* μ*-a.e.* $x \in X$:

(i) $h^{\boldsymbol{\phi}}_{1/k}(x) = e^{\delta/k}\mathcal{L}(\zeta_x)(1/k)$ *for all* $k \in \mathbb{N}$,
(ii) $\mathcal{L}(\chi_\sigma\zeta_{\phi_{1/k}(x)})(1/k) = \mathcal{L}(\zeta_{\phi_{1/k}(x)})(1/k)\cdot\zeta_x(\sigma)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ *and* $k \in \mathbb{N}$.

Then the C_0*-semigroup* $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ *is subnormal.*

Proof. Set $H^{\boldsymbol{\phi}}_u(x) = e^{\delta u}\mathcal{L}(\zeta_x)(u)$ for $x \in X$ and $u \in \mathbb{R}_+$. By (i), we have

$$h^{\boldsymbol{\phi}}_{1/k} = H^{\boldsymbol{\phi}}_{1/k} \quad \text{a.e. } [\mu], \qquad k \in \mathbb{N}. \tag{7.4}$$

As in the proof of Lemma 7.1, we infer from (ii) that

$$H^{\boldsymbol{\phi}}_{s+1/k} \circ \phi_{1/k} = H^{\boldsymbol{\phi}}_{1/k} \circ \phi_{1/k} \cdot H^{\boldsymbol{\phi}}_s \quad \text{a.e. } [\mu], \qquad k \in \mathbb{N},\ s \in \mathbb{R}_+. \tag{7.5}$$

We now show that for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$\begin{aligned}(\mu(X \setminus \phi_{1/k}(X)) = 0 \ \wedge\ h^{\boldsymbol{\phi}}_{n/k} = H^{\boldsymbol{\phi}}_{n/k} \text{ a.e. } [\mu])&\\ \Rightarrow\ h^{\boldsymbol{\phi}}_{(n+1)/k} = H^{\boldsymbol{\phi}}_{(n+1)/k} \text{ a.e. } [\mu].&\end{aligned} \tag{7.6}$$

Indeed, by Lemma 3.1(i), we have

$$h^{\phi}_{(n+1)/k} \circ \phi_{1/k} \overset{(6.4)}{=} h^{\phi}_{1/k} \circ \phi_{1/k} \cdot h^{\phi}_{n/k} \overset{(7.4)}{=} H^{\phi}_{1/k} \circ \phi_{1/k} \cdot H^{\phi}_{n/k} \overset{(7.5)}{=} H^{\phi}_{(n+1)/k} \circ \phi_{1/k} \quad \text{a.e. } [\mu].$$

Applying $\mu(X \setminus \phi_{1/k}(X)) = 0$ and Lemma 3.1(ii), we get $h^{\phi}_{(n+1)/k} = H^{\phi}_{(n+1)/k}$ a.e. $[\mu]$.

It follows from (7.4) and (7.6) via an induction argument that

$$h^{\phi}_{n} = H^{\phi}_{n} \quad \text{a.e. } [\mu], \quad n \in \mathbb{Z}_+, \tag{7.7}$$

the case $n = 0$ being covered by (5.3). Thus, by the measure transport theorem, we have

$$\begin{aligned} h^{\phi}_{n}(x) &= \int_0^\infty (e^{\delta - s})^n \, d\zeta_x(s) \\ &= \int_{(0,e^\delta]} t^n \, d\,(\zeta_x \circ \Phi^{-1})(t) \quad \text{a.e. } [\mu], \quad n \in \mathbb{Z}_+, \end{aligned} \tag{7.8}$$

where $\Phi\colon \mathbb{R}_+ \to (0, e^\delta]$ is given by $\Phi(s) = e^{\delta - s}$ for $s \in \mathbb{R}_+$. This means that for μ-a.e. $x \in X$, the sequence $\{h^{\phi}_n(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence whose representing measure $\zeta_x \circ \Phi^{-1}$ has compact support. Hence, by Lambert's criterion, the operator C_{ϕ_1} is subnormal and $\vartheta^1_x = \zeta_x \circ \Phi^{-1}$ for μ-a.e. $x \in X$. This in turn implies that $\vartheta^1_x(\{0\}) = 0$ for μ-a.e. $x \in X$, which, when combined with Proposition 5.2, leads to

$$\mu(X \setminus \phi_{1/k}(X)) = 0, \quad k \in \mathbb{N}. \tag{7.9}$$

Using again induction, (7.7), (7.6) and (7.9), we obtain

$$h^{\phi}_{n/k} = H^{\phi}_{n/k} \quad \text{a.e. } [\mu], \quad k \in \mathbb{N},\, n \in \mathbb{Z}_+. \tag{7.10}$$

