On the uniform convergence of double sine series

by

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Abstract. Let a single sine series $(*)$ $\sum_{k=1}^{\infty} a_k \sin kx$ be given with nonnegative coefficients $\{a_k\}$. If $\{a_k\}$ is a "mean value bounded variation sequence" (briefly, MVBVS), then a necessary and sufficient condition for the uniform convergence of series $(*)$ is that $ka_k \to 0$ as $k \to \infty$. The class MVBVS includes all sequences monotonically decreasing to zero. These results are due to S. P. Zhou, P. Zhou and D. S. Yu. In this paper we extend them from single to double sine series $(**)$ $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl} \sin kx \sin ly$, even with complex coefficients $\{c_{kl}\}$. We also give a uniform boundedness test for the rectangular partial sums of series $(**)$, and slightly improve the results on single sine series.

1. Introduction. Let $\{a_k : k = 1, 2, \ldots\}$ be a sequence of nonnegative real numbers (in symbols: $\{a_k\} \subset \mathbb{R}_+$), and consider the sine series

$$(1.1) \qquad \sum_{k=1}^{\infty} a_k \sin kx.$$

It is well-known (see, for example, [12, Vol. I, p. 182] that if the sequence $\{a_k\}$ is monotonically decreasing to zero, then series (1.1) converges for all x. On the other hand, according to the Cantor–Lebesgue theorem (see, for example, [1, p. 326]), if (1.1) converges on a set of positive measure, then $a_k \to 0$ as $k \to \infty$.

Chaundy and Joliffe [2] (see also [12, Vol. I, p. 182]) proved the following:

THEOREM A. *If $\{a_k\} \subset \mathbb{R}_+$ is decreasing, then series (1.1) converges uniformly in x if and only if*

$$(1.2) \qquad ka_k \to 0 \quad as\ k \to \infty.$$

The monotonicity condition in Theorem A was relaxed by a number of authors (see, for example, the references in [4] and [11]). The following notion was introduced in [11], and it seems to be the most general one at present.

[79]

A sequence $\{a_k\} \subset \mathbb{R}_+$ is said to belong to the class MVBVS (= *mean value bounded variation sequences*) if there exist constants C and $\lambda \geq 2$, both depending only on the sequence $\{a_k\}$, such that

$$(1.3) \qquad \sum_{k=m}^{2m} |\Delta a_k| \leq \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} a_k, \qquad m \geq \lambda,$$

where $[\cdot]$ means the floor integral part of a real number and

$$\Delta a_k := a_k - a_{k+1}, \qquad k = 1, 2, \ldots.$$

The following theorem was proved in [11, Theorem 5].

THEOREM B. *If* $\{a_k\} \subset \mathbb{R}_+$ *belongs to the class* MVBVS, *then series* (1.1) *converges uniformly in x if and only if condition* (1.2) *is satisfied.*

We note that in [11] Theorem B was proved in the following more general form: *If* $\{a_k\} \subset \mathbb{R}_+$ *belongs to the class* MVBVS, *then condition* (1.2) *is necessary and sufficient for the uniform convergence of series* (1.1), *and also for the continuity of its sum function.* The addendum on the continuity of the sum function is a variant of a theorem of Paley [8] with almost the same proof. We recall that Paley's theorem says that if the Fourier coefficients of a continuous function f are nonnegative, then the Fourier series of f converges uniformly.

A particular case of the notion of MVBVS was earlier introduced in [9] as follows. A sequence $\{a_k\} \subset \mathbb{R}_+$ is said to belong to the class NBVS (= *non-onesided bounded variation sequences*) if there exists a constant C, depending only on $\{a_k\}$, such that

$$(1.4) \qquad \sum_{k=m}^{2m} |\Delta a_k| \leq C(a_m + a_{2m}), \qquad m = 1, 2, \ldots.$$

The following theorem was also formulated in [11] and supported with a counterexample.

THEOREM C. *If* $\{a_k\} \subset \mathbb{R}_+$ *belongs to the class* NBVS, *then it belongs to* MVBVS. *The converse is not true.*

Clearly, if $\{a_k\} \subset \mathbb{R}_+$ is a decreasing sequence, then condition (1.4) is satisfied. Thus, Theorem A is an immediate corollary of Theorems B and C.

An analysis of the proofs in [9] and [11] reveals that the sufficiency part in Theorem B as well as Theorem C remain valid if we consider sequences $\{c_k : k = 1, 2, \ldots\}$ of complex numbers (in symbols: $\{c_k\} \subset \mathbb{C}$) and the sine series

$$(1.1') \qquad \sum_{k=1}^{\infty} c_k \sin kx,$$

under appropriately modified conditions.

THEOREM B′. *If* $\{c_k\} \subset \mathbb{C}$ *is such that*

(1.2′)
$$kc_k \to 0 \quad \text{as } k \to \infty,$$

and there exist constants C and $\lambda \geq 2$ for which

(1.3′)
$$\sum_{k=m}^{2m-1} |\Delta c_k| \leq \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} |c_k|, \quad m \geq \lambda,$$

then series (1.1′) *converges uniformly in x.*

THEOREM C′. *If* $\{c_k\} \subset \mathbb{C}$ *is such that there exists a constant C_1 for which*

(1.4′)
$$\sum_{k=m}^{2m-1} |\Delta c_k| \leq C_1(|c_m| + |c_{2m}|), \quad m = 1, 2, \ldots,$$

then condition (1.3′) *is satisfied for some constants C and $\lambda \geq 2$.*

We call the reader's attention to the upper limit of summation which is $2m - 1$ instead of $2m$ on the left-hand sides in (1.3′) and (1.4′) (cf. (1.3) and (1.4)). However, this slight modification does not affect the proofs at all.

