Elliptic functions, area integrals and the exponential square class on $B_1(0) \subseteq \mathbb{R}^n, n > 2$

by

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Abstract. For two strictly elliptic operators L_0 and L_1 on the unit ball in $\mathbb{R}^n$, whose coefficients have a difference function that satisfies a Carleson-type condition, it is shown that a pointwise comparison concerning Lusin area integrals is valid. This result is used to prove that if $L_1 u_1 = 0$ in $B_1(0)$ and $Su_1 \in L^\infty(S^{n-1})$ then $u_1|_{S^{n-1}} = f$ lies in the exponential square class whenever L_0 is an operator so that $L_0 u_0 = 0$ and $Su_0 \in L^\infty$ implies $u_0|_{S^{n-1}}$ is in the exponential square class; here S is the Lusin area integral. The exponential square theorem, first proved by Thomas Wolff for harmonic functions in the upper half-space, is proved on $B_1(0)$ for constant coefficient operator solutions, thus giving a family of operators for L_0. Methods of proof include martingales and stopping time arguments.

The subject treated here is the exponential square class for elliptic operator solutions in the unit ball in $\mathbb{R}^n$. The main result is given in Theorem 4, where it is proved that if solutions for one operator $L_0 u_0 = 0$ in $B_1(0)$ satisfy the exponential square result (i.e. if the Lusin area integral Su_0 is in $L^\infty(\partial B_1(0))$ then $u_0|_{\partial B_1(0)} = f$ lies in the exponential square class) then solutions u_1 to $L_1 u_1 = 0$ in $B_1(0)$ will also satisfy the exponential square theorem if the coefficients of L_1 satisfy a Carleson-type condition with vanishing trace with respect to the coefficients of L_0. The main ingredient in the proof is to establish a comparison of area integrals by methods similar to those of Fefferman, Kenig and Pipher [FKP] in their proof of Dahlberg's preservation of B^q for elliptic measures.

In order to ensure that the result is not vacuous, the exponential square theorem is proved on $B_1(0)$ for harmonic functions. The martingale argument of Chang, Wilson and Wolff [CWW] can be adapted to the geometry on $B_1(0)$. This result is established in part I in detail, using spherical coordinates. To establish the formula that served in place of the Calderón reproducing formula, and to establish the martingale proof, it seemed better to use spherical coordinates. A similar formula shows that solutions to constant coefficient equations also satisfy the exponential square result.

Part II contains the proof of Theorem 4. This is done by using rectangular coordinates, since there is no particular need for this result to be proved in spherical coordinates, and the formulas in rectangular coordinates are much simpler. The cone formulas are somewhat different for the two coordinate systems, but are shown to compare with each other in the appendix. The Main Lemma used to prove Theorem 4 establishes a local pointwise comparison of area integrals for the two operator solutions, a result that may be of some independent interest. The other key estimate used in the proof of Theorem 4 is that if $F(x)$ is the difference of the two solutions, $F(x) = u_1(x) - u_0(x)$, then $(\fint_R |F(x)|^2 dx)^{1/2} \le C\varepsilon(\operatorname{diam} R)M_{\omega_0}Su_1(q_0)$ for any $q_0 \in \operatorname{proj} R|_{\partial B_1(0)}$, where R is a Whitney-type region in $B_1(0)$ and $\varepsilon(\operatorname{diam} R)$ is the Carleson coefficient with an appropriate rate of decay as $\operatorname{diam} R \to 0$. This estimate can be used to prove a version of the good-λ inequality in Lemma 2.16 of [FKP]. There is a similar proof for a version of Lemma 7 in [CS2].

I. Exponential square on the unit ball in $\mathbb{R}^n$. Suppose $u(y)$ is defined on $B_1(0) \subseteq \mathbb{R}^n$ so that

$$\Delta u = 0 \text{ in } B_1(0), \quad u|_{\partial B_1(0)} = f \in L^1(S^{n-1}),$$

The *Lusin area integral* for u is

$$A_\gamma f(\theta_x) = A_\gamma u(\theta_x) = \left(\int_{\Gamma_\gamma^{\delta_0}(\theta_x)} |\nabla_y u(y)|^2 (\operatorname{dist}(y, S^{n-1}))^{2-n} \, dy \right)^{1/2},$$

where

$$\Gamma_\gamma^{\delta_0}(\theta_x) = \{(r,\theta) : \delta_0 < r < 1, \ |\theta - \theta_x| < \gamma(1-r)\}.$$

Then the following result is valid:

THEOREM 1. *For u and f as above, suppose $A_\gamma f(\theta_x) \in L^\infty(S^{n-1})$. Then there are constants $C_1, C_2 > 0$ independent of Q and f so that if $\ell(Q) \le \ell(Q_0)$ then*

$$\left(\frac{1}{|Q|} \int_Q \exp\left(\frac{C_1|f - f_Q|^2}{\|A_\gamma f\|_\infty^2} \right) dy \right)^{1/2} < C_2 < \infty.$$

Here Q is a surface cube on S^{n-1} of limited size; it will be the image of a cube $\widehat{Q}$ in $\mathbb{R}^{n-1}$ under the projection map $\widehat{P}$ defined below.

Theorem 1 can be proved by using a martingale result similar to the one in [CWW] along with other estimates as in [BM]. In fact, one can obtain a version of the Calderón reproducing formula so that

$$\int_{r=1-\varepsilon} u(r\theta_y)P_{1-r}(\theta_x - \theta_y) \, d\theta_y$$

equals a sum of eight integrals—three boundary integrals and five integrals over a solid spherical shell region R in $B_1(0)$. (See Lemma 1 below; P_{1-r} is

defined after the statement of Theorem 2.) Then $P_{1-r}(\theta)$ being an approximation to the identity implies that, as $r \to 1$, $\int_{r=1-\varepsilon} u(r\theta_y) P_{1-r}(\theta_x - \theta_y) \, d\theta_y \to f(\theta_x)$. This happens because u converges non-tangentially to the boundary function f.

