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Carleson measures associated with families of multilinear operators

by

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Abstract. We investigate the construction of Carleson measures from families of multilinear integral operators applied to tuples of L^∞ and BMO functions. We show that if the family R_t of multilinear operators has cancellation in each variable, then for BMO functions $b_1, \ldots, b_m$, the measure $|R_t(b_1, \ldots, b_m)(x)|^2 dxdt/t$ is Carleson. However, if the family of multilinear operators has cancellation in all variables combined, this result is still valid if b_j are L^∞ functions, but it may fail if b_j are unbounded BMO functions, as we indicate via an example. As an application of our results we obtain a multilinear quadratic $T(1)$ type theorem and a multilinear version of a quadratic $T(b)$ theorem analogous to those by Semmes [Proc. Amer. Math. Soc. 110 (1990), 721–726].

1. Introduction. A positive measure $d\mu(x,t)$ on $\mathbb{R}^{n+1}_+$ is called a *Carleson measure* if

$$\|d\mu\|_{\mathcal{C}} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|} d\mu(T(Q)) < \infty, \tag{1.1}$$

where $|Q|$ denotes the Lebesgue measure of the cube Q, $T(Q) = Q\times(0, l(Q)]$ denotes the *tent* over Q, and $l(Q)$ is the side length of Q. Carleson measures arose in the work of Carleson [2], [3] and turned out to be tools of fundamental importance in harmonic analysis; we mention for instance the great role they play in the study of the Cauchy integrals along Lipschitz curves [5], the $T(1)$ theorem [6], and in the study of the Kato problem [1].

There is a natural connection between Carleson measures, families of linear operators acting L^2, and BMO functions. Precisely, let $\{R_t\}_{t>0}$ be a family of integral operators

$$R_t(f)(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y)\, dy \tag{1.2}$$

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whose kernels K_t satisfy

$$|K_t(x,y)| \le \frac{At^{-n}}{(1+t^{-1}|x-y|)^{n+\delta}} \tag{1.3}$$

for some $\delta > 0$. Suppose that $R_t(1)(x) = 0$ for all $t > 0$ and that there is a positive constant B such that

$$\int_{\mathbb{R}^n}\int_0^\infty |R_t(f)(x)|^2 \frac{dx\,dt}{t} \le B^2\|f\|_2^2 \tag{1.4}$$

for all $f \in L^2(\mathbb{R}^n)$. Then given a function $b \in \mathrm{BMO}$, the measure

$$d\mu(x,t) := |R_t(b)(x)|^2 \frac{dx\,dt}{t}$$

is Carleson with norm

$$\|d\mu\|_{\mathcal{C}} \le C_{n,\delta}(A^2+B^2)\|b\|_{\mathrm{BMO}}^2.$$

Here the constant depends only on the dimension and on δ. A special case of this result is due to Fefferman and Stein [11]. Conversely, a result of Christ and Journé [4] says that given a family of operators R_t whose kernels K_t satisfy the size condition (1.3) and the regularity condition

$$|K_t(x,y) - K_t(x,y')| \le At^{-n}(t^{-1}|y-y'|)^\gamma, \tag{1.5}$$

if the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|}\int_Q\int_0^{l(Q)} |R_t(f)(x)|^2 \frac{dx\,dt}{t} < \infty \tag{1.6}$$

holds, then estimate (1.4) also holds.

In this work we investigate the construction of Carleson measures from families R_t of multilinear operators. The key question is: what is a reasonable analog of the cancellation condition $R_t(1) = 0$ in the multilinear case? We examine two kinds of cancellation conditions, a one-variable and a multi-variable condition. It is noteworthy that in one of these two cases BMO functions do not give rise to Carleson measures. To make our result precise we need some definitions.

We define a family $\{\Theta_t\}_{t>0}$ of multilinear operators by setting

$$\Theta_t(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{nm}} \theta_t(x,y_1,\dots,y_m)\prod_{i=1}^m f_i(y_i)\,dy_1\cdots dy_m, \tag{1.7}$$

where $f_1,\dots,f_m$ are initially functions in $C_0^\infty(\mathbb{R}^n)$ (smooth with compact support). For any $t > 0$, we suppose that the kernels $\theta_t(x,y_1,\dots,y_m)$ satisfy the size condition

$$|\theta_t(x,y_1,\dots,y_m)| \le A\prod_{i=1}^m \frac{t^{-n}}{(1+|x-y_i|/t)^{n+\delta}} \tag{1.8}$$

and the smoothness condition

$$|\theta_t(x, y_1, \ldots, y_i, \ldots, y_m) - \theta_t(x, y_1, \ldots, y_i', \ldots, y_m)| \le \frac{A}{t^{mn}} \frac{|y_i - y_i'|^\gamma}{t^\gamma} \tag{1.9}$$

for all $i = 1, \ldots, m$ and for all $x, y_i, y_i' \in \mathbb{R}^n$.

DEFINITION 1. We say that the family of operators Θ_t with kernels θ_t satisfies the *one-variable* $T(1)$ *cancellation condition* if for all x and y_i we have

$$\int_{\mathbb{R}^n} \theta_t(x, y_1, \ldots, y_m)\, dy_i = 0 \qquad \forall i = 1, \ldots, m. \tag{1.10}$$

We also say that Θ_t satisfies the *multi-variable* $T(1)$ *cancellation condition* if for all x we have

$$\int_{\mathbb{R}^{mn}} \theta_t(x, y_1, \ldots, y_m)\, dy_1 \cdots dy_m = 0. \tag{1.11}$$

We now state our main result:

THEOREM 1.1. *Consider the operator*

$$R_t(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} r_t(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i)\, dy_1 \cdots dy_m$$

whose kernel $r_t(x, y_1, \ldots, y_m)$ *satisfies the size condition* (1.8) *for some constant* $\delta > 0$. *Assume that for some fixed* p_i *satisfying* $2 \le p_i \le \infty$ *and*

$$\frac{1}{2} = \sum_{i=1}^m \frac{1}{p_i}$$

there is a constant B *such that*

$$\int_{\mathbb{R}^n} \int_0^\infty |R_t(f_1, \ldots, f_m)(x)|^2 \frac{dx\, dt}{t} \le B^2 \prod_{i=1}^m \|f_i\|_{p_i}^2, \tag{1.12}$$

for all $f_i \in L^{p_i}$.

(i) *If* r_t *satisfies the one-variable* $T(1)$ *cancellation condition, then there is a constant* $C_{n,\delta}$ *such that for all* $b_1, \ldots, b_m \in BMO(\mathbb{R}^n)$ *the measure*

$$d\mu(x, t) := |R_t(b_1, \ldots, b_m)(x)|^2 \frac{dx\, dt}{t} \tag{1.13}$$

is Carleson with constant

$$\|d\mu\|_{\mathcal{C}} \le C_{n,\delta}(A^2 + B^2)\|b_1\|_{\mathrm{BMO}}^2 \cdots \|b_m\|_{\mathrm{BMO}}^2. \tag{1.14}$$

(ii) *There is a constant* $C_{n,\delta}$ *such that for all* $b_1, \ldots, b_m \in L^\infty(\mathbb{R}^n)$, *the measure* $d\mu(x, t)$ *in* (1.1) *is Carleson with constant*

$$\|d\mu\|_{\mathcal{C}} \le C_{n,\delta}(A^2 + B^2)\|b_1\|_\infty^2 \cdots \|b_m\|_\infty^2.$$

In particular, this result holds if the R_t *satisfy the multi-variable* $T(1)$ *cancellation condition.*

(iii) *Under the multi-variable* $T(1)$ *cancellation condition, the measure in* (1.1) *may not be Carleson if at least one* b_i *is an unbounded BMO function.*

We recall the quadratic $T(b)$ theorem of Semmes in [23]: Let $\{R_t\}_{t>0}$ be a family of linear operators as in (1.2) whose kernels satisfy (1.3). Suppose that for all $t > 0$ we have the cancellation condition

$$R_t(b)(x) = 0$$

for some bounded complex-valued function b on $\mathbb{R}^n$ with $\operatorname{Re} b(x) \geq c_0 > 0$ for almost all $x \in \mathbb{R}^n$ (such functions are called *accretive*). Then (1.4) holds.