Analogously to (7.8), we infer from (7.10) that for μ-a.e. $x \in X$ and all $k \in \mathbb{N}$, the sequence $\{h^{\phi}_{n/k}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence (with the representing measure $\zeta_x \circ \Phi_k^{-1}$, where $\Phi_k\colon \mathbb{R}_+ \to (0, e^{\delta/k}]$ is given by $\Phi_k(s) = e^{(\delta - s)/k}$). Hence, by [4, Lemma 4.3], the C_0-semigroup $\{C_{\phi_t}\}_{t \in \mathbb{R}_+}$ is subnormal. ∎

Yet another way of proving Proposition 7.3 is to first establish subnormality of C_{ϕ_1} and equality $\vartheta^1_x = \zeta_x \circ \Phi^{-1}$ a.e. $[\mu]$ (as in the proof above), and then to apply Theorem 6.5.

8. C_0-groups. In this section we investigate subnormality of C_0-groups of composition operators on $L^2(\mu)$. It turns out that the criterion for subnormality of C_0-semigroups given in Theorem 6.5 (as well as the other results of Section 6) remains valid for C_0-groups without assuming that the transformations $\phi_{1/k}$ are injective and Σ-bimeasurable, and that $\mu \circ \phi_{1/k} \ll \mu$.

Consider the following general situation:

(8.1) (X, Σ, μ) is a σ-finite measure space with $\mu \neq 0$ and $\boldsymbol{\phi} = \{\phi_t\}_{t\in\mathbb{R}}$ is a family of Σ-measurable transformations of X such that every ϕ_t induces a bounded composition operator C_{ϕ_t} on $L^2(\mu)$ and $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ is a C_0-group.

As in (5.2), we define

$$h_t^{\boldsymbol{\phi}} = \frac{d\mu \circ \phi_t^{-1}}{d\mu}, \qquad t \in \mathbb{R}.$$

Since $C_{\phi_0} = C_{I_X}$ and $C_{\phi_{nt}} = C_{\phi_t}^n = C_{(\phi_t)^n}$ for $n \in \mathbb{Z}_+$, Lemma 3.3(i) leads to

$$h_0^{\boldsymbol{\phi}} = 1 \quad \text{a.e. } [\mu], \qquad h_n^{\phi_t} = h_{nt}^{\boldsymbol{\phi}} \quad \text{a.e. } [\mu] \text{ for all } t \in \mathbb{R} \text{ and } n \in \mathbb{Z}_+. \tag{8.2}$$

Let us formulate C_0-group analogs of Lemmata 6.1, 6.2 and 6.3.

Lemma 8.1. *Suppose that* (8.1) *holds and* $s, t, u \in \mathbb{R}$. *Then*

$$h_u^{\boldsymbol{\phi}}(\phi_s(\phi_t(x))) = h_u^{\boldsymbol{\phi}}(\phi_{s+t}(x)) \text{ and } h_u^{\boldsymbol{\phi}}(\phi_0(x)) = h_u^{\boldsymbol{\phi}}(x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Moreover, if the composition operator C_{ϕ_1} *is subnormal, then*

$$\vartheta^1_{\phi_s(\phi_t(x))} = \vartheta^1_{\phi_{s+t}(x)} \quad \text{and} \quad \vartheta^1_{\phi_0(x)} = \vartheta^1_x \quad \text{for } \mu\text{-a.e. } x \in X. \tag{8.3}$$

Proof. Argue as in the proof of Lemma 6.1. ■

Lemma 8.2. *Suppose that* (8.1) *holds. If* $s, t \in \mathbb{R}$ *and* $n \in \mathbb{Z}_+$, *then for* μ-*a.e.* $x \in X$,

$$h_{s+t}^{\boldsymbol{\phi}}(x) = h_t^{\boldsymbol{\phi}}(x) \cdot h_s^{\boldsymbol{\phi}}(\phi_{-t}(x)), \tag{8.4}$$

$$h_{s+t}^{\boldsymbol{\phi}}(\phi_t(x)) = h_t^{\boldsymbol{\phi}}(\phi_t(x)) \cdot h_s^{\boldsymbol{\phi}}(x), \tag{8.5}$$

$$1 = h_{-t}^{\boldsymbol{\phi}}(x) \cdot h_t^{\boldsymbol{\phi}}(\phi_t(x)), \tag{8.6}$$

$$h_{-n}^{\boldsymbol{\phi}}(x) = \frac{1}{h_n^{\boldsymbol{\phi}}(\phi_n(x))}. \tag{8.7}$$