Finally, we note that analogous theorems can be proved for the uniform boundedness of the partial sums of sine series. For example, the scheme of the proofs of Theorems B and B′ can be modified so as to provide the following:

THEOREM D. *If* $\{c_k\} \subset \mathbb{C}$ *is such that condition* (1.3′) *is satisfied and*

(1.5)
$$\sup_{k \geq 1} k|c_k| < \infty,$$

then

(1.6)
$$\sup_{m \geq 1} \sup_{x} \left| \sum_{k=1}^{m} c_k \sin kx \right| < \infty.$$

Conversely, if $\{c_k\} \subset \mathbb{R}_+$ *is such that conditions* (1.3′) *and* (1.6) *are satisfied, then condition* (1.5) *is also satisfied.*

2. Main results. We recall that the double series

(2.1)
$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \gamma_{kl}$$

of complex numbers is said to *converge in Pringsheim's sense* (see, for example, [12, Vol. II, pp. 301–302] if the finite limit of its *rectangular partial sums*

$$\sum_{k=1}^{m} \sum_{l=1}^{n} \gamma_{kl}, \quad m, n = 1, 2, \ldots,$$

exists as m and n tend to ∞ independently of each other. If, in addition, each "row series"

$$\sum_{k=1}^{\infty} \gamma_{kl}, \quad l = 1, 2, \ldots,$$

and each "column series"

$$\sum_{l=1}^{\infty} \gamma_{kl}, \quad k = 1, 2, \ldots,$$

converges, then (2.1) is said to *converge regularly* (see [3] and also [5]).

The *Cauchy convergence criterion for regular convergence* reads as follows (see [5]). The double series (2.1) converges regularly if and only if for every $\varepsilon > 0$ there exists a natural number $m_0 = m_0(\varepsilon)$ such that

$$(2.2) \quad \left| \sum_{k=m}^{M} \sum_{l=n}^{N} \gamma_{kl} \right| < \varepsilon \quad \text{if } \max\{m, n\} > m_0,\ 1 \leq m \leq M,\ 1 \leq n \leq N.$$

It is easy to see that it is enough to require (2.2) to hold in the special cases when either $m > m_0$ and $n = 1$, or $n > m_0$ and $m = 1$.

After these preliminaries, let $\{c_{kl} : k, l = 1, 2, \ldots\}$ be a double sequence of complex numbers (in symbols: $\{c_{kl}\} \subset \mathbb{C}$), and consider the double sine series

$$(2.3) \quad \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl} \sin kx \sin ly.$$

We use the standard notations for the difference operators:

$$\Delta_{10} c_{kl} := c_{kl} - c_{k+1,l}, \quad \Delta_{01} c_{kl} := c_{kl} - c_{k,l+1},$$
$$\Delta_{11} c_{kl} := \Delta_{01}(\Delta_{10} c_{kl}) = \Delta_{10}(\Delta_{01} c_{kl})$$
$$= c_{kl} - c_{k+1,l} - c_{k,l+1} + c_{k+1,l+1}, \quad k, l = 1, 2, \ldots.$$

We recall that a double sequence $\{c_{kl}\}$ is said to be *monotonically decreasing* if

$$(2.4) \quad c_{kl} \geq 0, \quad \Delta_{10} c_{kl} \geq 0, \quad \Delta_{01} c_{kl} \geq 0, \quad \Delta_{11} c_{kl} \geq 0$$

for all $k, l = 1, 2, \ldots$. It is clear that if

$$(2.5) \quad c_{kl} \to 0 \quad \text{as } k + l \to \infty,$$

then we have

$$(2.6) \quad \Delta_{10} c_{mn} = \sum_{l=n}^{\infty} \Delta_{11} c_{ml}, \quad \Delta_{01} c_{mn} = \sum_{k=m}^{\infty} \Delta_{11} c_{kn},$$

$$c_{mn} = \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} \Delta_{11} c_{kl}, \quad m, n = 1, 2, \ldots.$$

Consequently, under condition (2.5), the first three inequalities in (2.4) follow from the fourth one.

The second author proved in [6] that if $\{c_{jk}\} \subset \mathbb{C}$ is such that condition (2.5) is satisfied and $\{c_{jk}\}$ is of bounded variation in the sense that

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\Delta_{11} c_{kl}| < \infty,$$

then the double series (2.3) converges regularly for all (x, y). In particular, if the double sequence $\{c_{kl}\}$ of nonnegative numbers (in symbols: $\{c_{kl}\} \subset \mathbb{R}_+$) is monotonically decreasing to zero, then the double sine series (2.3) converges regularly for all (x, y).

On the other hand, according to the two-dimensional extension of the Cantor–Lebesgue theorem (see [7]), if series (2.3) converges regularly on a set of positive (planar) measure, then we necessarily have (2.5).

The two-dimensional extension of the classical theorem of Chaundy and Joliffe (see Theorem A in Section 1), proved by Zhak and Shneĭder [10], reads as follows.

THEOREM E. *If $\{c_{kl}\} \subset \mathbb{R}_+$ is a monotonically decreasing double sequence, then the regular convergence of the double sine series (2.3) is uniform in (x, y) if and only if*

$$(2.7) \qquad klc_{kl} \to 0 \quad as \ k + l \to \infty.$$

Our goal is to relax the monotonicity condition in Theorem E. To this end, we introduce the class MVBVDS motivated by (1.3′). A double sequence $\{c_{kl}\} \subset \mathbb{C}$ is said to belong to the class MVBVDS (= *mean value bounded variation double sequences*) if there exist constants C and $\lambda \geq 2$, both depending only on $\{c_{kl}\}$, such that

$$(2.8) \qquad \sum_{k=m}^{2m-1} |\Delta_{10} c_{kn}| \leq \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} |c_{kn}|, \quad m \geq \lambda, \ n = 1, 2, \ldots,$$

$$(2.9) \qquad \sum_{l=n}^{2n-1} |\Delta_{01} c_{ml}| \leq \frac{C}{n} \sum_{l=[\lambda^{-1}n]}^{[\lambda n]} |c_{ml}|, \quad n \geq \lambda, \ m = 1, 2, \ldots,$$

$$(2.10) \qquad \sum_{k=m}^{2m-1} \sum_{l=n}^{2n-1} |\Delta_{11} c_{kl}| \leq \frac{C}{mn} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} \sum_{l=[\lambda^{-1}n]}^{[\lambda n]} |c_{kl}|, \quad m, n \geq \lambda.$$

Our first main result is a two-dimensional extension of Theorems B and B′.