As an application of our main result, we obtain a multilinear extension of Semmes' theorem above.

THEOREM 1.2. *Let* $A, \delta, \gamma > 0$ *and suppose that* $\theta_t(x, y_1, \ldots, y_m)$ *satisfy the size condition* (1.8) *and the smoothness condition* (1.9). *Assume that there exist accretive functions* $b_1, \ldots, b_m$ *on* $\mathbb{R}^n$ *such that the cancellation condition*

$$\Theta_t(b_1, \ldots, b_m) = 0 \tag{1.15}$$

holds. Then for all $2 < p_i < \infty$ *satisfying* $1/2 = \sum_{i=1}^m 1/p_i$, *there is a positive constant* $C = C_{n,\delta,\gamma,p_i}$ *such that for all* $f_i \in L^{p_i}(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} \int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x)|^2 \, \frac{dx\,dt}{t} \leq C^2 A^2 \Big(1 + \prod_{i=1}^m \|b_i\|_\infty^2\Big) \prod_{i=1}^m \|f_i\|_{p_i}^2. \tag{1.16}$$

As in the linear case, the proof of the result above is based on a combination of a quadratic $T(1)$ estimate for the multilinear family of operators Θ_t (that could be seen as a continuous version of the Cotlar–Stein lemma [7], [19]) and a Carleson measure estimate based on the work of Coifman, McIntosh, and Meyer [5] on the Cauchy integral on Lipschitz curves.

An important step in the proof of the Theorem 1.2 is a multilinear quadratic $T(1)$ estimate which is stated below (Theorem 1.3). This theorem extends results of Maldonado [20] as well as of Maldonado and Naibo [21].

THEOREM 1.3. *Suppose that the kernel* $\theta_t(x, y_1, \ldots, y_m)$ *satisfies* (1.8) *and* (1.9) *for some constants* γ *and* δ. *Consider the "multilinear" square function*

$$S(f_1, \ldots, f_m) = \bigg(\int_0^\infty |\Theta_t(f_1, \ldots, f_m)|^2 \, \frac{dt}{t}\bigg)^{1/2}.$$

Suppose that either Θ_t *satisfies the one-variable* $T(1)$ *cancellation condition or it satisfies the multi-variable* $T(1)$ *cancellation condition. Then*

$$S : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

for all $1 < p_1, \dots, p_2, p < \infty$ *and* $1/p = 1/p_1 + \dots + 1/p_m$ *with norm*

$$\|S\|_{\mathrm{op}} \le C_{n,\delta,\gamma} A. \tag{1.17}$$

Moreover, if Θ_t *satisfies the one-variable* $T(1)$ *cancellation condition, then the endpoint estimates for some* (*but not all*) p_j *are infinite are valid.*

The paper is organized as follows. In the next section we state a basic lemma that will be used subsequently. In the third section we prove the Carleson measure estimate (Theorem 1.1). In the fourth section we prove the multilinear quadratic $T(1)$ theorems and in the fifth section we prove the multilinear quadratic $T(b)$ theorem.

Remark. During the preparation of this article, it came to our attention that Theorems 1.3 and 1.2 overlap recent results obtained by Hart [16] when $m = 2$. Hart's approach in [16], as well as in his subsequent work [17], is based on the Calderón reproducing formula adapted to the bilinear setting and is different than ours which is based on Carleson measures formed by multilinear operators.

2. Preliminary material. Throughout the proofs of our results we will be working with smooth functions with compact support (C_0^∞). Estimates for general L^{p_j} functions follow then by density.

Sometimes we will just use the notation $A \lesssim B$ to indicate the existence of a constant C that could depend on the dimension n, and on the numbers δ and γ introduced in equations (1.8) and (1.9), and on the norm of some Hardy–Littlewood operators that appears in our proofs.

We begin by recalling the definition of continuous Littlewood–Paley operators. To fix notation, we consider a smooth function Ψ whose Fourier transform is supported in the annulus $\{1/2 \le |\xi| \le 2\}$. We denote by $\Psi_s(x) = s^{-n}\Psi(x/s)$ the L^1 dilation of Ψ. We call the operator $Q_s(h) = h \star \Psi_s$ the *Littlewood–Paley operator* associated with Ψ. Throughout this paper we assume that Ψ has the additional property that

$$\int_0^\infty Q_s^2 \frac{ds}{s} := \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int_\epsilon^N Q_s^2 \frac{ds}{s} = I \tag{2.1}$$

where the convergence is in $\mathcal{S}' \setminus \mathcal{P}$; here $\mathcal{P}$ is the space of polynomials on $\mathbb{R}^n$.

We state a useful lemma, whose proof is straightforward and is omitted.

Lemma 1. *Let* $P_s(f)(x) = \int_{\mathbb{R}^n} \Phi_s(x-y) f(y)\, ds$, *where* $\Phi \in C_0^\infty(\mathbb{R}^n)$ *satisfies*

$$|\Phi(y)| \lesssim \frac{1}{(1+|y|)^{n+1}} \tag{2.2}$$

and

$$\int_{\mathbb{R}^n} \Phi(y)\,dy = 1. \tag{2.3}$$

Suppose also that $1 \le p_1, \ldots, p_m \le \infty$ *and* $0 < r < \infty$. *Then for all Carleson measures* $d\mu(x,t)$ *the* m*-linear operator*

$$\mathbb{P}_s(f_1, \ldots, f_m)(x) = \prod_{i=1}^{m} P_s(f_i)(x)$$

satisfies

$$\int_{\mathbb{R}^n} \int_0^\infty |\mathbb{P}_t(f_1, \ldots, f_m)(x)|^r \, d\mu(x,t) \lesssim \prod_{i=1}^{m} \|f_i\|_{p_i}^r \tag{2.4}$$

for all smooth functions $f_i \in L^{p_i}(\mathbb{R}^n)$, *where*

$$\frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}.$$

3. Carleson measure estimates. In this section we prove Theorem 1.1.

Fix a cube Q. To prove part (i) we need to show that for all $b_1, \ldots, b_m$ in $\mathrm{BMO}(\mathbb{R}^n)$,

$$\int_Q \int_0^{l(Q)} |R_t(b_1, \ldots, b_m)(x)|^2 \, \frac{dx\,dt}{t} \lesssim \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}}^2 |Q|. \tag{3.1}$$

Setting $b_{j,0} = (b_j - \mathrm{Avg}_Q\, b_j)\chi_{Q^*}$, $b_{j,1} = (b_j - \mathrm{Avg}_Q\, b_j)\chi_{(Q^*)^c}$, $b_{j,2} = \mathrm{Avg}_Q\, b_j$, where $Q^* = 2Q$, we introduce the decompositions

$$b_j = b_{j,0} + b_{j,1} + b_{j,2} \tag{3.2}$$

for all $j = 1, \ldots, m$. Using Minkowski's inequality, it suffices to show that

$$\int_Q \int_0^{l(Q)} |R_t(b_{1,j_1}, \ldots, b_{m,j_m})(x)|^2 \, \frac{dx\,dt}{t} \lesssim \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}}^2 |Q|, \tag{3.3}$$

where $j_i \in \{0,1,2\}$. Since r_t satisfies the multilinear one-variable $T(1)$ condition, each term of the form (3.3) that contains $b_{i,2}$ vanishes. So, we may focus attention on the situation where we have m_1 entries containing $b_{j,1}$ and m_2 entries entries containing $b_{j,2}$, with $m = m_1 + m_2$ and $0 \le m_1 \le m$. By a permutation of the variables we may assume that the first m_1 variables of θ_t correspond to $b_{j,1}$ and the remaining variables of θ_t correspond to $b_{j,2}$.