So it suffices to prove that $\int_{r=1-\varepsilon} u(r\theta_y) P_{1-r}(\theta_x - \theta_y) \, d\theta_y$ is in the exponential square class. This can be shown by the following steps. First all three boundary integrals I_i, $i = 1, 2, 3$, in Lemma 1 can be dominated by the area integral, i.e. $|I_i - I_{i,Q}| \le c\|A_\gamma f\|_\infty$ a.e. Next two of the integrals over R can be bounded directly by $cA_\gamma f(\theta_x)$ also. The remaining three integrals over R can be replaced by "dyadic" martingales with an error of $c\|A_\gamma f\|_\infty$ a.e. These martingales have their dyadic square functions bounded by $c\|A_\gamma f\|_\infty$ for a.e. θ_x, and so Theorem 2 below shows that the martingales are exponentially square integrable.

Note that each function bounded by $c\|A_\gamma f\|_\infty$ is in the exponential square class and any finite sum of exponentially square integrable functions is also exponentially square integrable; this implies that $\int_{r=1-\varepsilon} u P_{1-r}$ is in the exponential square class. The constants will be shown to be independent of ε, so $f(\theta_x)$ is in the exponential square class also.

The steps of this proof are well known by now ([CWW], [BM]). The intention in this paper is to make clear the elements of the proof that change for $B_1(0) \subseteq \mathbb{R}^n$, $n \ge 3$, and to show that the standard proof works in this slightly new setting.

To begin, a version of the dyadic martingale exponential square result must be established. To create σ-algebras $\mathcal{F}_m$ on S^{n-1}, with $\mathcal{F}_{m-1} \subseteq \mathcal{F}_m$, let S_0 be a fixed polar cap on the lower hemisphere of S^{n-1}. The cap must be large enough to contain $3Q$ for any surface cube Q, where $3Q$ is the image of $3\widehat{Q}$ under the projection operator $\widehat{P}$ defined by

$$\widehat{P}(x) = \widehat{P}(x_1, \ldots, x_{n-1}, -1) = \left(x_1, x_2, \ldots, x_{n-1}, -\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{1/2} \right)$$

so $\widehat{P}$ maps $\mathbb{R}^{n-1} \times \{-1\}$ onto the lower hemisphere of S^{n-1}.

Now $\widehat{Q}$ will denote a dyadic subcube of an initial cube $3\widehat{Q}_0$, where $\widehat{Q}_0$ is a cube of fixed size centered at $(0, \ldots, 0, -1)$ in $\mathbb{R}^{n-1} \times \{-1\}$, and $3\widehat{Q}_0$ is the cube concentric with $\widehat{Q}_0$ and with side length $\ell(3\widehat{Q}_0) = 3\ell(\widehat{Q}_0)$.

To define the σ-algebras, let $\mathcal{F}_m$ be the σ-algebra generated by the collection $\widehat{P}(\widehat{Q}_m)$, where $\widehat{Q}_m$ is a dyadic subcube of $3\widehat{Q}_0 \subseteq \mathbb{R}^{n-1} \times \{-1\}$ of side length $\ell(\widehat{Q}_m) = 2^{-m}\ell(\widehat{Q}_0)$. The projection function $\widehat{P}$ is 1-1 on $3\widehat{Q}_0$ and maps $\bigcup \widehat{Q}_m$ onto $\widehat{P}(3\widehat{Q}_0)$ in S_0, for each m. In this way the $\mathcal{F}_m$ are simply the usual dyadic σ-algebras in $\mathbb{R}^{n-1}$ mapped to S_0 by $\widehat{P}$. The distortion produced by $\widehat{P}$ in S_0 is bounded, so that $\ell(P(\widehat{Q})) = \ell(Q) \sim \ell(\widehat{Q})$ and

$|Q| \sim |\widehat{Q}|$. Here $\ell(Q)$ can be taken to be the diameter of Q, and $|Q|$ denotes the Lebesgue surface measure of Q.

It is easy to see that $\mathcal{F}_{m-1} \subseteq \mathcal{F}_m$ for all $m \geq 1$. Given the sequence $\{\mathcal{F}_m\}_{m=0}^{\infty}$ of σ-algebras, martingales are defined in the usual way, being a sequence $\{f_m\}_{m=0}^{\infty}$ of functions defined on S_0, so that each f_m is $\mathcal{F}_m$-measurable and $E(f_{m+1} \,|\, \mathcal{F}_m) = f_m$, $m = 0, 1, 2, \ldots$. If $\lim_{m \to \infty} f_m(\theta_x) = f(\theta_x)$ a.e. θ_x, then f is the martingale limit function. Also any L^1 function $f(\theta_x)$ can be taken to generate a martingale. By a mild abuse of terminology the term dyadic will continue to be used here; this dyadic martingale is

$$f_m = \sum_{Q_m^j \text{ are generators of } \mathcal{F}_m} \frac{1}{|Q_m^j|} \int_{Q_m^j} f(\theta) \, d\theta \cdot \chi_{Q_m^j},$$

where $|Q_m^j| = \int_{Q_m^j} d\theta_y$. The $\{f_m\}_{m=0}^{\infty}$ have $f(\theta_x)$ as their martingale limit function.

The *dyadic martingale square function* for any dyadic martingale $\{g_m\}$ is defined by

$$S_m = Sg_m(\theta) = \Big(\sum_{j=1}^{m} \|d_j \chi_{Q_{j-1}}\|_{\infty}^2 \Big)^{1/2}, \quad d_j = g_j - g_{j-1}.$$

If g_m has a martingale limit function g then

$$Sg(\theta) = \Big(\sum_{j=1}^{\infty} \|d_j \chi_{Q_{j-1}}\|_{\infty}^2 \Big)^{1/2}.$$

But $Sg(\theta)$ is well defined even if g is not known.