Proof. We argue essentially as in the proof of Lemma 6.2. As there, we see that (6.5) is valid for all $s, t \in \mathbb{R}$. Fix $s, t \in \mathbb{R}$. Since $C_{\phi_{-t}\circ\phi_t} = C_{\phi_t} C_{\phi_{-t}} = C_{I_X}$, we infer from Lemma 3.3(iv) that

$$h_s^{\boldsymbol{\phi}} \circ \phi_{-t} \circ \phi_t = h_s^{\boldsymbol{\phi}} \quad \text{a.e. } [\mu]. \tag{8.8}$$

Thus, the measure transport theorem yields

$$\begin{aligned}\int_\sigma h_{s+t}^{\boldsymbol{\phi}}\, d\mu &= \int_X \chi_\sigma \, d\mu \circ \phi_{s+t}^{-1} \overset{(6.5)}{=} \int_X \chi_\sigma \, d\mu \circ \phi_s^{-1} \circ \phi_t^{-1} \\ &= \int_X \chi_\sigma \circ \phi_t \, d\mu \circ \phi_s^{-1} = \int_X \chi_\sigma \circ \phi_t \, h_s^{\boldsymbol{\phi}} \, d\mu\end{aligned}$$

$$\overset{(8.8)}{=} \int_X \chi_\sigma \circ \phi_t \, h_s^{\boldsymbol{\phi}} \circ \phi_{-t} \circ \phi_t \, d\mu = \int_\sigma h_s^{\boldsymbol{\phi}} \circ \phi_{-t} \, d\mu \circ \phi_t^{-1}$$

$$= \int_\sigma h_t^{\boldsymbol{\phi}} h_s^{\boldsymbol{\phi}} \circ \phi_{-t} \, d\mu, \qquad \sigma \in \Sigma,$$

which, together with σ-finiteness of μ, gives (8.4). The equality (8.5) follows from (8.4) via Lemma 3.1(i) and (8.8). In turn, the equality (8.6) is a consequence of (8.2) and (8.4). Finally, (8.7) follows from (8.6). ■

LEMMA 8.3. *If* (8.1) *holds and the C_0-group $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ is subnormal, then*

(a) *$\vartheta_x^1(\{0\}) = 0$ for μ-a.e. $x \in X$,*
(b) *$\xi_t \, d\vartheta^1_{\phi_t(x)} = h_t^{\boldsymbol{\phi}}(\phi_t(x)) \, d\vartheta_x^1$ for μ-a.e. $x \in X$ and every $t \in \mathbb{R}_+$.*

Proof. Argue as in the proof of Lemma 6.3 using Lemma 8.2 in place of Lemma 6.2. ■

We are now in a position to prove a C_0-group analog of Theorem 6.5.

THEOREM 8.4. *Assume that* (8.1) *holds. Then the C_0-group $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ is subnormal if and only if the operator C_{ϕ_1} is subnormal and*

(b*) *$\xi_{1/k} \, d\vartheta^1_{\phi_{1/k}(x)} = h_{1/k}^{\boldsymbol{\phi}}(\phi_{1/k}(x)) \, d\vartheta_x^1$ for μ-a.e. $x \in X$ and every $k \in \mathbb{N}$.*

Proof. According to Lemma 8.3, it is enough to prove the "if" part of the conclusion. We preserve the notation from the proof of Theorem 6.5. Arguing exactly as there (skipping Step 1) and employing Lemma 8.2 in place of Lemma 6.2, we see that for μ-a.e. $y \in X$,

$$h_{n/k}^{\boldsymbol{\phi}}(\phi_{j/k}(y)) = \int_0^\infty s^n \, d(\vartheta^1_{\phi_{j/k}(y)} \circ \xi_k)(s),$$

where $n = mk + j$ (the case $j = 0$ follows from (5.4) and Lemma 3.1(i)). By Lemmas 3.1(i) and 8.1 this implies that for μ-a.e. $x \in X$,