We begin with the case where $m_2 = 0$. Then

$$\int_{\mathbb{R}^n}\int_0^\infty |R_t(b_{1,1},\dots,b_{m,1})|^2\,\frac{dx\,dt}{t} \le B^2\prod_{i=1}^m \|(b_i - \operatorname{Avg}_Q b_i)\chi_{Q^*}\|_{p_i}^2 \lesssim B^2\prod_{i=1}^m \|b_i\|_{\mathrm{BMO}}^2|Q|. \tag{3.4}$$

We now consider the case where $m_2 > 0$. In this case, we simply estimate $R_t(b_1,\dots,b_m)(x)$ using the size condition (1.8) which permits us to obtain

$$\begin{aligned}|R_t(b_{1,1},\dots,b_{m_1,1},b_{m_1+1,2},\dots,b_{m,2})(x)|&\\ \lesssim A\prod_{i=1}^{m_1}\Big(\int_{\mathbb{R}^n}\frac{(b_i-\operatorname{Avg}_Q b_i)\chi_{Q^*}(y_i)}{t^n(1+t^{-1}|x-y_i|)^{n+\delta}}\,dy_i\Big)&\\ \times\prod_{i=m_1+1}^{m_2}\Big(\int_{\mathbb{R}^n}\frac{(b_i-\operatorname{Avg}_Q b_i)\chi_{(Q^*)^c}(y_i)}{t^n(1+t^{-1}|x-y_i|)^{n+\delta}}\,dy_i\Big)&\\ =: A\Big(\prod_{i=1}^{m_1}P_{1,t}(b_i)(x)\Big)\Big(\prod_{i=m_1+1}^{m}P_{2,t}(b_i)(x)\Big).&\end{aligned} \tag{3.5}$$

For $P_{2,t}(b_i)$ we use the easy estimate

$$|P_{2,t}(b_i)(x)| \lesssim \Big(\frac{t}{l(Q)}\Big)^{\delta}\|b_i\|_{\mathrm{BMO}}, \tag{3.6}$$

which can be found for instance in [24, p. 160]. As for $P_{1,t}(b_i)(x)$, we control it by the Hardy–Littlewood maximal function:

$$|P_{1,t}(b_i)(x)| \lesssim M((b_i - \operatorname{Avg}_Q b_i)\chi_{Q^*}(x)). \tag{3.7}$$

Combining these estimates we deduce

$$\begin{aligned}&\int_Q\int_0^{l(Q)} |R_t(b_{1,1},\dots,b_{m_1,1},b_{m_1+1,2},\dots,b_{m,2})(x)|^2\,\frac{dx\,dt}{t}\\ &\quad\lesssim A^2\int_Q\int_0^{l(Q)}\Big(\prod_{i=1}^{m_1}M((b_i-\operatorname{Avg}_Q b_i)\chi_{Q^*})(x)\Big)^2\\ &\qquad\qquad\times\frac{t^{2m_2}}{l(Q)^{2m_2}}\prod_{i=m_1+1}^{m_2}\|b_i\|_{\mathrm{BMO}}^2\,\frac{dx\,dt}{t}\\ &\quad\lesssim A^2\Big\|\prod_{i=1}^{m_1}M((b_i-\operatorname{Avg}_Q b_i)\chi_{Q^*})\Big\|_{L^2(Q)}^2\prod_{i=m_1+1}^{m}\|b_i\|_{\mathrm{BMO}}^2\\ &\quad\lesssim A^2\prod_{i=1}^{m_1}\|M((b_i-\operatorname{Avg}_Q b_i)\chi_{Q^*})\|_{L^{p_i}(\mathbb{R}^n)}^2\prod_{i=m_1+1}^{m}\|b_i\|_{\mathrm{BMO}}^2\end{aligned}$$

$$\lesssim A^2 \prod_{i=1}^{m_1} \|(b_i - \operatorname{Avg}_Q b_i)\chi_{Q^*}\|^2_{L^{p_i}(\mathbb{R}^n)} \prod_{i=m_1+1}^{m} \|b_i\|^2_{\mathrm{BMO}}$$
$$\lesssim A^2 \prod_{i=1}^{m_1} \|b_i\|^2_{\mathrm{BMO}}(|Q|^{\frac{1}{p_1}+\cdots+\frac{1}{p_{m_1}}})^2 \prod_{i=m_1+1}^{m} \|b_i\|^2_{\mathrm{BMO}}$$
$$= CA^2|Q| \prod_{i=1}^{m} \|b_i\|^2_{\mathrm{BMO}},$$

where $p_i \geq 2$ are numbers that satisfy $1/p_1 + \cdots + 1/p_{m_1} = 1/2$. We used here the characterization of BMO in terms of an L^p mean oscillation.

We now turn to part (ii) of the theorem. Here we use the decompositions

$$b_j = \chi_{Q^*} b_j + \chi_{(Q^*)^c} b_j = b_{j,0} + b_{j,1} \tag{3.8}$$

where $Q^* = 2Q$. To estimate the quantity

$$\int_0^{l(Q)} \int_Q |R_t(b_{1,0}, \ldots, b_{m,0})(x)|^2 \,\frac{dx\,dt}{t} \tag{3.9}$$

we use (1.12), which shows that (3.9) is at most

$$B^2 \|b_{1,0}\|^2_{L^{p_1}} \cdots \|b_{m,0}\|^2_{L^{p_m}} \leq B^2 (|Q^*|^{1/p_1+\cdots+1/p_m})^2 \prod_{i=1}^{m} \|b_i\|^2_{L^\infty}$$
$$\leq cB^2 \prod_{i=1}^{m} \|b_i\|^2_{L^\infty},$$

and this yields the desired conclusion for this term. If at least one $b_{j,0}$ is replaced by $b_{j,1}$ in (3.9), then we use the estimate

$$|R_t(b_{1,r_1}, \ldots, b_{m,r_m})|^2 \leq A\Big[\prod_{i,r_i=0} M(b_{i,r_i})\Big]\Big[\prod_{i,r_i=1} P_{2,t}(b_{i,r_i})\Big], \tag{3.10}$$

where $P_{2,t}$ is defined earlier, and the second product has at least one term, and we make use of (3.6) which makes the dt/t integral converge.

We now turn to part (iii) of the theorem. We give an example indicating that one may not replace L^∞ with BMO in Theorem 1.1(ii) under the multi-variable multilinear $T(1)$ condition. We select a smooth polynomially decaying function $\psi(y, z)$ on $\mathbb{R}^2$ with the following properties ([1]):

(a) $\psi(-y, z) = -\psi(y, z)$.

(b) Let $\psi^0(y) = \int_{\mathbb{R}} \psi(y, z)\,dz$. Then $\int_0^1 |\widehat{\psi^0}(\xi)|^2\,d\xi > 0$.

Condition (a) implies that $\psi(y, z)$ has mean value zero with respect to the variable y and thus over $\mathbb{R}^2$. Condition (b) in particular implies that ψ^0

([1]) The function $\psi(y, z) = y(1 + |y|^2 + |z|^2)^{-5/2}$ has the specified properties.

is not the zero function, hence $\int_{\mathbb{R}} \psi(y,z)\,dz = \psi^0(y) \neq 0$ for some $y \in \mathbb{R}$. Hence the kernel

$$\theta_t(x,y_1,y_2) = \frac{1}{t^2}\psi\left(\frac{x-y_1}{t}, \frac{x-y_2}{t}\right)$$

satisfies the multi-variable $T(1)$ cancellation condition and also the one-variable $T(1)$ cancellation condition with respect to the y_1 variable, but not with respect to the y_2 variable.