THEOREM 2. *If $\{f_m\}_{m=0}^{\infty}$ is a dyadic martingale as defined above on $Q \subseteq S_0 \subseteq S^{n-1}$ with limit function $f(\theta_x)$, suppose $Sf \in L^{\infty}(Q)$. Then there are constants $C_1, C_2 > 0$ independent of Q and f so that*

$$\frac{1}{|Q|} \int_Q \exp\left\{ \frac{C_1 |f(\theta_x) - f_Q|^2}{\|Sf\|_{\infty}^2} \right\} d\theta_x < C_2 < \infty.$$

The proof is the same argument used in [CWW] and [BM] to establish this result for dyadic martingales in $\mathbb{R}^n$.

To establish the result of Theorem 1 using Theorem 2, an essential ingredient is to find a form of the Calderón reproducing formula. The following calculation uses the Laplacian on a sphere; the spherical form of the Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \sum \sum \theta_j \theta_k \frac{\partial^2}{\partial \theta_j \partial \theta_k} + \frac{n-1}{r} \frac{\partial}{\partial r}$$

$$+ \frac{1}{r^2} \sum \frac{\partial^2}{\partial \theta_j^2} - \frac{n-1}{r^2} \sum \theta_k \frac{\partial}{\partial \theta_k}.$$

The sums are all taken from 1 to $n-1$ for each index. Here $y_j = r\theta_j$ for $j = 1, \ldots, n$, $\theta_j = \cos\psi_j$ for $j = 1, \ldots, n-1$ and $\theta_n = (1 - \sum_{j=1}^{n-1}\theta_j^2)^{1/2}$. The angles ψ_j are the angles with respect to each coordinate axis. (See Helm's book [H].)

To produce a substitute for the Calderón reproducing formula, following the method in Bañuelos and Moore [BM], take $k(\theta)$ defined on S^{n-1} to be smooth, radial, of compact support, with $\int k(\theta)\,d\theta = 1$, and let

$$k_{1-r}(\theta) = \frac{1}{(1-r)^{n-1}}\,k\left(\frac{\theta}{1-r}\right).$$

Then define

$$P_{1-r}(\theta) = \frac{\partial}{\partial r}\left\{\frac{1}{(1-r)^{n-2}}\,k\left(\frac{\theta}{1-r}\right)\right\} = \frac{1}{(1-r)^{n-1}}\,P\left(\frac{\theta}{1-r}\right).$$

This gives

$$P(\theta) = (n-2)k(\theta) + \sum \theta_k \frac{\partial}{\partial \theta_k}k(\theta)$$

as long as $\left(\theta_k \frac{\partial}{\partial \theta_k}k(\theta)\right)_{1-r}$ is understood to be

$$\frac{1}{(1-r)^{n-1}}\,\frac{\theta_j}{1-r}\,\frac{\partial k(\theta/(1-r))}{\partial(\theta_j/(1-r))}.$$

Then

$$\frac{\partial P_{1-r}}{\partial r}(\widehat{\theta}) = \frac{\partial}{\partial r}\left\{\frac{1}{(1-r)^{n-1}}P\left(\frac{\theta_x - \theta_y}{1-r}\right)\right\} = \frac{n-1}{(1-r)^n}\,P\left(\frac{\theta_x - \theta_y}{1-r}\right)$$

$$+ \frac{1}{(1-r)^{n-1}}\sum \frac{\partial}{\partial r}\left(\frac{\theta_{x,j} - \theta_{y,j}}{1-r}\right)\frac{\partial P(\widehat{\theta})}{\partial \widehat{\theta}_j}$$

$$= \frac{n-1}{(1-r)^n}\,P(\widehat{\theta}) + \frac{1}{(1-r)^n}\sum \widehat{\theta}_j\,\frac{\partial P(\widehat{\theta})}{\partial \widehat{\theta}_j}$$

$$= \frac{1}{(1-r)^n}\sum \frac{\partial}{\partial \widehat{\theta}_j}(\widehat{\theta}_j P(\widehat{\theta})),$$

where

$$\widehat{\theta}_j = \frac{\theta_{x,j} - \theta_{y,j}}{1-r}, \quad \widehat{\theta} = \frac{\theta_x - \theta_y}{1-r}.$$

Notice that

$$\int k_{1-r}(\theta_x - \theta_y)\,d\theta_x = 1, \quad \int P_{1-r}(\theta_x - \theta_y)\,d\theta_x = 1$$

and $\int \frac{\partial}{\partial r}P_{1-r}\,d\theta_x = 0$.

Integrating from $r = 1 - \varepsilon$ to $r = \delta_0$ gives

$$\int_{r=\delta_0} uP_{1-r} - \int_{r=1-\varepsilon} uP_{1-r} = \int_{1-\varepsilon}^{\delta_0} \frac{\partial}{\partial r} \int_{\text{supp } P_{1-r}} u(r\theta) P_{1-r}(\widehat{\theta}) \, d\theta \, dr$$

$$= \int_{1-\varepsilon}^{\delta_0} \int_{\text{supp } P_{1-r}} \left\{ \frac{\partial u}{\partial r} P_{1-r} + u \frac{\partial P_{1-r}}{\partial r} \right\} d\theta \, dr$$

$$= \iint_R \frac{\partial u}{\partial r} \frac{\partial}{\partial r} \left(\frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \right) d\theta \, dr$$

$$+ \iint_R u \frac{1}{(1-r)^n} \sum \frac{\partial}{\partial \widehat{\theta}_j} (\widehat{\theta}_j P(\widehat{\theta})) \, d\theta \, dr = I + II,$$

where

$$R = \{(r,\theta) : 1 - \varepsilon > r > \delta_0, \theta_x \in \beta Q_0\},$$

and $\beta > 1$ is chosen so that $\text{supp } k_{1-r}(\theta_x - *)$ lies inside

$$\{(r,\theta) : \delta < r < 1, \theta \in \beta Q_0\} \quad \text{if } \theta_x \in Q_0.$$

Here δ is a fixed radius, but δ_0 depends on the cube Q under consideration. For all cubes Q, $\delta \leq \delta_0/2$. For the purpose of Lemma 1 below, consider Q_0 to be fixed.