THEOREM 1. *If* $\{c_{kl}\} \subset \mathbb{C}$ *belongs to the class* MVBVDS *and condition* (2.7) *is satisfied, then the regular convergence of the double sine series* (2.3) *is uniform in* (x, y).

Conversely, if $\{c_{kl}\} \subset \mathbb{R}_+$ *belongs to the class* MVBVDS *and the regular convergence of* (2.3) *is uniform in* (x, y), *then condition* (2.7) *is satisfied.*

In particular, the following corollary of Theorem 1 follows immediately: *If* $\{c_{kl}\} \subset \mathbb{R}_+$ *belongs to the class* MVBVDS, *then the regular convergence of the double sine series* (2.3) *is uniform in* (x, y) *if and only if condition* (2.7) *is satisfied.*

To our best knowledge, an extension of Paley's theorem mentioned in Section 1 (see [8]) from single to double Fourier series with nonnegative Fourier coefficients is not known. Thus, regarding the extension of the addendum to Theorem B in Section 1, on continuity of the sum function of (2.3), we raise the following problem:

PROBLEM. Is the following claim true or false? If $\{c_{kl}\} \subset \mathbb{R}_+$ belongs to the class MVBVDS, then condition (2.7) is necessary or sufficient for the continuity of the sum function of the double sine series (2.3).

The introduction of the following class of double sequences of complex numbers is motivated by (1.4′): $\{c_{kl}\} \subset \mathbb{C}$ is said to belong to the class NBVDS (*non-onesided bounded variation double sequences*) if there exists a constant C, depending only on $\{c_{kl}\}$, such that

$$(2.11) \qquad \sum_{k=m}^{2m-1} |\Delta_{10} c_{kn}| \le C(|c_{mn}| + |c_{2m,n}|),$$

$$(2.12) \qquad \sum_{l=n}^{2n-1} |\Delta_{01} c_{ml}| \le C(|c_{mn}| + |c_{m,2n}|),$$

$$(2.13) \qquad \sum_{k=m}^{2m-1} \sum_{l=n}^{2n-1} |\Delta_{11} c_{kl}| \le C(|c_{mn}| + |c_{2m,n}| + |c_{m,2n}| + |c_{2m,2n}|),$$
$$m, n = 1, 2, \ldots.$$

Our second main result is a two-dimensional extension of Theorems C and C′.

THEOREM 2. *If* $\{c_{kl}\} \subset \mathbb{C}$ *belongs to the class* NBVDS, *then it belongs to* MVBVDS. *The converse is not true.*

Clearly, if $\{c_{kl}\} \subset \mathbb{R}_+$ is a monotonically decreasing double sequence, then conditions (2.11)–(2.13) are satisfied. Thus, Theorem E, which was proved by Zhak and Shneĭder [10] in another way, is an immediate corollary of Theorems 1 and 2.

So far, we have presented sufficient conditions for the regular convergence of the double sine series (2.3) to be uniform in (x, y). It is of some interest to observe that the conditions in Theorem 1 can be easily reformulated so as to guarantee the uniform boundedness of the *rectangular sums*

$$(2.14) \quad s(m, M; n, N; x, y) := \sum_{k=m}^{M} \sum_{l=n}^{N} c_{kl} \sin kx \sin ly,$$
$$1 \le m \le M,\ 1 \le n \le N.$$

The reformulated Theorem 1 reads as follows.

THEOREM 3. *If $\{c_{kl}\} \subset \mathbb{C}$ belongs to the class* MVBVDS *and*

$$(2.15) \qquad\qquad \sup_{k,l \ge 1} kl|c_{kl}| < \infty,$$

then

$$(2.16) \qquad \sup_{1 \le m \le M,\, 1 \le n \le N}\ \sup_{x,y} |s(m, M; n, N; x, y)| < \infty.$$

Conversely, if $\{c_{kl}\} \subset \mathbb{R}_+$ belongs to the class MVBVDS *and condition* (2.16) *is satisfied, then* (2.15) *also holds.*

The following corollary of Theorem 3 is immediate: *If $\{c_{kl}\} \subset \mathbb{R}_+$ belongs to the class* MVBVDS, *in particular, if $\{c_{kl}\}$ is a monotonically decreasing double sequence, then a necessary and sufficient condition for the uniform boundedness of the rectangular sums* (2.14) *is that the double sequence $\{klc_{kl}\}$ is bounded.*

We note that each of the theorems and corollaries in this section can be easily extended to multiple sine series with complex or nonnegative real coefficients, respectively. However, the technical details become more cumbersome.

3. Auxiliary results. To prove Theorems 1–3, we will need four lemmas, which may also be useful in other contexts. In addition, we present a proof of Theorem C$'$.

LEMMA 1. *If $\{c_{kl}\} \subset \mathbb{C}$ is such that conditions* (2.7) *and* (2.10) *are satisfied, then*

$$(3.1) \qquad mn \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}| \to 0 \quad \text{as } m + n \to \infty \text{ and } m, n \ge 2\lambda.$$

Proof. Given $\varepsilon > 0$, by (2.7) there exists a natural number $m_1 = m_1(\varepsilon)$ such that

$$(3.2) \qquad kl|c_{kl}| < \varepsilon \quad \text{for all } m + n > m_1,\ m, n \ge 2\lambda.$$

Without loss of generality, we may assume that $m_1 \ge 4\lambda$.