Now consider the bilinear integral operator given by

$$R_t(f,g)(x) = \int_{\mathbb{R}^2} \frac{1}{t^2}\psi\left(\frac{x-y}{t}, \frac{x-z}{t}\right) f(y)g(z)\,dy\,dz.$$

Given $0 < \epsilon < 1/100$ take

$$b_1^\epsilon(y) = e^{2\pi i y/\epsilon} - e^{-2\pi i y/\epsilon}, \qquad b_2(y) = \log|y|,$$

and $I_\epsilon = [-\epsilon, \epsilon]$. Then b_2 is a BMO function whose average over I_ϵ is $a_\epsilon \approx \log \epsilon^{-1}$ and b_1 is an odd function bounded by 2. Write

$$b_2 = b_{2,0} + b_{2,1} + b_{2,2},$$

where

$$\begin{aligned} b_{2,0}(y) &= (\log|y| - a_\epsilon)\chi_{2I_\epsilon}, \\ b_{2,1}(y) &= (\log|y| - a_\epsilon)\chi_{(2I_\epsilon)^c}, \qquad b_{2,2} = a_\epsilon. \end{aligned}$$

We have

$$R_t(b_1^\epsilon, b_2) = R_t(b_1^\epsilon, b_{2,0}) + R_t(b_1^\epsilon, b_{2,1}) + R_t(b_1^\epsilon, b_{2,2}).$$

We split $b_1^\epsilon = b_1^\epsilon\chi_{2I_\epsilon} + b_1^\epsilon\chi_{(2I_\epsilon)^c}$ and we use the estimates in the proof of Theorem 1.1(ii), which yield (without using any cancellation properties of ψ)

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{\epsilon} |R_t(b_1^\epsilon, b_{2,0})(x)|^2 \,\frac{dx\,dt}{t} + \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{\epsilon} |R_t(b_1^\epsilon, b_{2,1})(x)|^2 \,\frac{dx\,dt}{t} \le C',$$

where $C' \ge C_n(A+B)^2\|b_1^\epsilon\|_{\mathrm{BMO}}^2\|b_2\|_{\mathrm{BMO}}^2$ and C' is independent of ϵ. The cancellation of ψ was only used to annihilate the terms containing at least one average but no average appears in the preceding estimate. Using these observations, in order to prove that the measure $|R_t(b_1^\epsilon, b_2)|^2 dx\,dt/t$ is not Carleson, it will suffice to show that

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{\epsilon} |R_t(b_1^\epsilon, b_{2,2})(x)|^2 \,\frac{dx\,dt}{t} \ge c|\log \epsilon^{-1}|^2 \uparrow \infty \qquad \text{as } \epsilon \to 0+,$$

which, since $b_{2,2} = a_\epsilon$, is equivalent to showing that

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{\epsilon} |R_t(b_1^\epsilon, 1)(x)|^2 \,\frac{dx\,dt}{t} \ge c. \tag{3.11}$$

A simple calculation gives that

$$R_t(b_1^\epsilon,1)(x) = e^{2\pi ix/\epsilon}\widehat{\psi^0}(t/\epsilon) - e^{-2\pi ix/\epsilon}\widehat{\psi^0}(-t/\epsilon)$$
$$= 2\cos(2\pi x/\epsilon)\widehat{\psi^0}(t/\epsilon),$$

where the last identity follows from the fact that $\widehat{\psi^0}$ is odd, which is a consequence of the fact that ψ is odd in the first variable. Using this identity for $R_t(b_1^\epsilon,1)(x)$, we find that the expression on the left in (3.11) is at least

$$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{\epsilon} 4\cos^2(2\pi x/\epsilon)|\widehat{\psi^0}(t/\epsilon)|^2\,\frac{dx\,dt}{t}$$

and by a change of variables the above is bounded from below by a constant independent of ϵ since

$$\int_0^1 |\widehat{\psi^0}(\xi)|^2\,d\xi > 0.$$

4. $T(1)$ square function estimates. In this section we prove Theorem 1.3. This theorem will be a necessary tool in the proof of Theorem 1.2. We fix functions f_j in $C_0^\infty(\mathbb{R}^n)$.

Proof of Theorem 1.3. Using the Littlewood–Paley operator Q_s introduced in Section 2, we rewrite the operator Θ_t in a more convenient form:

$$\begin{aligned}
(4.1)\qquad & \Theta_t(f_1,\ldots,f_m)(x)\\
&= \Theta_t\Big(\int_0^\infty Q_s^2 f_1\,\frac{ds}{s},\ldots,f_m\Big)(x)\\
&= \int_{\mathbb{R}^{2n}}\Big(\int_0^\infty (Q_s^2 f_1)(y_1)\,\frac{ds}{s}\Big)\prod_{i=2}^m f_i(y_i)\theta_t(x,y_1,\ldots,y_m)\,dy_1\cdots dy_m\\
&= \int_0^\infty\Big(\int_{\mathbb{R}^{2n}} (Q_s^2 f_1)(y)\prod_{i=2}^m f_i(z)\theta_t(x,y_1,\ldots,y_m)\,dy_1\cdots dy_m\Big)\frac{ds}{s}\\
&= \int_0^\infty \Theta_t(Q_s^2 f_1,\ldots,f_m)(x)\,\frac{ds}{s}.
\end{aligned}$$

We will study of the boundedness of this operator via duality. For this purpose we introduce a function $h(x,t)\in L^{p'}(dx,L^2(dt/t))$ with

$$\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty |h(x,t)|^2\,\frac{dt}{t}\Big)^{p'/2}dx\Big)^{1/p'} = \|h\|_{p',2}\le 1.$$

Using this function we obtain

$$\|S(f_1,\dots,f_m)\|_p=\Big(\int_{\mathbb{R}^n}\Big|\int_0^\infty|\Theta_t(f_1,\dots,f_m)(x)|^2\,\frac{dt}{t}\Big|^{p/2}dx\Big)^{1/p}$$
$$=\sup_{\|h\|_{p',2}\le 1}\Big|\int_{\mathbb{R}^n}\int_0^\infty\Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\Big|. \tag{4.2}$$

Now for a fixed h with $\|h\|_{p',2}\le 1$ we have

$$\Big|\int_{\mathbb{R}^n}\int_0^\infty\Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\Big| \tag{4.3}$$
$$=\Big|\int_{\mathbb{R}^n}\int_0^\infty\Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\Big|$$
$$=\Big|\int_{\mathbb{R}^n}\int_0^\infty\Big(\int_0^\infty\Theta_t(Q_s^2f_1,\dots,f_m)(x)\,\frac{ds}{s}\Big)h(x,t)\,\frac{dt}{t}\,dx\Big|$$
$$=\Big|\int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty\Theta_t(Q_s^2f_1,\dots,f_m)(x)h(x,t)\,\frac{ds}{s}\,\frac{dt}{t}\,dx\Big|$$
$$\le\Big|\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_0^\infty|\Theta_t(Q_s^2f_1,\dots,f_m)(x)|^2w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{1/2}$$
$$\times\Big(\int_0^\infty\int_0^\infty|h(x,t)|^2w(t,s)\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{1/2}dx\Big|,$$

where $w(t,s)$ is a positive symmetric function which satisfies

$$A_w:=\sup_{t>0}\int_0^\infty w(t,s)\,\frac{ds}{s}=\sup_{s>0}\int_0^\infty w(t,s)\,\frac{dt}{t}<\infty \tag{4.4}$$

and will be defined later. Combining (4.2) and (4.3) we deduce that

$$\|S(f_1,\dots,f_m)\|_p\le\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_0^\infty|\Theta_t(Q_s^2f_1,\dots,f_m)|^2w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{p/2}dx\Big)^{1/p}$$
$$\times\sup_h\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_0^\infty|h(x,t)|^2w(t,s)\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{p'/2}dx\Big)^{1/p'}$$
$$\le\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_0^\infty|\Theta_t(Q_s^2f_1,\dots,f_m)(x)|^2w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{p/2}dx\Big)^{1/p}$$
$$\times\sup_h\Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty|h(x,t)|^2\Big(\sup_{t>0}\int_0^\infty w(t,s)\,\frac{ds}{s}\Big)\,\frac{dt}{t}\Big)^{p'/2}dx\Big)^{1/p'}$$

$$\leq \bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty\int_0^\infty |\Theta_t(Q_s^2 f_1,\dots,f_m)(x)|^2 w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p/2}dx\bigg)^{1/p}$$

$$\times \sup_h A_\omega^{1/2}\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty |h(x,t)|^2\,\frac{dt}{t}\bigg)^{p'/2}dx\bigg)^{1/p'}$$

$$\leq A_\omega^{1/2}\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty\int_0^\infty |\Theta_t(Q_s^2 f_1,\dots,f_m)|^2 w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p/2}dx\bigg)^{1/p},$$

where the supremum is taken over all h satisfying $\|h\|_{p',2}\leq 1$.