Next, to produce integrals in forms that can be dealt with by the martingale replacement method (as in [BM]), or that can be estimated directly, integration by parts is used on both terms. The first integral becomes

$$I = \int_{r=\delta_0} \left(\frac{\partial}{\partial r} u \right) \left(\frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \right) d\theta$$

$$- \int_{r=1-\varepsilon} \left(\frac{\partial}{\partial r} u \right) \left(\frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \right) d\theta - \iint_R \left(\frac{\partial^2}{\partial r^2} u(r\theta) \right) \frac{1}{(1-r)^{n-2}} k(\theta) \, d\theta \, dr$$

$$= \int_{r=\delta_0} \left(\frac{\partial}{\partial r} u \right) \left(\frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \right) d\theta - \int_{r=1-\varepsilon} \left(\frac{\partial}{\partial r} u \right) \left(\frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \right) d\theta$$

$$+ (n-1) \iint_R \frac{1}{(1-r)^{n-2}} \frac{1}{r} \left(\frac{\partial}{\partial r} u \right) k(\widehat{\theta}) \, d\theta \, dr$$

$$+ \iint_R \frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \frac{1}{r^2} \sum \frac{\partial^2}{\partial \theta_j^2} u(r\theta) \, d\theta \, dr$$

$$- \iint_R \frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \frac{1}{r^2} \sum \sum \theta_j \theta_k \frac{\partial^2}{\partial \theta_j \partial \theta_k} u(r\theta) \, d\theta \, dr$$

$$- (n-1) \iint_R \frac{1}{(1-r)^{n-2}} k(\widehat{\theta}) \frac{1}{r^2} \sum \theta_k \frac{\partial}{\partial \theta_k} u(r\theta).$$

The expression substituted for $-\frac{\partial^2}{\partial r^2}u(r\theta)$ has been taken from the Laplacian as given above.

Now using integration by parts again on the two integrals in the last expression that involve second derivatives in u gives

$$I = \int_{r=\delta_0} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\,k(\widehat{\theta})\right)d\theta - \int_{r=1-\varepsilon} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\,k(\widehat{\theta})\right)d\theta$$

$$+ (n-1)\iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r}\left(\frac{\partial}{\partial r}u\right)k(\widehat{\theta})\,d\theta\,dr$$

$$- \iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\left(\sum \frac{\partial}{\partial\theta_j}k\left(\frac{\theta_{x,j}-\theta_{y,j}}{1-r}\right)\right)\frac{\partial}{\partial\theta_j}u(r\theta)\,d\theta\,dr$$

$$+ n\iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\,k(\widehat{\theta})\sum \theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\,d\theta\,dr$$

$$+ \iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\sum\sum\left(\frac{\partial}{\partial\theta_j}k\left(\frac{\theta_{x,j}-\theta_{y,j}}{1-r}\right)\right)\theta_j\theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\,d\theta\,dr$$

$$- (n-1)\iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\,k(\widehat{\theta})\sum \theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\,d\theta\,dr$$

$$= \int_{r=\delta_0} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\,k(\widehat{\theta})\right)d\theta - \int_{r=1-\varepsilon} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\,k(\widehat{\theta})\right)d\theta$$

$$- \iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\sum\left(\frac{\partial}{\partial\theta_j}k\left(\frac{\theta_{x,j}-\theta_{y,j}}{1-r}\right)\right)\frac{\partial}{\partial\theta_j}u(r\theta)\,d\theta\,dr$$

$$+ \iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r^2}\sum\sum\left(\frac{\partial}{\partial\theta_j}k\left(\frac{\theta_{x,j}-\theta_{y,j}}{1-r}\right)\right)\theta_j\theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\,d\theta\,dr$$

$$+ (n-1)\iint_R \frac{1}{(1-r)^{n-2}}\,\frac{1}{r}\left(\frac{\partial}{\partial r}u\right)k(\widehat{\theta})\,d\theta\,dr$$

$$+ \iint_R \frac{1}{(1-r)^2}\,\frac{1}{r^2}\,k(\widehat{\theta})\left(\sum \theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\right)d\theta\,dr$$

All sums are from 1 to $n-1$, $\theta = \theta_y$ and $\theta_j = \theta_{y,j}$.

The boundary terms can be dominated by the Lusin area integral for u and the first two integrals over R in the last expression can be estimated by using the method of replacing them by a martingale with an error bounded by a constant times the Lusin area integral of u. The essential properties for doing this are that the kernels $\frac{\partial}{\partial\theta_j}k(\widehat{\theta})$ have $\int \frac{\partial}{\partial\theta_j}k(\widehat{\theta})\,d\theta_x = 0$ and that the dilation factors, $1/(1-r)^{n-2}$, are the right magnitude for domination by the area integral.

However, the last integral can be bounded directly by $A_\gamma u(\theta_x)$. Breaking $R = \{(r,\theta) : 1 - \varepsilon > r > \delta_0,\ \theta \in \beta Q_0\}$ into dyadic approach regions T_Q (annular segments of approximately the same geometric shape as R, whose dimension compares with their distance from S^{n-1}), one easily sees that

$$\left| \iint\limits_R \frac{1}{(1-r)^{n-2}} \frac{1}{r} k(\widehat{\theta}) \frac{\partial u}{\partial r}\, d\theta\, dr \right|$$

$$= \left| \sum_{Q \subset S^{n-1}} \iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \frac{1}{(1-r)^{n-2}} \frac{1}{r} k(\widehat{\theta}) \frac{\partial u}{\partial r}\, d\theta\, dr \right|$$

$$\leq \sum_{Q \subset S^{n-1}} \left(\iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \left| \frac{\partial u}{\partial r} \right|^2 (1-r)^{2-n}\, d\theta\, dr \right)^{1/2}$$

$$\times \left(\iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \frac{1}{r^2} \frac{1}{(1-r)^{n-2}} \left(k\left(\frac{\theta_x - \theta_y}{1 - r} \right) \right)^2 d\theta\, dr \right)^{1/2}.$$