$$h_{n/k}^{\boldsymbol{\phi}}(x) = h_{n/k}^{\boldsymbol{\phi}}(\phi_{j/k}(\phi_{-j/k}(x))) = \int_0^\infty s^n \, d(\vartheta^1_{\phi_{j/k}(\phi_{-j/k}(x))} \circ \xi_k)(s)$$

$$= \int_0^\infty s^n \, d(\vartheta_x^1 \circ \xi_k)(s).$$

Summarizing, we have proved that for μ-a.e. $x \in X$ and every $k \in \mathbb{N}$, the sequence $\{h_{n/k}^{\boldsymbol{\phi}}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence. Thus, by [4, Lemma 4.3], the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal. An application of Proposition 2.1 completes the proof. ■

The next result is a C_0-group analogue of Proposition 7.3. Its proof is shorter, even though we assume much less about the transformations $\phi_{1/k}$.

PROPOSITION 8.5. *Assume that* (8.1) *holds and there exists* $\delta \in \mathbb{R}$ *and a family* $\{\zeta_x\}_{x\in X}$ *of probability Borel measure on* $\mathbb{R}_+$ *satisfying the following two conditions for* μ*-a.e.* $x \in X$:

(i) $h^{\boldsymbol{\phi}}_{1/k}(x) = e^{\delta/k}\mathfrak{L}(\zeta_x)(1/k)$ *for all* $k \in \mathbb{N}$,
(ii) $\mathfrak{L}(\chi_\sigma\,\zeta_x)(1/k) = \mathfrak{L}(\zeta_x)(1/k)\cdot\zeta_{\phi_{-1/k}(x)}(\sigma)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ *and* $k \in \mathbb{N}$.

Then the C_0*-group* $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ *is subnormal.*

Proof. As in the proof of Proposition 7.3, we see that (7.4) holds and

$$H^{\boldsymbol{\phi}}_{s+1/k} = H^{\boldsymbol{\phi}}_{1/k}\cdot H^{\boldsymbol{\phi}}_{s}\circ\phi_{-1/k} \quad \text{a.e. } [\mu], \quad k\in\mathbb{N},\ s\in\mathbb{R}_+. \tag{8.9}$$

Using induction on n, we now show that (7.10) holds. Indeed, the case $n = 0$ is obvious. If (7.10) is valid for a fixed $n \in \mathbb{Z}_+$, then by Lemma 3.1(i) and the induction hypothesis, we have

$$\begin{aligned} h^{\boldsymbol{\phi}}_{(n+1)/k} &\overset{(8.4)}{=} h^{\boldsymbol{\phi}}_{1/k}\cdot h^{\boldsymbol{\phi}}_{n/k}\circ\phi_{-1/k} \\ &\overset{(7.4)}{=} H^{\boldsymbol{\phi}}_{1/k}\cdot H^{\boldsymbol{\phi}}_{n/k}\circ\phi_{-1/k} \overset{(8.9)}{=} H^{\boldsymbol{\phi}}_{(n+1)/k} \quad \text{a.e. } [\mu], \quad k\in\mathbb{N}, \end{aligned}$$

which completes the induction argument. As in the proof of Proposition 7.3, we deduce from (7.10) that the C_0-semigroup $\{C_{\phi_t}\}_{t\in\mathbb{R}_+}$ is subnormal. We finish the proof by applying Proposition 2.1. ∎

Below we show that conditions (i) and (ii) of Proposition 8.5 are always satisfied by a measurable family $\{P(x,\cdot)\}_{x\in X}$ of probability Borel measures attached to a subnormal C_0-group $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ via conditions 1° to 4° preceding Lemma 7.1.

PROPOSITION 8.6. *Assume that* (8.1) *holds and the* C_0*-group* $\{C_{\phi_t}\}_{t\in\mathbb{R}}$ *is subnormal. Let* $P\colon X\times\mathfrak{B}(\mathbb{R}_+)\to[0,1]$ *satisfy conditions* 1° *to* 4° *preceding Lemma* 7.1. *Then for every* $t\in\mathbb{R}_+$ *and* μ*-a.e.* $x\in X$,

$$\mathfrak{L}(\chi_\sigma P(x,\cdot))(t) = \mathfrak{L}(P(x,\cdot))(t)\,P(\phi_{-t}(x),\sigma), \qquad \sigma\in\mathfrak{B}(\mathbb{R}_+), \tag{8.10}$$

$$h^{\boldsymbol{\phi}}_{s+t}(x) = h^{\boldsymbol{\phi}}_{t}(x)\cdot h^{\boldsymbol{\phi}}_{s}(\phi_{-t}(x)), \qquad s\in\mathbb{R}_+. \tag{8.11}$$