Let M be the Hardy–Littlewood maximal function. In view of the result in [13, pp. 247–248] there is a function $w(t,s)$ satisfying (4.4) such that

$$|\Theta_t(Q_s f_1,\dots,f_m)(x)| \lesssim A w(t,s)\prod_{i=1}^m M(f_i)(x). \tag{4.5}$$

Assume for the moment that (4.5) holds. Define p^* by setting

$$\frac{1}{p^*}=\frac{1}{p_2}+\cdots+\frac{1}{p_m}$$

so that

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p^*}.$$

In view of (4.5) we obtain

$$\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty\int_0^\infty |\Theta_t(Q_s^2 f_1,\dots,f_m)(x)|^2 w(t,s)^{-1}\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p/2}dx\bigg)^{1/p}$$

$$\lesssim A\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty\int_0^\infty |M(Q_s f_1)(x)|^2\Big|\prod_{i=2}^m M(f_i)(x)\Big|^2 w(t,s)\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p/2}dx\bigg)^{1/p}$$

$$= A\bigg(\int_{\mathbb{R}^n}\Big|\prod_{i=2}^m M(f_i)(x)\Big|^p\bigg(\int_0^\infty\int_0^\infty |M(Q_s f_1)(x)|^2 w(t,s)\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p/2}dx\bigg)^{1/p}$$

$$\leq \Big\|\prod_{i=2}^m M(f_i)\Big\|_{p^*}\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty\int_0^\infty |M(Q_s f_1)(x)|^2 w(t,s)\,\frac{ds}{s}\,\frac{dt}{t}\bigg)^{p_1/2}dx\bigg)^{1/p_1}$$

$$\lesssim A\prod_{i=2}^m \|f_i\|_{p_i}\bigg(\int_{\mathbb{R}^n}\bigg\{\int_0^\infty |M(Q_s f_1)(x)|^2\bigg(\sup_s\int_0^\infty w(t,s)\,\frac{dt}{t}\bigg)\frac{ds}{s}\bigg\}^{p_1/2}dx\bigg)^{1/p_1}$$

$$\lesssim A\,A_w^{1/2}\prod_{i=2}^m \|f_i\|_{p_i}\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty |M(Q_s f_1)(x)|^2\,\frac{ds}{s}\bigg)^{p_1/2}dx\bigg)^{1/p_1}$$

$$\lesssim A\,A_w^{1/2}\prod_{i=2}^m \|f_i\|_{p_i}\bigg(\int_{\mathbb{R}^n}\bigg(\int_0^\infty |(Q_s f_1)(x)|^2\,\frac{ds}{s}\bigg)^{p_1/2}dx\bigg)^{1/p_1}\lesssim A\,A_w^{1/2}\prod_{i=1}^m \|f_i\|_{p_i},$$

where we used the Fefferman–Stein [10] vector-valued maximal function inequality and the Littlewood–Paley theorem. We conclude that

$$\|S(f_1,\dots,f_m)\|_p \lesssim A\prod_{i=1}^m \|f_i\|_{p_i} \tag{4.6}$$

whenever $1<p_i<\infty$ and

$$\frac{1}{p}=\sum_{i=1}^m \frac{1}{p_i}.$$

But (4.6) is a restatement of (1.17) that we were supposed to prove.

We now discuss the proof of (4.5). To begin, we introduce some notation. Observe that

$$\begin{aligned}
&\Theta_t(Q_s f_1,\dots,f_m)(x)\\
&\quad=\int_{\mathbb{R}^{mn}} \theta_t(x,y_1,\dots,y_m)(Q_s f_1)(y_1)\prod_{i=2}^m f_i(y_i)\,dy_1\cdots dy_m\\
&\quad=\int_{\mathbb{R}^{mn}} \theta_t(x,y_1,\dots,y_m)\Big(\int_{\mathbb{R}^n}\psi_s(y_i-u)f_1(u)\,du\Big)\prod_{i=2}^m f_i(y_i)\,dy_1\cdots dy_m\\
&\quad=\int_{\mathbb{R}^{mn}} \Big\{\int_{\mathbb{R}^n}\theta_t(x,y_1,\dots,y_m)\psi_s(y_1-u)\,dy_1\Big\}f_1(u)\prod_{i=2}^m f_i(y_i)\,du\,dy_2\cdots dy_m\\
&\quad=\int_{\mathbb{R}^{mn}} L_{t,s}(x,u,y_2,\dots,y_m)f_1(u)\prod_{i=2}^m f_i(y_i)\,du\,dy_2\cdots dy_m.
\end{aligned}$$

The result will follow from a pointwise estimate for $L_{t,s}(x,u,y_2,\dots,y_m)$. We will make use of the following integral identities (whose proof can be found in [13, pp. 229]):

$$\int_{\mathbb{R}^N}\frac{s^{-N}\min(2,t^{-1}|u|^\gamma)}{(1+s^{-1}|u|)^{N+1}}\,du \le C_N\Big(\frac{s}{t}\Big)^{\frac12\min(\gamma,1)} \qquad \text{if } s\le t, \tag{4.7}$$

$$\int_{\mathbb{R}^N}\frac{t^{-N}\min(2,s^{-1}|u|)}{(1+t^{-1}|u|)^{N+\delta}}\,du \le C_N\Big(\frac{t}{s}\Big)^{\frac12\min(\delta,1)} \qquad \text{if } t\le s. \tag{4.8}$$

We begin with the case $s\le t$. Using the fact that $Q_s(1)=0$ for all $s>0$, and that $\psi_s(y)$ satisfies the same size estimate as $\theta_t(x,y_1,\dots,y_m)$ with $\gamma=\delta=1$, we get

$$\begin{aligned}
|L_{s,t}(x,u,\dots,y_m)| &= \Big|\int_{\mathbb{R}^n}\theta_t(x,y_1,\dots,y_m)\psi_s(y_1-u)\,dy_1\Big| \\
&= \Big|\int_{\mathbb{R}^n}[\theta_t(x,y_1,\dots,y_m)-\theta_t(x,u,\dots,y_m)]\psi_s(y_1-u)\,dy_1\\
&\qquad + \theta_t(x,u,\dots,y_m)\int_{\mathbb{R}^n}\psi_s(y_1-u)\,dy_1\Big|
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&= \Big| \int_{\mathbb{R}^n} [\theta_t(x, y_1, \dots, y_m) - \theta_t(x, u, \dots, y_m)] \psi_s(y_1 - u)\, dy_1 \Big| \\
&\le CA \int_{\mathbb{R}^n} \frac{\min(2, (t^{-1}|y_1 - u|)^\gamma)}{t^{2n}} \frac{s^{-n}}{(1 + s^{-1}|y_1 - u|)^{n+1}} dy_1 \\
&\le C_n A \frac{1}{t^{2n}} \left(\frac{s}{t}\right)^{\frac{1}{2}\min(\gamma,1)} \\
&\le C_n A \min\left(\frac{1}{t}, \frac{1}{s}\right)^n \min\left(\frac{t}{s}, \frac{s}{t}\right)^{\frac{1}{2}\min(\gamma,\delta,1)}.
\end{aligned}
$$

Now, for $t \le s$ we have

$$
\begin{aligned}
(4.10)\qquad & |L_{s,t}(x, u, \dots, y_m)| \\
&= \Big| \int_{\mathbb{R}^n} \theta_t(x, y_1, \dots, y_m) \psi_s(y_1 - u)\, dy_1 \Big| \\
&= \Big| \int_{\mathbb{R}^n} \theta_t(x, y_1, \dots, y_m) [\psi_s(y_1 - u) - \psi(x - u)]\, dy_1 \Big| \\
&\le CAt^{-(m-1)n} \left(\prod_{i=2}^m \frac{1}{(1 + t^{-1}|x - y_i|)^{n+\delta}} \right) \\
&\qquad \times \int_{\mathbb{R}^n} \frac{t^{-n}}{(1 + t^{-1}|x - y_1|)^{n+\delta}} \frac{\min(2, s^{-1}|x - y_1|)}{s^n}\, dy_1 \\
&\le C_n A \min\left(\frac{1}{t}, \frac{1}{s}\right)^n \min\left(\frac{t}{s}, \frac{s}{t}\right)^{\frac{1}{2}\min(\gamma,\delta,1)} \prod_{i=2}^m \frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^{n+\delta}}.
\end{aligned}
$$