In T_Q we have $\left(k\left(\frac{\theta_x - \theta_y}{1-r} \right) \right)^2 \leq c$, and $1/r^2 \leq 1/\delta_0^2$, so this implies that

$$\left(\iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \frac{1}{r^2} \frac{1}{(1-r)^{n-2}} \left(k\left(\frac{\theta_x - \theta_y}{1 - r} \right) \right)^2 d\theta\, dr \right)^{1/2}$$

$$\geq c \left\{ \frac{|T_Q|}{\delta_0^2 \inf (1-r)^{n-2}} \right\}^{1/2} \leq c' \ell(Q)$$

since $|T_Q|$ compares with $(1-r)^n$ if $r\theta \in T_Q$. The $\inf (1-r)^{n-2}$ is also taken over $r\theta \in T_Q$. So

$$\left| \iint\limits_R \frac{1}{(1-r)^{n-2}} \frac{1}{r} k(\widehat{\theta}) \frac{\partial u}{\partial r}\, d\theta\, dr \right|$$

$$\leq c' \sum_{Q \subset S^{n-1}} \left(\iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \left| \frac{\partial u}{\partial r} \right|^2 (1-r)^{2-n}\, d\theta\, dr \right)^{1/2} \ell(Q)$$

$$\leq c'' \left(\sum_{Q \subset S^{n-1}} \iint\limits_{T_Q \cap \operatorname{supp} k(\widehat{\theta})} \left| \frac{\partial u}{\partial r} \right|^2 (1-r)^{2-n}\, d\theta\, dr \right)^{1/2} \left(\sum_{T_Q \cap \Gamma_\gamma^{\delta_0}(\theta_x) \neq \emptyset} \ell(Q)^2 \right)^{1/2}$$

$$\leq c(\delta_0) \left(\int\limits_{\Gamma_\gamma^{\delta}(\theta_x)} |\nabla u|^2 (1-r)^{2-n}\, d\theta\, dr \right)^{1/2} \sim A_\gamma u(\theta_x)$$

as long as $\theta_x \in Q_0$ since βQ_0 stays well inside the lower half of S^{n-1} (see the Appendix).

For the integral II, integration by parts gives

$$II = -\iint\limits_R \frac{1}{(r-1)^n} \sum \left(\frac{\partial}{\partial \widehat{\theta}_j} u(r\theta)\right) \widehat{\theta}_j P(\widehat{\theta})$$

$$= \iint\limits_R \frac{1}{(r-1)^{n-1}} \sum \left(\frac{\partial}{\partial \theta_j} u(r\theta)\right) \widehat{\theta}_j P(\widehat{\theta}),$$

where $\widehat{\theta}_j$ and $\widehat{\theta}$ are as before. The latter integral should be easy to estimate using the martingale method, described above. $\widehat{\theta}_j P(\widehat{\theta})$ is odd in the θ_j variable so that $\int \widehat{\theta}_j P(\widehat{\theta})\, d\theta_x = 0$. Also in this integrand $1/(1-r)^{n-1}$ is the correct dilation for later estimations by Lipschitz norms.

The Lusin area integral equals

$$A_\gamma u(\theta_x) = \left(\int\limits_{\Gamma_\gamma^\delta(\theta_x)} |\nabla_y u|^2 (\mathrm{dist}(y, S^{n-1}))^{2-n}\, dy \right)^{1/2}$$

$$= \left(\int\limits_{\Gamma_\gamma^\delta(\theta_x)} \left\{ \left(\frac{\partial}{\partial r} u\right)^2 + \frac{1}{r^2} \sum \left(\frac{\partial}{\partial \theta_j} u\right)^2 - \frac{1}{r^2}\left(\sum \theta_j \frac{\partial}{\partial \theta_j} u\right)^2 \right\} \right.$$

$$\left. \times \frac{r^{n-1}}{(1-r)^{n-2}}\, dr\, d\sigma(\theta) \right)^{1/2}$$

in spherical coordinates, where

$$\Gamma_\gamma^\delta(\theta_x) = \{(r, \theta) : \delta < r < 1,\ |\theta - \theta_x| < \gamma(1-r)\}.$$

$\Gamma_\gamma^\delta(\theta_x)$ is not a Euclidean cone, but it can be compared with Euclidean cones of different apertures as long as $\delta > 0$ is fixed. Here, as above, $\delta \le \delta_0/2$. Moreover, $A_\gamma u(\theta_x)$ can be shown to compare to

$$\left(\int\limits_{\Gamma_\gamma^\delta(\theta_x)} \left\{ \left(\frac{\partial}{\partial r} u\right)^2 + \frac{1}{r^2} \sum \left(\frac{\partial}{\partial \theta_j} u\right)^2 \right\} \frac{r^{n-1}}{(1-r)^{n-2}}\, dr\, d\theta \right)^{1/2}$$

by using the elementary estimate

$$\left(\sum \theta_j \frac{\partial}{\partial \theta_j} u\right)^2 \le \alpha(n) \sum \left(\frac{\partial}{\partial \theta_k} u\right)^2, \quad \text{where} \quad \alpha(n) = 1 - \theta_n^2 < 1,$$

because the region on S^{n-1} under consideration will always be well inside the lower hemisphere. Also $d\sigma(\theta) = d\theta/|\theta_n| \sim d\theta$ in the region under consideration.