Making use of (2.10) and (3.2) gives

$$(3.3) \quad \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}| = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{2^i m \le k < 2^{i+1} m} \sum_{2^j n \le l < 2^{j+1} n} |\Delta_{11} c_{kl}|$$

$$\le C \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{i+j} mn} \sum_{k=[\lambda^{-1} 2^i m]}^{[\lambda 2^i m]} \sum_{l=[\lambda^{-1} 2^j n]}^{[\lambda 2^j n]} |c_{kl}|$$

$$\le \frac{C\varepsilon}{mn} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{i+j}} \sum_{k=[\lambda^{-1} 2^i m]}^{[\lambda 2^i m]} \sum_{l=[\lambda^{-1} 2^j n]}^{[\lambda 2^j n]} \frac{1}{kl}.$$

We have

$$(3.4) \quad \sum_{k=[\lambda^{-1} 2^i m]}^{[\lambda 2^i m]} \frac{1}{k} \le \frac{1}{[\lambda^{-1} 2^i m]} + \int_{[\lambda^{-1} 2^i m]}^{[\lambda 2^i m]} \frac{dx}{x}$$

$$\le 1 + \ln \frac{[\lambda 2^i m]}{[\lambda^{-1} 2^i m]} \le 1 + \ln(2\lambda^2), \quad i = 0, 1, 2, \dots,$$

since

$$\frac{[\lambda 2^i m]}{[\lambda^{-1} 2^i m]} \le \frac{\lambda 2^i m}{\lambda^{-1} 2^i m - 1} \le 2\lambda^2 \quad \text{for } m \ge 2\lambda.$$

Now (3.3), (3.4) and its counterpart with $1/l$ in place of $1/k$ yield

$$mn \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}| \le C\varepsilon (1 + \ln(2\lambda^2))^2$$

whenever $m + n \ge m_1$ and $m, n \ge 2\lambda$. Since $\varepsilon > 0$ is arbitrary, this proves (3.1). ∎

LEMMA 2. *Under the conditions of Lemma 1, we have*

$$(3.5) \quad mn \sum_{k=m}^{\infty} \sup_{l \ge n} |\Delta_{10} c_{kl}| \to 0$$

and

$$(3.6) \quad mn \sum_{l=n}^{\infty} \sup_{k \ge m} |\Delta_{01} c_{kl}| \to 0 \quad \text{as } m + n \to \infty, \, m, n \ge 2\lambda.$$

Proof. By (2.6),

$$\sup_{l \ge n} |\Delta_{10} c_{kl}| = \sup_{l \ge n} \left| \sum_{l_1 = l}^{\infty} \Delta_{11} c_{kl_1} \right| \le \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}|, \quad k, n = 1, 2, \dots.$$

Thus, we have

$$\sum_{k=m}^{\infty} \sup_{l \ge n} |\Delta_{10} c_{kl}| \le \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}|$$

and an application of Lemma 1 yields (3.5). A similar argument gives (3.6). ∎

LEMMA 3. *If* $\{c_{kl}\} \subset \mathbb{R}_+$ *belongs to the class* MVBVDS, *then*

$$(3.7) \qquad mnc_{mn} \le (8C + 2) \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{t=[n/2\lambda]}^{[\lambda n]} c_{kl}, \qquad m, n \ge 2\lambda,$$

where the constants C *and* λ *are from* (2.7)–(2.9).

Proof. Let $m \ge 2\lambda$ and ν be arbitrary natural numbers. Applying inequality (2.8) for $\mu = m + 1, m + 2, \ldots, 2m$ gives

$$(3.8) \qquad c_{m\nu} = \sum_{k=m}^{\mu-1} \Delta_{10} c_{k\nu} + c_{\mu\nu} \le \sum_{k=m}^{\mu-1} |\Delta_{10} c_{k\nu}| + c_{\mu\nu}$$

$$\le \sum_{k=[\mu/2]}^{2[\mu/2]-1} |\Delta_{10} c_{k\nu}| + c_{\mu\nu} \le \frac{C}{[\mu/2]} \sum_{k=[\lambda^{-1}[\mu/2]]}^{[\lambda[\mu/2]]} c_{k\nu} + c_{\mu\nu}$$

$$\le \frac{2C}{m} \sum_{k=[m/2\lambda]}^{[\lambda m]} c_{k\nu} + c_{\mu\nu}, \qquad m < \mu \le 2m, \, \nu \ge 1.$$

Next, let $n \ge 2\lambda$ and μ be arbitrary natural numbers. An analogous argument, using inequality (2.9) for $\nu = n + 1, n + 2, \ldots, 2n$, gives

$$(3.9) \qquad c_{\mu n} \le \frac{2C}{n} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{\mu l} + c_{\mu\nu}, \qquad \mu \ge 1, \, n < \nu \le 2n.$$

Finally, let $m, n \ge 2\lambda$ be arbitrary natural numbers. A double version of the above argument, using (2.10) for $\mu = m + 1, m + 2, \ldots, 2m$ and $\nu = n + 1, n + 2, \ldots, 2n$, gives

$$(3.10) \qquad c_{mn} = \sum_{k=m}^{\mu-1} \sum_{l=n}^{\nu-1} \Delta_{11} c_{kl} + c_{m\nu} + c_{\mu n} - c_{\mu\nu}$$

$$\le \sum_{k=m}^{\mu-1} \sum_{l=n}^{\nu-1} |\Delta_{11} c_{kl}| + c_{m\nu} + c_{\mu n}$$

$$\le \sum_{k=[\mu/2]}^{2[\mu/2]-1} \sum_{l=[\nu/2]}^{2[\nu/2]-1} |\Delta_{11} c_{kl}| + c_{m\nu} + c_{\mu n}$$

$$\le \frac{C}{[\mu/2][\nu/2]} \sum_{k=[\lambda^{-1}[\mu/2]]}^{[\lambda[\mu/2]]} \sum_{l=[\lambda^{-1}[\nu/2]]}^{[\lambda[\nu/2]]} c_{kl} + c_{m\nu} + c_{\mu n}$$

$$\le \frac{4C}{mn} \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{kl} + c_{m\nu} + c_{\mu n},$$
$$m < \mu \le 2m, \, n < \nu \le 2n.$$