We also have the following integral inequality (see Appendix K.1 in [12]):

$$
\begin{aligned}
(4.11)\qquad |L_{s,t}(x, u, \dots, y_m)| &\le \int_{\mathbb{R}^n} |\theta_t(x, y_1, \dots, y_m) \psi_s(y_1 - u)|\, dy_1 \\
&\le \frac{CA \min(1/t, 1/s)^n}{(1 + \min(1/t, 1/s)|x - u|)^{n+\min(\delta,1)}} \left(\prod_{i=2}^m \frac{1}{(1 + t^{-1}|x - y_i|)^{n+\delta}} \right).
\end{aligned}
$$

Combining the estimates (4.9)–(4.11), we obtain

$$
\begin{aligned}
(4.12)\quad |L_{s,t}(x, u, \dots, y_m)| &\lesssim \frac{\min(1/t, 1/s)^{\frac{1}{2}\min(\delta,\gamma,1)(1-\beta)} A \min(1/t, 1/s)^n}{((1 + \min(1/t, 1/s)|x - u|)^{n+\min(\delta,1)})^\beta} \\
&\qquad \times \left(\prod_{i=2}^m \frac{1}{(1 + t^{-1}|x - y_i|)^{n+\delta}} \right)
\end{aligned}
$$

for all $0 < \beta < 1$. If we choose $\beta = \big(n + \frac{1}{2}\min(\delta, 1)\big)(n + \min(\delta, 1))^{-1}$, then estimate (4.12) implies that

$$(4.13)\quad |\Theta_t(Q_s f_1,\dots,f_m)(x)| \lesssim A\min\Big(\frac{1}{t},\frac{1}{s}\Big)^{\epsilon}\Big(\int_{\mathbb{R}^n}\frac{\min(1/t,1/s)^n}{((1+\min(1/t,1/s)|x-u|)^{n+\min(\delta,1)})^{\beta}}|f(u)|\,du\Big)$$
$$\times\prod_{i=2}^{m}\Big(\int_{\mathbb{R}^n}\frac{t^{-n}}{(1+t^{-1}|x-y_i|)^{n+\delta}}|f_i(y_i)|\,dy_i\Big)$$
$$\lesssim \min\Big(\frac{t}{s},\frac{s}{t}\Big)^{\epsilon}\prod_{i=1}^{m}M(f_i)(x),$$

where we set

$$\epsilon=\frac{1}{4}\min(\delta,\gamma,1)\frac{\min(\delta,1)}{n+\min(\delta,1)}$$

and we used the well-known inequality

$$(4.14)\qquad \int_{\mathbb{R}^n}|h(w)|\frac{a^{-n}}{(1+a^{-1}|x-w|)}\,dw\lesssim M(h)(x).$$

As the function $w(t,s)=\min(t/s,s/t)^{\epsilon}$ satisfies (4.4), estimate (4.5) easily follows.

This completes the proof of the endpoint estimate

$$L^p\times L^\infty\times\dots\times L^\infty\to L^p$$

in Theorem 1.3 under the one-variable $T(1)$ cancellation condition. Boundedness at the remaining points as claimed by the theorem follows by symmetry and multilinear interpolation [14].

It remains to obtain the same estimate under the multi-variable $T(1)$ cancellation condition.

So suppose now that θ_t satisfies this condition. Just as in the preceding situation, we start by using duality and Littlewood–Paley operators to put the operator in a more tractable form. Then

$$(4.15)\qquad \|S(f_1,\dots,f_m)\|_p = \sup_{\|h\|_{p',2}\le 1}\Big|\int_{\mathbb{R}^n}\int_0^\infty \Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\Big|.$$

Setting $\Theta_t(Q^2_{s_1}f_1,\dots,Q^2_{s_m}f_m)(x)=\widetilde{\Theta}_{t,\vec{s}}(x)$, we have

$$(4.16)\quad \Big|\int_{\mathbb{R}^n}\int_0^\infty \Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\Big|$$
$$=\Big|\int_{\mathbb{R}^n}\int_0^\infty\Big(\int_0^\infty\cdots\int_0^\infty\widetilde{\Theta}_{t,\vec{s}}(x)\,\frac{ds_1}{s_1}\cdots\frac{ds_m}{s_m}\Big)h(x,t)\,\frac{dt}{t}\,dx\Big|$$
$$=\Big|\int_{\mathbb{R}^n}\int_0^\infty\cdots\int_0^\infty\widetilde{\Theta}_{t,\vec{s}}(x)h(x,t)\,\frac{ds_1}{s_1}\cdots\frac{ds_m}{s_m}\,\frac{dt}{t}\,dx\Big|$$

$$\leq \Big| \int_{\mathbb{R}^n} \Big(\int_0^\infty \cdots \int_0^\infty |\widetilde{\Theta}_{t,\vec{s}}(x)|^2 w(s_1, \ldots, s_m, t)^{-1} \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \frac{dt}{t} \Big)^{1/2}$$
$$\times \Big(\int_0^\infty \cdots \int_0^\infty |h(x,t)|^2 w(s_1, \ldots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \frac{dt}{t} \Big)^{1/2} dx \Big|,$$

where

$$w(s_1, \ldots, s_m, t) = \prod_{i=1}^m \min\Big(\frac{t}{s_i}, \frac{s_i}{t} \Big)^\epsilon$$

for some $\epsilon > 0$. This function is symmetric in all variables and satisfies

$$A_1 = \sup_t \int_0^\infty \cdots \int_0^\infty w(s_1, \ldots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} < \infty, \tag{4.17}$$

$$A_2 = \sup_{s_1>0} \ldots \sup_{s_m>0} \int_0^\infty w(s_1, \ldots, s_m, t) \frac{dt}{t} < \infty. \tag{4.18}$$

In particular, in view of (4.17), we have

$$\Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty \cdots \int_0^\infty |h(x,t)|^2 w(s_1, \ldots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \frac{dt}{t} \Big)^{p'/2} dx \Big)^{1/p'}$$
$$\leq \Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty |h(x,t)|^2 \sup_{t>0} \int_0^\infty \cdots \int_0^\infty w(s_1, \ldots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \Big)^{p'/2} dx \Big)^{1/p'}$$
$$\lesssim \|h\|_{p',2}.$$

Proceeding exactly as in the case of one variable, we reduce matters to showing that

$$|\Theta_t(Q_{s_1} f_1, \ldots, Q_{s_m} f_m)(x)| \lesssim w(s_1, \ldots, s_m, t) \prod_{i=1}^m M(f_i)(x). \tag{4.19}$$

Inserting this estimate in (4.16) we obtain

$$\Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty \cdots \int_0^\infty |\Theta_t(Q_{s_1}^2 f_1, \ldots, Q_{s_m}^2 f_m)(x)|^2 \tag{4.20}$$
$$\times w(s_1, \ldots, s_m, t)^{-1} \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \frac{dt}{t} \Big)^{p/2} dx \Big)^{1/p}$$
$$\leq \Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty \cdots \int_0^\infty \Big| \prod_{i=1}^m M(Q_{s_i} f_i)(x) \Big|^2$$
$$\times w(s_1, \ldots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} \frac{dt}{t} \Big)^{p/2} dx \Big)^{1/p}$$

$$\leq \bigg(\int_{\mathbb{R}^n} \bigg(\prod_{i=1}^{m} \int_0^\infty |M(Q_{s_i} f_i)(x)|^2 \, \frac{ds_i}{s_i} \bigg)^{p/2} \\ \times \bigg(\sup_{s_1>0} \ldots \sup_{s_m>0} \int_0^\infty w(s_1, \ldots, s_m, t) \, \frac{dt}{t} \bigg)^{p/2} dx \bigg)^{1/p}.$$

The proof is now easily completed by applying Hölder's inequality, the vector-valued maximal function Fefferman–Stein inequality [10], and the Littlewood–Paley theorem.