Altogether the formula for approximating the boundary function $f(\theta_x)$, using the non-tangential convergence of $u(r\theta)$ to $f(\theta_x)$, is given in Lemma 1:

LEMMA 1.

$$\int\limits_{r=1-\varepsilon} u(r\theta_y)P_{1-r}(\theta_x - \theta_y)\, d\theta_y = \int\limits_{r=\delta_0} u(r\theta_y)P_{1-r}(\theta_x - \theta_y)\, d\theta_y$$

$$- \int\limits_{r=\delta_0} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\, k(\widehat{\theta})\right) d\theta - \int\limits_{r=1-\varepsilon} \left(\frac{\partial}{\partial r}u\right)\left(\frac{1}{(1-r)^{n-2}}\, k(\widehat{\theta})\right) d\theta$$

$$- \iint\limits_{R} \frac{1}{(1-r)^{n-2}}\, \frac{1}{r^2} \sum \left(\frac{\partial}{\partial\theta_j}k\left(\frac{\theta_x - \theta_y}{1-r}\right)\right)\frac{\partial}{\partial\theta_j}u(r\theta)\, d\theta\, dr$$

$$+ \iint\limits_{R} \frac{1}{(1-r)^{n-2}}\, \frac{1}{r^2} \sum\sum \left(\frac{\partial}{\partial\theta_j}k\left(\frac{\theta_x - \theta_y}{1-r}\right)\right)\theta_j\theta_k\frac{\partial}{\partial\theta_k}u(r\theta)\, d\theta\, dr$$

$$+ (n-1)\iint\limits_{R} \frac{1}{(1-r)^{n-2}}\, \frac{1}{r}\left(\frac{\partial}{\partial r}u\right)k(\widehat{\theta})\, d\theta\, dr$$

$$+ \iint\limits_{R} \frac{1}{(1-r)^{n-1}} \sum \left(\frac{\partial u}{\partial\theta_j}\right)\widehat{\theta}_j P(\widehat{\theta})\, d\theta\, dr.$$

The last integral over R has been shown to be dominated by $cA_\gamma f(\theta_x)$. The three boundary integrals will be proved to have the same upper bound. The remaining two integrals over R can be shown to differ from dyadic martingales by an error that is bounded by $CA_\gamma f(\theta_x)$; the dyadic martingales have square functions that are bounded by $CA_\gamma f(\theta_x)$. The constants C are independent of ε and θ_x. The method that will be used to prove these facts is essentially the one in [CWW] and [BM], with some relatively minor changes due to the geometry of $B_1(0)$.

To estimate the boundary terms in the formula of Lemma 1, first $\int_{r=\delta_0} uP_{1-r} - (\int_{r=\delta_0} uP_{1-r})_Q$ is bounded by $c\|A_\gamma f\|_\infty$ because

$$\left|\int\limits_{r=\delta_0} u(r\theta_y)\frac{1}{(1-r)^{n-1}} P\left(\frac{\theta_x - \theta_y}{1-r}\right) d\theta_y\right.$$

$$\left.- \oint\limits_{Q}\int\limits_{r=\delta_0} u(r\theta_y)P_{1-r}(\theta_z - \theta_w)\, d\theta_w\, d\theta_z\right|$$

$$\leq \frac{1}{|Q|}\int\limits_{Q}\left|\int\limits_{r=\delta_0} u(r\theta_y)P\left(\frac{\theta_x - \theta_y}{1-r}\right) d\theta_y - \int\limits_{r=\delta_0} u(r\theta_w)P_{1-r}(\theta_z - \theta_w)\, d\theta_w\right| d\theta_z$$

$$= \text{(letting } \underline{\theta}_x = \theta_x - \theta_y \text{ and } \underline{\theta}_z = \theta_z - \theta_w)$$

$$\frac{1}{|Q|}\int\limits_{Q}\left|- \int\limits_{r=\delta_0} u(r(\theta_x - \underline{\theta}_x))P_{1-r}(\underline{\theta}_x)\, d\underline{\theta}_x\right.$$

$$\left.+ \int\limits_{r=\delta_0} u(r(\theta_z - \underline{\theta}_z))\cdot P_{1-r}(\underline{\theta}_z)\, d\underline{\theta}_z\right| d\theta_z$$

$$\leq \frac{1}{|Q|} \int\limits_{Q} \int\limits_{r=\delta_0} P_{1-r}(\underline{\theta}_z)|u(r(\theta_x - \underline{\theta}_x)) - u(r(\theta_z - \underline{\theta}_z))| \, d\underline{\theta}_z \, d\theta_z$$

$$\leq \frac{1}{|Q|} \int\limits_{Q} \int\limits_{r=\delta_0} \frac{1}{(1-r)^{n-1}} P\left(\frac{\theta_z}{1-r}\right)|\nabla_\theta u(\theta^*)| \, |\theta_x - \theta_z + \underline{\theta}_z - \underline{\theta}_x|$$

$$\leq \sup_{\theta^* \in Q} |\nabla_\theta u(\theta^*)| \cdot c\ell(Q) \leq C\|A_\gamma f\|_\infty.$$

The last inequality follows from an estimate for harmonic functions that appears in Stein [S]. The result is stated here (Lemma 2) for derivatives in spherical coordinates.

The second to the last inequality is valid since $\operatorname{supp} P(\theta_x - \theta_y) \subseteq 3Q$ when $\theta_x \in Q$ and also $|\theta_x - \underline{\theta}_x - (\theta_z - \underline{\theta}_z)| = |\theta_y - \theta_w| \leq c\ell(Q) \leq c(1 - \delta_0)$ by definition of δ_0, taking $\delta_0 = 1 - \left(\frac{r_0 + 12\sqrt{n-1}}{r_0 \gamma}\right)\ell(Q)$ as below. Here r_0 is a bound for the distortion produced by $\widehat{P}$.

LEMMA 2 (Stein). *If u is harmonic in $\Gamma_\gamma^\delta(\theta_x)$ then for any point $r\theta_y \in \Gamma_{\alpha\gamma}^{\delta/2}(\theta_x)$ both*

$$|1 - r|\left|\frac{\partial u}{\partial r}\right|, |1 - r| \, |\nabla_\theta u| \leq cA_\gamma f(\theta_x), \quad c = c(\alpha).$$

Proof. Use the mean value property of the harmonic functions $\partial u/\partial r$ and $\partial u/\partial \theta_j$, $j = 1, \ldots, n-1$, and Cauchy–Schwarz. ∎

Lemma 2 and the fact that $\int k_{1-r}(\theta) \, d\theta = 1$ give an upper bound of $c\|A_\gamma f\|_\infty$ for both boundary integrals $\int_{r=\varrho}(1-r)\frac{\partial u}{\partial r}k_{1-r}(\theta_x - \cdot)$, $\varrho = \delta_0$ and $1 - \varepsilon$.