If we add up all the inequalities in (3.10), then we obtain

$$mnc_{mn} \leq 4C \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{kl} + m \sum_{\nu=n+1}^{2n} c_{m\nu} + n \sum_{\mu=m+1}^{2m} c_{\mu n}.$$

Now making use of (3.8) and (3.9) yields

$$mnc_{mn} \leq 4C \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{kl} + 2C \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{\nu=n+1}^{2n} c_{k\nu}$$

$$+ 2C \sum_{\mu=m+1}^{2m} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{\mu l} + 2 \sum_{\mu=m+1}^{2m} \sum_{\nu=n+1}^{2n} c_{\mu\nu}$$

$$\leq (8C+2) \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{kl},$$

which is (3.7). ∎

In Section 1, we have noted that the first statement in Theorem C has not been explicity proved in [11]. Instead, it is only indicated there that the proof is similar to that of [11, Proposition 3]. For the reader's convenience, below we present a proof of the more general Theorem C′. This proof moreover illustrates the key ingredients of the proof of our Theorem 2 in Section 4.

Proof of Theorem C′. Let $m \geq 6$. We apply inequality (1.4′) for all integers between $m - [m/6]$ and $m + [m/6]$. Adding up these inequalities yields

$$(3.11) \qquad \sum_{i=m-[m/6]}^{m+[m/6]} \sum_{k=i}^{2i-1} |\Delta c_k| \leq C \sum_{i=m-[m/6]}^{m+[m/6]} (|c_i| + |c_{2i}|).$$

It is easy to see that the term $|\Delta c_k|$ with $m \leq k \leq 2m - 1$ occurs on the left-hand side of (3.11) at least

$$\left[\frac{m}{6}\right] + 1 \geq \frac{m}{6} \qquad \text{times.}$$

Furthermore, on the right-hand side of (3.11), the smallest index of the term $|c_i|$ is

$$m - \left[\frac{m}{6}\right] \geq \frac{m}{3},$$

while the largest index of the term $|c_{2i}|$ does not exceed

$$2\left(m + \left[\frac{m}{6}\right]\right) \leq 3m;$$

and over the range $m - [m/6] \leq i \leq m + [m/6]$, we have

$$\max i \leq m + \left[\frac{m}{6}\right] < 2\left(m - \left[\frac{m}{6}\right]\right) \leq \min 2i.$$

Keeping these observations in mind, from (3.11) it follows that

$$(3.12) \qquad \frac{m}{6} \sum_{k=m}^{2m-1} |\Delta c_k| \leq C \sum_{i=[m/3]}^{3m} |c_i|.$$

Thus, we have shown that inequality (1.3′) is satisfied with $6C$ in place of C and $\lambda = 3$. Consequently, $\{c_k\}$ belongs to the class MVBVS. ∎

For the second statement in Theorem C, the counterexample given in [11, Proposition 4] is recalled in the following

LEMMA 4. *Let* $\{b_k\} \subset \mathbb{R}_+$ *be an arbitrary decreasing sequence. Define*

$$c_k := \begin{cases} 0 & \text{for } 2^i \leq k < 2^i + i, \\ b_k & \text{for } 2^i + i \leq k < 2^{i+1} - i, \\ 0 & \text{for } 2^{i+1} - i \leq k < 2^{i+1}, \ i = 1, 2, \dots. \end{cases}$$

Then the sequence $\{c_k\}$ *belongs to the class* MVBVS, *but not to* NBVS.

4. Proofs of Theorems 1–3

Proof of Theorem 1. Part 1. By conditions (2.8) and (2.9), Theorem C′ applies to the single sequences

$$\sum_{k=1}^{\infty} c_{kl} \sin kx, \quad l = 1, 2, \dots, \qquad \sum_{l=1}^{\infty} c_{kl} \sin ly, \quad k = 1, 2, \dots,$$

and guarantees that each row series and each column series of the double series (2.3) converge uniformly in (x, y).

In order to prove the uniform regular convergence of (2.3), we consider the rectangular sums

$$s(m, M; n, N; x, y) := \sum_{k=m}^{M} \sum_{l=n}^{N} c_{kl} \sin kx \sin ly, \quad 1 \leq m < M, \ 1 \leq n < N.$$

It is enough to consider the case where $0 < x, y < \pi$. Let

$$(4.1) \qquad \mu(x) := \left[\frac{1}{x}\right] \quad \text{and} \quad \nu(y) := \left[\frac{1}{y}\right].$$

We distinguish the following four cases.

CASE (a): $1 \le m < M \le \mu(x)$ *and* $1 \le n < N \le \nu(y)$. A trivial estimate gives

$$(4.2) \qquad |s(m, M; n, N; x, y)| \le xy \sum_{k=m}^{M} \sum_{l=n}^{N} kl|c_{kl}|$$

$$\le \frac{1}{\mu(x)\nu(y)} \sum_{k=1}^{\mu(x)} \sum_{l=1}^{\nu(y)} kl|c_{kl}|.$$

By (2.7), the double sequence $\{klc_{kl}\}$ boundedly converges to zero. Therefore, the arithmetic mean on the right-hand side of (4.2) converges to zero as $\mu(x), \nu(y) \to \infty$.

CASE (b): $\mu(x) < m < M$ *and* $1 \le n < N \le \nu(y)$. This time, we estimate as follows:

$$(4.3) \qquad |s(m, M; n, N; x, y)| = \sum_{l=n}^{N} |\sin ly| \cdot \left| \sum_{k=m}^{M} c_{kl} \sin kx \right|$$

$$\le \sum_{l=n}^{N} ly \left| \sum_{k=m}^{M} c_{kl} \sin kx \right|$$

$$\le \frac{1}{\nu(y)} \sum_{l=n}^{\nu(y)} l \left| \sum_{k=m}^{M} c_{kl} \sin kx \right|.$$