It suffices to obtain estimate (4.19). We begin by observing that the kernel of the operator $\Theta_t(Q_{s_1} f_1, \ldots, Q_{s_m} f_m)(x)$ is given by

$$L_{t,s_1,\ldots,s_m}(x, u_1, \ldots, u_m) \\ = \int_{\mathbb{R}^{mn}} \theta_t(x, y_1, \ldots, y_m) \prod_{i=1}^{m} \psi_{s_i}(y_i - u_i) \, dy_1 \cdots dy_m. \tag{4.21}$$

We estimate this operator using two different kinds of decompositions. If $t \leq r, s_1, \ldots, s_m$, we can use the cancellation properties of θ_t and ψ_{s_i} and rewrite $L_{t,s_1,\ldots,s_m}$ as

$$\begin{aligned} &L_{t,s_1,\ldots,s_m}(x, u_1, \ldots, u_m) \\ &\quad = \int_{\mathbb{R}^{mn}} \theta_t(x, y_1, \ldots, y_m) \prod_{i=1}^{m} \psi_{s_i}(y_i - u_i) \, d\vec{y} \\ &\quad = \int_{\mathbb{R}^{mn}} [\theta_t(x, y_1, \ldots, y_m) - \theta_t(x, u_1, y_2, \ldots, y_m)] \prod_{i=1}^{m} \psi_{s_i}(y_i - u_i) \, d\vec{y} \\ &\quad\quad + \int_{\mathbb{R}^{mn}} [\theta_t(x, u_1, y_2 \ldots, y_m) - \theta_t(x, u_1, u_2, \ldots, y_m)] \prod_{i=1}^{m} \psi_{s_i}(y_i - u_i) \, d\vec{y} \\ &\quad\quad \vdots \\ &\quad\quad + \int_{\mathbb{R}^{mn}} [\theta_t(x, u_1, \ldots, u_{m-1}, y_m) - \theta_t(x, u_1, \ldots, u_m)] \prod_{i=1}^{m} \psi_{s_i}(y_i - u_i) \, d\vec{y} \\ &\quad = \sum_{i=1}^{m} L^{i,I}_{t,s_1,\ldots,s_m}(x, u_1, \ldots, u_m), \end{aligned}$$

noting that the term containing the $\theta_t(x, u_1, \ldots, u_m)$ vanishes since it is multiplied by the product of the ψ_j's which have mean value zero. (We set $d\vec{y} = dy_1 \cdots dy_m$.) We will refer to this kind of kernel pieces as *type I*. These terms are estimated using (4.7) and (4.8); the ith term of the sum is

$$(4.22)\quad |L^{i,I}_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m)|$$
$$\lesssim A \int_{\mathbb{R}^{nm}} \frac{\min(2,(t^{-1}|y_i-u_i|)^\gamma}{t^{mn}} \prod_{k=1}^m \frac{s_k^{-n}}{(1+s_k^{-1}|y_k-u_k|)^{n+1}}\,dy_1\cdots dy_m$$
$$\lesssim A\frac{1}{t^{n(m-1)}}\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^n \min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^{\frac12\min(\delta,\gamma,1)}.$$

Now, if $r, s_1,\dots,s_m \le t$, using the cancellation conditions again, setting $d\vec{y} = dy_1\cdots dy_m$ we obtain

$$L_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m)$$
$$= \int_{\mathbb{R}^{mn}} \theta_t(x,y_1,\dots,y_m)\prod_{i=1}^m \psi_{s_i}(y_i-u_i)\,dy_1\cdots dy_m$$
$$= \int_{\mathbb{R}^{mn}} \theta_t(x,y_1,\dots,y_m)\prod_{i=1}^m [\{\psi_{s_i}(y_i-u_i)-\psi_{s_i}(x-u_i)\}+\psi_{s_i}(x-u_i)]\,d\vec{y}$$
$$= \sum_{i=1}^{2^m-1} L^{i,II}_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m),$$

where each term $L^{i,II}_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m)$ has the form

$$L^{i,II}_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m)$$
$$= \int_{\mathbb{R}^{mn}} \theta_t(x,y_1,\dots,y_m)\prod_{k=1}^k [\psi_{s_{i_k}}(y_{i_k}-u_{i_k})-\psi_{s_{i_k}}(x-u_{i_k})]$$
$$\times \prod_{l=1}^l \psi_{s_{i_l}}(x-u_{i_l})\,dy_1\cdots dy_m,$$

where $k+l=m$ and $\{i_k\}$ are new indices obtained by permuting the original ones. For simplicity, we may suppose that $i_k = i$ since the other terms are a permutation of this case. This kind of kernel pieces will be referred to as *type II*. These terms are estimated by

$$(4.23)\quad |L^{i,II}_{t,s_1,\dots,s_m}(x,u_1,\dots,u_m)|$$
$$\le \int_{\mathbb{R}^{mn}} |\theta_t(x,y_1,\dots,y_m)|\prod_{i=1}^k |\psi_{s_i}(y_i-u_i)-\psi_{s_i}(x-u_i)|$$
$$\times \prod_{j=k+1}^m |\psi_{s_j}(x-u_j)|\,dy_1\cdots dy_m$$

$$\lesssim A\Big(\prod_{i=1}^{k}\int_{\mathbb{R}^n}\frac{t^{-n}}{(1+t^{-1}|x-y_i|)^{n+\delta}}\,\frac{\min(2,s_i^{-1}|x-y_i|)}{s_i^n}\,dy_i\Big)$$

$$\lesssim A\prod_{i=1}^{k}\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^n\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^{1/2\min(\delta,\gamma,1)}.$$

Combining all these estimates with

$$(4.24)\qquad |L_{t,s_1,\ldots,s_m}(x,u_1,\ldots,u_m)| \le \int_{\mathbb{R}^{mn}}|\theta_t(x,y_1,\ldots,y_m)|\prod_{i=1}^{m}|\psi_{s_i}(y_i-u_i)|\,dy_1\cdots dy_m \lesssim A\prod_{i=1}^{m}\frac{\min(t/s_i,s_i/t)^n}{(1+\min(t/s_i,s_i/t)|x-u_i|)^{n+\min(\delta,1)}}$$

we obtain (after some manipulations)

$$(4.25)\qquad |\Theta_t(Q_{s_1}f_1,\ldots,Q_{s_m}f_m)(x)| \lesssim A\prod_{i=1}^{m}\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^{\epsilon}\prod_{i=1}^{m}\Big(\int_{\mathbb{R}^n}\frac{\min(t/s_i,s_i/t)^n|f_i(u_i)|}{((1+\min(t/s_i,s_i/t)|x-u_i|)^{n+\min(\delta,1)})^{\beta}}\,du_i\Big) \lesssim A\Big(\prod_{i=1}^{m}\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^{\epsilon}\Big)\prod_{i=1}^{m}M(f_i)(x)$$

for some positive constants $\epsilon,\beta>0$. Noticing that

$$w(t,s_1,\ldots,s_m)=\prod_{i=1}^{m}\min\Big(\frac{t}{s_i},\frac{s_i}{t}\Big)^{\epsilon}$$

satisfies (4.13) and (4.14) leads to the result.

5. The multilinear $T(b)$ theorem for square functions. In this section we prove Theorem 1.2. We will obtain the estimate

$$(5.1)\qquad \int_{\mathbb{R}^n}\int_0^{\infty}|\Theta_t(f_1,\ldots,f_m)(x)|^2\,\frac{dx\,dt}{t}\lesssim\prod_{i=1}^{m}\|f_i\|_{p_i}^2$$

for operators $\Theta_t(f_1,\ldots,f_m)$ satisfying the cancellation condition

$$\Theta_t(b_1,\ldots,b_m)=0$$

for some accretive functions $b_1,\ldots,b_m$.