The three integrals

$$\iint\limits_R \frac{1}{r^2(1-r)^{n-2}} \sum_{j=1}^{n-1} \frac{\partial u}{\partial \theta_j} \frac{\partial k}{\partial \theta_j}, \quad \iint\limits_R \frac{1}{r^2(1-r)^{n-2}} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \frac{\partial u}{\partial \theta_j} \theta_j \theta_k \frac{\partial k}{\partial \theta_k}$$

and

$$\iint\limits_R \frac{1}{(1-r)^{n-1}} \sum_{j=1}^{n-1} \frac{\partial u}{\partial \theta_j} \bar{\theta}_j P(\bar{\theta})$$

are left to deal with. These will be replaced by dyadic martingales.

First each integral is written as a sum of smaller parts:

$$\iint\limits_R \frac{1}{r^2(1-r)^{n-2}} \sum \frac{\partial u}{\partial \theta_i} \frac{\partial k}{\partial \theta_i} = \sum_{i=1}^{n-1} \sum_{Q \in \mathcal{Q}} \iint\limits_{T_Q} \frac{1}{r^2(1-r)^{n-2}} \frac{\partial u}{\partial \theta_i} \frac{\partial k}{\partial \theta_i}$$

where $\mathcal{Q}$ is the collection of all dyadic subcubes $Q \subseteq P(3Q_0) \subseteq S_0$ with $\operatorname{supp} k_{1-r}(\theta_x - \cdot) \cap Q \neq \emptyset$. The regions T_Q are segments of spherical shells that

are obtained by projecting Q radially into the region $c/2^{k+1} < 1 - r < c/2^k$; here c depends on β and r_0, and $\ell(\widehat{P}^{-1}(Q)) = 1/2^k$.

Now each $\sum_{Q \in \mathcal{Q}} \lambda_Q$ is further subdivided into sums

$$\sum_{i=1}^{m} \sum_{Q \in \mathcal{Q}^i} \lambda_Q = \sum_{i=1}^{m} \Lambda_{\varepsilon,k}^{x_i},$$

where $\mathcal{Q}^i = \mathcal{Q}^{x_i}$ is a collection of cubes from $\mathcal{Q}$. These cubes are chosen so that if $Q \in \mathcal{Q}^i$ then there is a dyadic cube $Q'' \in \mathcal{Q}$ so that $3Q \subseteq Q'' + x_i$ and $\ell(\widehat{P}^{-1}(Q'')) = 2^3 \ell(\widehat{P}^{-1}(Q))$. The construction of the $\mathcal{Q}^i$ is accomplished in $\mathbb{R}^{n-1}$, dealing with the (truly) dyadic cubes $\widehat{P}^{-1}(Q)$ that were originally used to define the "dyadic" surface cubes. The following Lemma 3 guarantees that such a subdivision of $\mathcal{Q}$ exists:

LEMMA 3 ([CWW], [BM]). *There are a finite set of points* $\{x_i\}_{i=1}^M$ *in* $\mathbb{R}^{n-1}$ *and a disjoint collection* $\mathcal{Q}^{x_i}$ *of cubes so that*

  (i) $\bigcup_{i=1}^{M} \bigcup_{Q \in \mathcal{Q}^{x_i}} \widehat{P}^{-1}(Q) = \bigcup_{Q \in \mathcal{Q}} \widehat{P}^{-1}(Q) = \widehat{P}^{-1}(3Q_0)$.
 (ii) *For each* $Q \in \mathcal{Q}^{x_i}$ *there is a cube* $Q'' \in \mathcal{Q}$ *so that* $\ell(\widehat{P}^{-1}(Q'')) = 2^3 \ell(\widehat{P}^{-1}(Q))$ *and* $\widehat{P}^{-1}(3Q) \subseteq \widehat{P}^{-1}(Q'' + \theta_{x_i}) = \widehat{P}^{-1}(Q'') + x_i$.
(iii) *If* $Q_1, Q_2 \in \mathcal{Q}^{x_i}$ *and* $Q_1 \neq Q_2$, *then* $Q_1'' \neq Q_2''$.

The lemma is proved exactly as in [CWW] dealing with the $\widehat{P}^{-1}(Q)$, $\widehat{P}^{-1}(Q'')$ and the proof is consequently omitted.

There are three crucial properties that hold for the integrals $\lambda_Q = \int_{T_Q} I_j(\theta_x - \theta_y) \, dr \, d\theta_y$ and these properties are the ones that allow the martingale square functions to be bounded by $cA_\gamma f$. They are

$$\operatorname{supp} \lambda_Q \subseteq 3Q, \quad \int_{S_0} \lambda_Q = 0,$$

$$\|\lambda_Q\|_{\operatorname{Lip}\alpha} \, \ell(Q)^\alpha \le c \Big(\int_{T_Q} |\nabla u|^2 (1-r)^{2-n} \Big)^{1/2}.$$

The first property of dyadic cubes in S^{n-1} means $\operatorname{supp} \lambda_Q \subseteq \widehat{P}(3\widehat{Q})$ if $\widehat{Q} \subseteq \mathbb{R}^{n-1}$ and $Q = \widehat{P}(\widehat{Q})$. To have this hold one needs $\operatorname{supp} k_{1-r}(\theta_x - \theta_y) \subseteq Q$ together with all neighboring dyadic surface cubes of Q. For this the distortion produced by the projection operator $\widehat{P}$ must be taken into account. This is accomplished by taking

$$\operatorname{supp} k(\theta) = \operatorname{supp} P(\theta) \subseteq \{\theta \in S_0 : |\theta - (0,\ldots,0,-1)| < \beta\},$$

where $\beta = r_0 \gamma / (r_0 + 12\sqrt{n-1})$. This choice of β in relation to γ is also designed for the proof of Theorem 3 below. Here r_0 is a constant such that

$r_0\ell(\widehat{P}^{-1}(Q)) < \ell(Q) < (1/r_0)\ell(\widehat{P}^{-1}(Q))$ for all cubes $Q \in \mathcal{Q}$ and

$$T_Q = \left\{ (r, \theta) : \theta \in Q, \; \frac{\ell(\widehat{P}^{-1}(Q))}{2\beta} < 1 - r \le \frac{\ell(\widehat{P}^{-1}(Q))}{\beta} \right\}$$

for any $Q \in \mathcal{Q}$.