Summation by parts gives

$$(4.4) \qquad \sum_{k=m}^{M} c_{kl} \sin kx = \sum_{k=m}^{M-1} \widetilde{D}_k(x)\Delta_{10}c_{kl} + c_{Ml}\widetilde{D}_M(x) - c_{ml}\widetilde{D}_{m-1}(x),$$

where $\widetilde{D}_k(x)$ is the conjugate Dirichlet kernel (see, for example, [12, Vol. I, p. 49]):

$$\widetilde{D}_k(x) := \sum_{i=1}^{k} \sin ix = \frac{\cos \frac{x}{2} - \cos \left(k + \frac{1}{2} \right)x}{2 \sin \frac{x}{2}}.$$

Using the familiar estimate

$$|\widetilde{D}_k(x)| \le \pi/x, \quad k = 1, 2, \ldots, \ 0 < x < \pi,$$

it follows from (4.3) and (4.4) that

$$(4.5) \quad |s(m, M; n, N; x, y)| \le \frac{\pi}{x\nu(y)} \sum_{l=n}^{\nu(y)} l \left\{ \sum_{k=m}^{M-1} |\Delta_{10}c_{kl}| + |c_{Ml}| + |c_{ml}| \right\}$$

$$\le \frac{\pi m}{\nu(y)} \sum_{l=n}^{\nu(y)} l \left\{ \sum_{k=m}^{\infty} |\Delta_{10}c_{kl}| + 2 \sup_{k \ge m} |c_{kl}| \right\}.$$

By Lemma 2 (cf. (3.5)), the arithmetic mean on the right-hand side of (4.5) converges to zero as $m, n \to \infty$.

CASE (c): $1 \le m < M \le \mu(x)$ *and* $\nu(y) < n < N$. This is the symmetric counterpart of Case (b), and it can be handled analogously. The corresponding estimate is

$$(4.6) \qquad |s(m, M; n, N; x, y)| \le \frac{\pi n}{\mu(x)} \sum_{k=1}^{\mu(x)} k \Big\{ \sum_{l=n}^{\infty} |\Delta_{01} c_{kl}| + 2 \sup_{l \ge n} |c_{kl}| \Big\}.$$

By Lemma 2 (cf. (3.6)), the right-hand side of (4.6) converges to zero as $m, n \to \infty$.

CASE (d): $\mu(x) < m < M$ *and* $\nu(y) < n < N$. This time, we perform double summation by parts to obtain

$$s(m, M; n, N; x, y) = \sum_{k=m}^{M-1} \sum_{l=n}^{N-1} \widetilde{D}_k(x) \widetilde{D}_l(y) \Delta_{11} c_{kl}$$

$$+ \sum_{k=m}^{M-1} \widetilde{D}_k(x) \widetilde{D}_N(y) \Delta_{10} c_{kN} - \sum_{k=m}^{M-1} \widetilde{D}_k(x) \widetilde{D}_{n-1}(y) \Delta_{10} c_{kn}$$

$$+ \sum_{l=n}^{N-1} \widetilde{D}_M(x) \widetilde{D}_l(y) \Delta_{01} c_{Ml} - \sum_{l=n}^{N-1} \widetilde{D}_{m-1}(x) \widetilde{D}_l(y) \Delta_{01} c_{ml}$$

$$+ c_{MN} \widetilde{D}_M(x) \widetilde{D}_N(y) - c_{mN} \widetilde{D}_{m-1}(x) \widetilde{D}_N(y)$$

$$- c_{Mn} \widetilde{D}_M(x) \widetilde{D}_{n-1}(y) + c_{mn} \widetilde{D}_{m-1}(x) \widetilde{D}_{n-1}(y),$$

whence it follows that

$$(4.7) \qquad |s(m, M; n, N; x, y)| \le \frac{\pi^2}{xy} \Big\{ \sum_{k=m}^{M-1} \sum_{l=n}^{N-1} |\Delta_{11} c_{kl}|$$

$$+ \sum_{k=m}^{M-1} (|\Delta_{10} c_{kN}| + |\Delta_{10} c_{kn}|) + \sum_{l=n}^{N-1} |\Delta_{01} c_{Ml}| + |\Delta_{01} c_{ml}|)$$

$$+ |c_{MN}| + |c_{mN}| + |c_{Mn}| + |c_{mn}| \Big\}$$

$$\le \pi^2 mn \Big\{ \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}| + 2 \sum_{k=m}^{\infty} \sup_{l \ge n} |\Delta_{10} c_{kl}|$$

$$+ 2 \sum_{l=n}^{\infty} \sup_{k \ge m} |\Delta_{01} c_{kl}| + 4 \sup_{k \ge m, l \ge n} |c_{kl}| \Big\}.$$

By Lemmas 1 and 2, the right-hand side of (4.7) converges to zero as $m, n \to \infty$.

After these preliminaries, we can prove the uniform regular convergence of the double sine series (2.3) as follows. By (2.8) and (2.9), Theorem C′ implies the uniform convergence of each row series and each column series of (2.3).

By (2.7), for any $\varepsilon > 0$ there exists a natural number $m_1 = m_1(\varepsilon) \geq 2\lambda$ such that

$$(4.8) \qquad kl|c_{kl}| < \varepsilon \quad \text{for all } k, l > m_1;$$

consequently, there exists $m_2 = m_2(\varepsilon) \geq 2\lambda$ such that

$$(4.9) \qquad \left| \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} kl c_{kl} \right| < \varepsilon \quad \text{for all } m, n > m_2.$$

By Lemma 1, there exists $m_3 = m_3(\varepsilon) \geq 2\lambda$ such that

$$(4.10) \qquad mn \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} |\Delta_{11} c_{kl}| < \varepsilon \quad \text{for all } m, n > m_3.$$