Proof. Fix $t\in\mathbb{R}_+$. First note that (7.1) holds in the present context (the proof follows that of Proposition 7.2, with Lemma 6.3 replaced by Lemma 8.3). Using (7.3), (8.3) and Lemma 3.1(i), we see that

$$P(\phi_t(\phi_{-t}(x)),\cdot) = P(x,\cdot) \quad \text{for } \mu\text{-a.e. } x\in X.$$

Making the substitution $x \rightsquigarrow \phi_{-t}(x)$ in (7.1) for μ-a.e. $x\in X$ (which is possible due to Lemma 3.1(i)), we get (8.10). Then, arguing as in the proof of the implication (i)⇒(ii) of Lemma 7.1, we infer (8.11) from (8.10). ∎

9. Examples. In what follows, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Let $\|\cdot\|$ be a norm on $\mathbb{K}^{\varkappa}$ ($\varkappa \in \mathbb{N}$) induced by an inner product (accordingly, real or complex). Denote by $\mathcal{R}_{\|\cdot\|}$ the class of all (density) functions $\varrho\colon \mathbb{K}^{\varkappa} \to [0,\infty)$ of the form

$$\varrho(x) = \sum_{m=0}^{\infty} a_m \|x\|^{2m}, \qquad x \in \mathbb{K}^{\varkappa},$$

where the a_m are nonnegative real numbers and $a_k > 0$ for some $k \geq 1$. A function $\varrho \in \mathcal{R}_{\|\cdot\|}$ is said to be of *polynomial type* if there exists $k \geq 2$ such that $a_m = 0$ for all $m \geq k$. Denote by $\nu_{\varkappa}^{\mathbb{K}}$ the Lebesgue measure on $\mathbb{K}^{\varkappa}$. Note that $\nu_{\varkappa}^{\mathbb{C}} = \nu_{2\varkappa}^{\mathbb{R}}$.

We begin by characterizing subnormality of C_0-semigroups of composition operators on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$ induced by C_0-semigroups of linear transformations of $\mathbb{K}^{\varkappa}$.

Theorem 9.1. *Let $\|\cdot\|$ be a norm on $\mathbb{K}^{\varkappa}$ induced by an inner product, ϱ be a member of $\mathcal{R}_{\|\cdot\|}$, and A be a linear transformation of $\mathbb{K}^{\varkappa}$ such that for every $t \in \mathbb{R}_+$, the composition operator $C_{e^{tA}}$ is bounded on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$ (respectively, on $L^2((1/\varrho)\, d\nu_{\varkappa}^{\mathbb{K}})$). Then $\{C_{e^{tA}}\}_{t\in\mathbb{R}_+}$ and $\{C_{e^{tA}}^*\}_{t\in\mathbb{R}_+}$ are C_0-semigroups. Moreover, the C_0-semigroup $\{C_{e^{tA}}\}_{t\in\mathbb{R}_+}$ (respectively, $\{C_{e^{tA}}^*\}_{t\in\mathbb{R}_+}$) is subnormal if and only if A is a normal operator in $(\mathbb{K}^{\varkappa}, \|\cdot\|)$.*

Proof. By [4, Corollary 5.2 and Remark 5.3], $\{C_{e^{tA}}\}_{t\in\mathbb{R}_+}$ is a C_0-semigroup. Together with [33, Corollary 1.10.6], this implies that the family $\{C_{e^{tA}}^*\}_{t\in\mathbb{R}_+}$ is a C_0-semigroup as well. In view of [40, Theorem 2.5 and Section 3], the C_0-semigroup $\{C_{e^{tA}}\}_{t\in\mathbb{R}_+}$ (respectively, $\{C_{e^{tA}}^*\}_{t\in\mathbb{R}_+}$) is subnormal if and only if e^{tA} is a normal operator in $(\mathbb{K}^{\varkappa}, \|\cdot\|)$ for every $t \in \mathbb{R}_+$. The latter holds if and only if A is a normal operator in $(\mathbb{K}^{\varkappa}, \|\cdot\|)$ (this is a very particular case of the Stone theorem [34, Theorem 13.37]). ■

Our next goal is to determine when $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$ (respectively, on $L^2((1/\varrho)\, d\nu_{\varkappa}^{\mathbb{K}})$).