Fix a smooth compactly supported and nonnegative function Φ with integral equal to 1. Let $\Phi_t(x)=t^{-n}\Phi(x/t)$. Let $\mathbb{P}_t$ be given by

$$(5.2)\qquad \mathbb{P}_t(f_1,\ldots,f_m)(x):=\prod_{i=1}^{m}(\Phi_t\star f_i)(x),$$

which satisfies the same size condition as Θ_t. We introduce the operator

$$\Xi_t(f_1,\dots,f_m)=\Theta_t(\Phi_t\star f_1,\dots,\Phi_t\star f_m) \tag{5.3}$$

and we observe that

$$\Xi_t(1,\dots,1)=\Theta_t(\Phi_t\star 1,\dots,\Phi_t\star 1)=\Theta_t(1,\dots,1). \tag{5.4}$$

The key idea is to decompose $\Theta_t(f_1,\dots,f_m)$ as the sum

$$(\Theta_t-\Theta_t(1,\dots,1)\mathbb{P}_t)(f_1,\dots,f_m)+\Theta_t(1,\dots,1)\mathbb{P}_t(f_1,\dots,f_m). \tag{5.5}$$

The first term in (5.5) satisfies

$$\begin{aligned}(\Theta_t-\Theta_t(1,\dots,1)\mathbb{P}_t)&(1,\dots,1)\\ &=\Theta_t(1,\dots,1)-\Theta_t(1,\dots,1)\mathbb{P}_t(1,\dots,1)=0.\end{aligned} \tag{5.6}$$

Since the kernel of this operator satisfies the same estimates as the kernel of Θ_t, we apply Theorem 1.3 to conclude that this part satisfies (5.1) with bound at most $(C_{n,\delta,\gamma}A)^2$.

To obtain the required estimates for the term $\Theta_t(1,\dots,1)\mathbb{P}_t(f_1,\dots,f_m)(x)$ we use the Carleson measure estimates. In fact, in view of Lemma 1, we reduce this problem to showing that

$$|\Theta_t(1,\dots,1)|^2\,\frac{dx\,dt}{t} \tag{5.7}$$

is a Carleson measure. As $\Theta_t(b_1,\dots,b_m)=0$, we have

$$\begin{aligned}\Theta_t(1,\dots,1)&\mathbb{P}_t(b_1,\dots,b_m)\\ &=(\mathbb{P}_t(b_1,\dots,b_m)\Theta_t(1,\dots,1)-\Xi_t(b_1,\dots,b_m))\\ &\quad+(\Xi_t(b_1,\dots,b_m)-\Theta_t(b_1,\dots,b_m)).\end{aligned} \tag{5.8}$$

If we could show that

$$|\Theta_t(b_1,\dots,b_m)(x)-\Xi_t(b_1,\dots,b_m)(x)|^2\,\frac{dx\,dt}{t} \tag{5.9}$$

and

$$|\Xi_t(b_1,\dots,b_m)(x)-\mathbb{P}_t(b_1,\dots,b_m)(x)\Theta_t(1,\dots,1)(x)|^2\,\frac{dx\,dt}{t} \tag{5.10}$$

are Carleson measures, then the measure

$$|\mathbb{P}_t(b_1,\dots,b_m)\Theta_t(1,\dots,1)|^2\,\frac{dx\,dt}{t} \tag{5.11}$$

would also be a Carleson measure. Since each piece of $\mathbb{P}_t$ is a positive operator and the functions $b_1,\dots,b_m$ are accretive, we deduce

$$|\mathbb{P}_t(b_1,\dots,b_m)|\ge\mathbb{P}_t(\mathrm{Re}(b_1),\dots,\mathrm{Re}(b_m))\ge\mathbb{P}_t(c_1,\dots,c_m)=\prod_{i=1}^m c_i, \tag{5.12}$$

which implies

$$|\Theta_t(1,\dots,1)|^2 \le \frac{1}{\prod_{i=1}^m c_i^2}|\mathbb{P}_t(b_1,\dots,b_m)\Theta_t(1,\dots,1)|^2. \tag{5.13}$$

This would show that $|\Theta_t(1,\dots,1)|^2 \frac{dx\,dt}{t}$ is a Carleson measure if so are (5.9) and (5.10).

As $b_1,\dots,b_m$ are bounded functions, the assertions that (5.9) and (5.10) are Carleson measures follow from the facts that the operators

$$\Pi_t^1(f_1,\dots,f_m) := (\Theta_t - \Xi_t)(f_1,\dots,f_m)(x), \tag{5.14}$$

$$\Pi_t^2(f_1,\dots,f_m) := (\Xi_t - \Theta_t(1,\dots,1)\mathbb{P}_t)(f_1,\dots,f_m)(x) \tag{5.15}$$

satisfy the hypothesis of Theorem 1.1.

CASE 1: The operator $\Pi_t^1(f_1,\dots,f_m)$. Note that

$$\Xi_t(f_1,\dots,f_m) = \int_{\mathbb{R}^{mn}} \xi_t(x,y_1,\dots,y_m)\prod_{i=1}^m f_i(y_i)\,dy_1\cdots dy_m \tag{5.16}$$

has the kernel

$$\xi_t(x,y_1,\dots,y_m) = \int_{\mathbb{R}^{mn}} \theta_t(x,u_1,\dots,u_m)\prod_{i=1}^m \Phi_t(u_i-y_i)\,du_1\cdots du_m, \tag{5.17}$$

which is easily seen to satisfy the size conditions. Thus we conclude that

$$\Pi_t^1 = \Theta_t - \Xi_t$$

satisfies the required size estimates and additionally we have

$$\Pi_t^1(1,\dots,1) = \Theta_t(1,\dots,1) - \Xi_t(1,\dots,1) = 0$$

in view of (5.4). Applying Theorem 1.3, we deduce that Π_t^1 is a bounded operator

$$L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n, L^2(\mathbb{R}^+))$$

with operator norm controlled by $C_{n,\delta,\gamma}A$; here $1/p_1+\cdots+1/p_m = 1/2$. Then, by Theorem 1.1, we conclude that

$$|\Pi_t^1(b_1,\dots,b_m)(x)|^2\,\frac{dx\,dt}{t} \tag{5.18}$$

is a Carleson measure with constant controlled by $C_{n,\delta,\gamma}A^2\prod_{i=1}^m \|b_i\|_\infty^2$.

CASE 2: The operator $\Pi_t^2(f_1,\dots,f_m)$. The kernel of Π_t^2 is given by

$$\pi_t^2(x,y_1,\dots,y_m) := \xi_t(x,y_1,\dots,y_m) - \Theta_t(1,\dots,1)(x)\prod_{i=1}^m \Phi_t(x-y_i) \tag{5.19}$$

and this is easily seen to satisfy the required size conditions, since both $\xi_t(x,y_1,\dots,y_m)$ and $\prod_{i=1}^m \Phi_t(x-y_i)$ do so and since $\Theta_t(1,\dots,1)(x)$ is a bounded function for each $t > 0$. Moreover, using (5.4) we have

$$\begin{aligned}(5.20)\qquad \Pi_t^2(1,\dots,1)(x) \\ = \Xi_t(1,\dots,1)(x) - \Theta_t(1,\dots,1)(x)\mathbb{P}_t(1,\dots,1)(x) = 0,\end{aligned}$$

and one more application of Theorem 1.3 permits us to conclude that Π_t^2 maps

$$L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n, L^2(\mathbb{R}^+))$$

continuously with the same norm estimate. Using Theorem 1.1 we obtain the desired conclusion, that is, (5.18) is a Carleson measure with constant controlled by

$$C_{n,\delta,\gamma} A^2 \prod_{i=1}^m \|b_i\|_\infty^2.$$

Combining all these, we obtain (5.1) with bound

$$C_{n,\delta,\gamma}^2 A^2 \Big(1 + \prod_{i=1}^m \|b_i\|_\infty^2\Big).$$

This concludes the proof of Theorem 1.2.

6. Final remarks and further comments. As in Semmes [23], it is possible to modify the preceding proof to the case of para-accretive functions. This modification is straightforward and the details are left to the interested reader.

Under stronger regularity for the kernel of Θ_t, Hart [16] proves a vector-valued version of a bilinear Calderón–Zygmund theorem (in the spirit of Grafakos and Torres [15] and Kenig and Stein [18]) which allows him to obtain boundedness for the quadratic square function formed by Θ_t via interpolation in the full range of exponents starting from an initial estimate. The same idea could be adapted in our situation and this would yield the boundedness of the quadratic square function formed by Θ_t in Theorem 1.2 from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p where $1 < p_j < \infty$, $1/m < p < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, and also from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$.

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