By Lemma 2, we also have

$$(4.11) \qquad mn \sum_{k=m}^{\infty} \sup_{l \geq n} |\Delta_{10} c_{kl}| < \varepsilon$$

and

$$(4.12) \qquad mn \sum_{l=n}^{\infty} \sup_{k \geq m} |\Delta_{01} c_{kl}| < \varepsilon \quad \text{for all } m, n \geq m_3;$$

consequently, there exists $m_4 = m_4(\varepsilon) \geq 2\lambda$ such that

$$(4.13) \qquad \frac{m}{N} \sum_{l=n}^{N} l \sum_{k=m}^{\infty} |\Delta_{10} c_{kl}| < \varepsilon \quad \text{for all } m, n > m_4 \text{ and } N > n;$$

and

$$(4.14) \qquad \frac{n}{M} \sum_{k=m}^{M} k \sum_{l=n}^{\infty} |\Delta_{01} c_{kl}| < \varepsilon \quad \text{for all } m, n > m_4 \text{ and } M > m.$$

Let $m_0 := \max\{m_1, m_2, m_3, m_4\}$. We claim that

$$(4.15) \quad |s(m, M; n, N; x, y)| < (9\pi^2 + 6\pi + 1)\varepsilon \quad \text{for all } (x, y)$$
$$\text{whenever } M > m > m_0 \text{ and } N > n > m_0.$$

To justify this claim, given $0 < x, y < \pi$, we distinguish nine cases listed under (i)–(ix). We recall that $\mu(x)$ and $\nu(y)$ were defined in (4.1).

(i) $m_0 < m < M \leq \mu(x)$ and $m_0 < n < N \leq \nu(y)$. This was treated in Case (a). By (4.2), (4.8) and (4.9), we have (4.15) with ε in place of $(9\pi^2 + 6\pi + 1)\varepsilon$.

(ii) $\mu(x) < m < M$ *and* $m_0 < n < N \leq \nu(y)$. This was treated in Case (b). By (4.5), (4.9) and (4.13), we have (4.15) with $3\pi\varepsilon$.

(iii) $m_0 < m \leq \mu(x) < M$ *and* $m_0 < n < N \leq \nu(y)$. This can be split into two subcases which belong to Case (a) and Case (b), respectively. Combining it with the previous cases (i) and (ii) gives (4.15) with $(3\pi+1)\varepsilon$.

(iv) $m_0 < m < M \leq \mu(x)$ *and* $\nu(y) < n < N$. This was indicated in Case (c), and it is the symmetric counterpart of (ii). Thus, we have (4.15) again with $3\pi\varepsilon$.

(v) $\mu(x) < m < M$ *and* $\nu(y) < n < N$. This was treated in Case (d). By (4.7), (4.8), (4.10)–(4.12), we deduce (4.15) with $\pi^2(1+2+2+4)\varepsilon = 9\pi^2\varepsilon$.

(vi) $m_0 < m \leq \mu(x) < M$ *and* $\nu(y) < n < N$. This can be split into two cases which belong to Case (c) and Case (d), respectively. Combining it with case (v) and case (ix) below gives (4.15) with $(9\pi^2 + 3\pi + 1)\varepsilon$.

(vii) $m_0 < m < M \leq \mu(x)$ *and* $m_0 < n \leq \nu(y) < N$. This can be split into two cases which belong to Case (a) and Case (c), respectively; and it is the symmetric counterpart of (iii). Thus, we have (4.15) again with $(3\pi+1)\varepsilon$.

(viii) $\mu(x) < m < M$ *and* $m_0 < n \leq \nu(y) < N$. This can be split into four subcases which fall under Cases (a)–(d), respectively. Combining it with the previous cases (i), (ii), (iv) and (v) gives (4.15) with the upper bound as indicated.

(ix) $m_0 < m \leq \mu(x) < M$ *and* $m_0 < n \leq \nu(y) < N$. This is Case (c), and it is the symmetric counterpart of (iii). Thus, we have (4.15) again with $(3\pi + 1)\varepsilon$.

Now the justification of (4.15) is complete. What we have proved so far is that the inequality for $|s(m, M; n, N; x, y)|$ in (4.15) holds when $\min\{m, n\} > m_0$. In order to complete the proof of the uniform regular convergence of the double sine series (2.3), we need to prove an analogous inequality when just $\max\{m, n\}$ is large enough.

To this end, fix $\eta > 0$. We choose $\varepsilon > 0$ in (4.15) to be

$$\varepsilon := \frac{\eta}{2(9\pi^2 + 6\pi + 1)},$$

and focus on the first m_0 row series and the first m_0 column series in (2.3). Each of them converges uniformly in (x, y). Therefore, there exists a natural number $n_1 = n_1(\eta)$ such that

(4.16) $\quad |s(m, M; n, N; x, y)| < \eta/2 \quad$ for all (x, y) whenever

$$\text{either } n_1 < m \leq M \text{ and } 1 \leq n \leq N \leq m_0,$$
$$\text{or } 1 \leq m \leq M \leq m_0 \text{ and } n_1 < n \leq N.$$

Let $n_0 := \max\{m_0, n_1\}$. Putting together (4.15) and (4.16) yields

$|s(m, M; n, N; x, y)| < \eta \quad$ for all (x, y) whenever

$\quad \text{either } n_0 < m \leq M \text{ and } 1 \leq n \leq N, \text{ or } 1 \leq m \leq M \text{ and } n_0 < n \leq N.$

Taking into account the remark made after (2.2) in Section 2 and that $\eta > 0$ is arbitrary, this proves the uniform regular convergence of the double sine series (2.3).

This completes the proof of Part 1.

Part 2. By (2.8), (2.9) and the uniform convergence of each row series and each column series (2.3), we may apply the necessity part of Theorem B to deduce (2.7) when either $k \to \infty$ and l is fixed, or vice versa.