Proposition 9.2. *Let $\|\cdot\|$ be a norm on $\mathbb{K}^{\varkappa}$ induced by an inner product, ϱ be a member of $\mathcal{R}_{\|\cdot\|}$, and A be a linear transformation of $\mathbb{K}^{\varkappa}$. Denote by μ any of the measures $\varrho\, d\nu_{\varkappa}^{\mathbb{K}}$ or $(1/\varrho)\, d\nu_{\varkappa}^{\mathbb{K}}$.*

- (i) *If ϱ is of polynomial type, then $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $L^2(\mu)$.*
- (ii) *If ϱ is not of polynomial type, then $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $L^2(\mu)$ if and only if $A + A^* = 0$ (A^* is defined in $(\mathbb{K}^{\varkappa}, \|\cdot\|)$), or equivalently if $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of unitary operators on $L^2(\mu)$.*

Proof. (i) Apply [40, Proposition 2.2 and Section 3], [4, Corollary 5.2 and Remark 5.3] and Remark 2.2.

(ii) As in (i), we see that $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $L^2(\mu)$ if and only if

$$\|e^{tA}\| \le 1, \quad t \in \mathbb{R}. \tag{9.1}$$

If (9.1) holds then

$$\|x\| = \|e^{-tA}e^{tA}x\| \le \|e^{tA}x\| \le \|x\|, \quad x \in \mathbb{K}^{\varkappa},\ t \in \mathbb{R},$$

which implies that e^{tA} is a unitary operator in $(\mathbb{K}^{\varkappa}, \|\cdot\|)$ for every $t \in \mathbb{R}$ (equivalently, $A + A^* = 0$, cf. [16, Section I.3.15]). It is now a routine matter to verify that $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of unitary operators on $L^2(\mu)$. ∎

Example 9.3. Let $|\cdot|$ be the Euclidean norm on $\mathbb{K}^{\varkappa}$, ϱ be a member of $\mathcal{R}_{|\cdot|}$, and A be a linear transformation of $\mathbb{K}^{\varkappa}$. Consider first the case when the density function ϱ is not of polynomial type. If A is given by a nonzero diagonal matrix with nonnegative real entries, then $\|e^{-tA}\| \le 1$ for all $t \in \mathbb{R}_+$ and $\|e^{-tA}\| > 1$ for all real $t < 0$. Hence, by [40, Proposition 2.2 and Section 3], [4, Corollary 5.2 and Remark 5.3] and Theorem 9.1, $\{C_{e^{tA}}\}_{t\in\mathbb{R}_+}$ is a subnormal C_0-semigroup of bounded operators on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$ and $C_{e^{tA}} = C_{e^{-tA}}^{-1}$ is an unbounded closed densely defined operator in $L^2(\varrho\ d\nu_{\varkappa}^{\mathbb{K}})$ for all real $t < 0$ (use Proposition 4.1(vii)). In turn, if $\varkappa = 2$ and A is given by the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, then by Proposition 9.2(ii), $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of unitary operators on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$.

Assume now that ϱ is of polynomial type (A is still an arbitrary linear transformation of $\mathbb{K}^{\varkappa}$). It follows from Proposition 9.2(i) that $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $L^2(\varrho\, d\nu_{\varkappa}^{\mathbb{K}})$. By Theorem 9.1, this C_0-group is subnormal if and only if A is a normal operator in $(\mathbb{K}^{\varkappa}, |\cdot|)$. If $\mathbb{K} = \mathbb{C}$, $\varkappa = 2$ and A is given by the matrix $\pi\left[\begin{smallmatrix} \mathrm{i} & 1 \\ 0 & -\mathrm{i} \end{smallmatrix}\right]$, then the C_0-group $\{C_{e^{tA}}\}_{t\in\mathbb{R}}$ is not subnormal, though C_{e^A} is a unitary operator (because $e^A = -\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$). This example (which is based on an example due to R. Mathias, cf. [1]) was discussed in [4, Example 5.4].

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Piotr Budzyński
Katedra Zastosowań Matematyki
Uniwersytet Rolniczy w Krakowie
Al. Mickiewicza 24/28
30-059 Kraków, Poland
E-mail: piotr.budzynski@ar.krakow.pl

Jan Stochel
Instytut Matematyki
Uniwersytet Jagielloński
ul. Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Jan.Stochel@im.uj.edu.pl

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