It remains to prove (2.7) when both k and l tend to ∞. To this end, we will apply Lemma 3. Let $m, n \geq 4\lambda$ and set

$$x(m) := \frac{\pi}{2\lambda m} \quad \text{and} \quad y(n) := \frac{\pi}{2\lambda n}.$$

Clearly, for all k in the range $[m/2\lambda] \leq k \leq [\lambda m]$ we have

$$kx(m) \leq \lambda m \frac{\pi}{2\lambda m} = \frac{\pi}{2}$$

and

$$kx(m) \geq \left(\frac{m}{2\lambda} - 1 \right) \frac{\pi}{2\lambda m} \geq \frac{m}{4\lambda} \frac{\pi}{2\lambda m} = \frac{\pi}{8\lambda^2},$$

since $m \geq 4\lambda$. Likewise,

$$\frac{\pi}{8\lambda^2} \leq ly(n) \leq \frac{\pi}{2} \quad \text{whenever} \quad \left[\frac{n}{2\lambda} \right] \leq l \leq [\lambda n].$$

By these inequalities and the assumption that all c_{kl} are nonnegative, it follows from (3.7) that

$$
\begin{aligned}
(4.17) \quad \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} & c_{kl} \sin kx(m) \sin ly(n) \\
& \geq \left(\sin \frac{\pi}{8\lambda^2} \right)^2 \sum_{k=[m/2\lambda]}^{[\lambda m]} \sum_{l=[n/2\lambda]}^{[\lambda n]} c_{kl} \\
& \geq \frac{1}{8C+2} \left(\sin \frac{\pi}{8\lambda^2} \right)^2 mn c_{mn}.
\end{aligned}
$$

Since the regular convergence of the double sine series (2.3) is supposed to be uniform in (x, y), the fulfillment of (2.7) when $k, l \to \infty$ is an immediate consequence of (4.17).

This completes the proof of Part 2.

Now the proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Part 1. By Theorem C′, conditions (2.11) and (2.12) imply (2.8) and (2.9), respectively. Thus, in order to prove that the double sequence $\{c_{kl}\}$ in question belongs to the class MVBVDS, it remains to check (2.10).

To this end, let $m, n \geq 6$ be given. We apply inequality (2.13) for all integers i and j in the ranges

$$(4.18) \qquad m - \left[\frac{m}{6}\right] \leq i \leq m + \left[\frac{m}{6}\right] \quad \text{and} \quad n - \left[\frac{n}{6}\right] \leq j \leq n + \left[\frac{n}{6}\right].$$

Adding up all these inequalities results in

$$(4.19) \qquad \sum_{i=m-[m/6]}^{m+[m/6]} \sum_{j=n-[n/6]}^{n+[n/6]} \sum_{k=i}^{2i-1} \sum_{l=j}^{2j-1} |\Delta_{11} c_{kl}|$$

$$\leq C \sum_{i=m-[m/6]}^{m+[m/6]} \sum_{j=n-[n/6]}^{n+[n/6]} (|c_{ij}| + |c_{2i,j}| + |c_{i,2j}| + |c_{2i,2j}|).$$

We observe that for the integers k and l in the ranges

$$m \leq k \leq 2m - 1 \quad \text{and} \quad n \leq l \leq 2n - 1,$$

the term $|\Delta c_{kl}|$ occurs on the left-hand side of (4.19) at least

$$\left(\left[\frac{m}{6}\right] + 1\right)\left(\left[\frac{n}{6}\right] + 1\right) \geq \frac{mn}{36} \quad \text{times.}$$

Furthermore, on the right-hand side of (4.19), the smallest indices of the term $|c_{ij}|$ are

$$m - \left[\frac{m}{6}\right] \geq \frac{m}{3} \quad \text{and} \quad n - \left[\frac{n}{6}\right] \geq \frac{n}{3};$$

while the largest indices of the term $|c_{2i,2j}|$ do not exceed

$$2\left(m + \left[\frac{m}{6}\right]\right) \leq 3m \quad \text{and} \quad 2\left(n + \left[\frac{n}{6}\right]\right) \leq 3n;$$

and over the ranges indicated in (4.18), we have

$$\max i \leq m + \left[\frac{m}{6}\right] < 2\left(m - \left[\frac{m}{6}\right]\right) \leq \min 2i,$$

and analogous inequalities for j.

Taking these observations into account, from (4.19) it follows that

$$(4.20) \qquad \frac{mn}{36} \sum_{k=m}^{2m-1} \sum_{l=n}^{2n-1} |\Delta_{11} c_{kl}| \leq C \sum_{k=[m/3]}^{3m} \sum_{l=[n/3]}^{3n} |c_{kl}|.$$

Thus, we have shown that the required inequality (2.13) is satisfied with $36C$ in place of C and $\lambda = 3$. Consequently, $\{c_{kl}\}$ belongs to the class MVBVDS.

Part 2. Take the sequence $\{c_k\}$ given in Lemma 4 and define the double sequence $\{c_{kl}\}$ by setting $c_{kl} := c_k c_l$, $k, l = 1, 2, \ldots$. Since $\{c_k\}$ belongs to

the class MVBVS, we have

$$(4.21) \qquad \sum_{k=m}^{2m-1} |\Delta_{10} c_{kn}| = c_n \sum_{k=m}^{2m-1} |\Delta c_k|$$

$$\leq c_n \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} c_k = \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} c_{kn},$$

that is, condition (2.8) is satisfied. Condition (2.9) can be proved analogously.

It is also clear that

$$(4.22) \qquad \sum_{k=m}^{2m-1} \sum_{l=n}^{2n-1} |\Delta_{11} c_{kl}| = \sum_{k=m}^{2m-1} |\Delta c_k| \sum_{l=n}^{2n-1} |\Delta c_l|$$

$$\leq \frac{C}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} c_k \cdot \frac{C}{n} \sum_{l=[\lambda^{-1}n]}^{[\lambda n]} c_l = \frac{C^2}{mn} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} \sum_{l=[\lambda^{-1}n]}^{[\lambda n]} c_{kl},$$

which is (2.10) with C^2 in place of C. To sum up, we have proved that $\{c_{kl}\}$ belongs to the class MVBVDS.

On the other hand, $\{c_{kl}\}$ cannot belong to NBVDS. It is easy to see that none of conditions (2.11)–(2.13) is satisfied. Again, the justification hinges on the fact that we can factorize both sides of the inequalities in question exactly in the same way as in the cases of (4.21) and (4.22).

The proof of Theorem 2 is complete. ∎

Proof of Theorem 3. It goes along the same lines as that of Theorem 1 with appropriate modifications. ∎

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