The regions T_Q with $\ell(\widehat{P}^{-1}(Q)) = \ell(\widehat{P}^{-1}(Q_0))/2^k$ are not of uniform volume. However, for Q_1, Q_2 with $\ell(\widehat{P}^{-1}(Q_1)) = \ell(\widehat{P}^{-1}(Q_2))$, $|T_{Q_1}| \sim |T_{Q_2}|$ with constants depending on the function $\widehat{P}(x)$. Also the T_Q cover a region in the lower hemisphere of $B_1(0)$ so for any $\varepsilon > 0$ the region R in Lemma 1 will be contained in $\bigcup_{Q \in \mathcal{Q}} T_Q$.

Now,

$$\Lambda^i_{\varepsilon,k}(\theta_x) = \sum_{Q \in \mathcal{Q}^i} \lambda_Q(\theta_x) = \sum_{\ell(Q) \ge 1/2^{m+2}} (\lambda_Q + \lambda_{Q,\varepsilon}),$$

where $1/2^{m+3} < \varepsilon \le 1/2^{m+2}$ and $\lambda_{Q,\varepsilon} = \int_{T_Q \cap \{r > 1 - \varepsilon\}} I_k(\theta_x)$ (if $\varepsilon = 1/2^{m+2}$, $\lambda_{Q,\varepsilon}$ disappears). By using the shifted grid of cubes in $\mathcal{Q}^i$, $\Lambda^i_{\varepsilon,k}$ can be approximated by the function $E(\Lambda^i_{\varepsilon,k} \,|\, \mathcal{F}_{m-1})$, the conditional expectation of $\Lambda^i_{\varepsilon,k}$ with respect to the σ-algebra $\mathcal{F}_{m-1}$. Notice that

$$E(\Lambda^i_{\varepsilon,k} \,|\, \mathcal{F}^i_{m-1}) = E(\Lambda^i_{1/2^{m+2},k} \,|\, \mathcal{F}^i_{m-1})$$

because $\int_{Q_{m-1}} \lambda_{Q,\varepsilon} = 0$ for any Q_{m-1} in the generating set for $\mathcal{F}^i_{m-1}$. This happens because $\operatorname{supp} \lambda_{Q,\varepsilon} \subseteq$ some $Q_{m-1} \in \mathcal{F}^i_{m-1}$ by Lemma 3.

Now, with $\sum'_Q := \sum_{c(\delta_0) \ge \ell(\widehat{P}^{-1}(Q)) > 1/2^{m+2}}$,

$$|\Lambda^i_{\varepsilon,k}(\theta_x) - E(\Lambda^i_{\varepsilon,k} \,|\, \mathcal{F}^i_{m-1})| = \left| \sum_Q{}' \lambda^i_Q - \frac{1}{|Q_{m-1}|} \int_{Q_{m-1}} \sum_Q{}' \lambda^i_Q \right|$$

$$\le \sum_Q{}' \frac{1}{|Q_{m-1}|} \int_{Q_{m-1}} |\lambda^i_Q(\theta_x) - \lambda^i_Q(\theta_z)| \, d\theta_z$$

$$\le \sum_Q{}' \frac{1}{|Q_{m-1}|} \int_{Q_{m-1}} \int_{T_Q} \frac{1}{r^2(1-r)^{n-2}} \left| \frac{\partial u}{\partial \theta_j}(r\theta_w) \right|$$

$$\cdot \left| \frac{\partial k}{\partial \theta_j}\left(\frac{\theta_x - \theta_w}{1-r} \right) - \frac{\partial k}{\partial \theta_j}\left(\frac{\theta_z - \theta_w}{1-r} \right) \right| dr \, d\theta_w \, d\theta_z$$

$$\le \sum \frac{1}{|Q_{m-1}|} \int_{Q_{m-1}} \int_{T_Q} \frac{1}{r^2(1-r)^{n-1}} \left| \frac{\partial u}{\partial \theta_j}(r\theta_w) \right|$$

$$\cdot \left| \nabla_{\bar{\theta}} \frac{\partial k}{\partial \theta_j}(\theta^*) \right| \cdot \left| \frac{\theta_x - \theta_z}{1-r} \right| dr \, d\theta_w \, d\theta_z$$

$$\leq C \sum_{T_Q} \int \frac{1}{(1-r)^n} \, |\nabla_\theta u| \sup_{Q_{m-1}} |\theta_x - \theta_z|$$

$$\leq C \sum \ell(Q_{m-1}) \Big(\int_{T_Q} |\nabla_\theta u|^2 (1-r)^{2-n} \Big)^{1/2} \Big(\int_{T_Q} \frac{1}{(1-r)^{n+2}} \Big)^{1/2}$$

$$\leq C \Big({\sum_Q}' \int_{T_Q} |\nabla_\theta u|^2 (1-r)^{2-n} \Big)^{1/2} \Big({\sum_Q}' \Big(\frac{\ell(Q_{m-1})}{\ell(Q)} \Big)^2 \Big)^{1/2} = C A_\gamma^\delta f(\theta_x).$$

The last inequality follows because the cone $\Gamma_\gamma^\delta(\theta_x)$ contains any T_Q where $\operatorname{supp} \lambda_Q^i \cap Q_{m-1} \neq \emptyset$ and $\theta_x \in Q_{m-1}$. Taking $\gamma = (1 + 12\sqrt{n-1}/r_0)\beta$ is enough to guarantee this.

A similar calculation shows that each integrand in $\Lambda_{\varepsilon,k}$, $k = 4, 5, 6$, has the same upper bound for $|\Lambda_{\varepsilon,k}^i(\theta_x) - E(\Lambda_{\varepsilon,k}^i \mid \mathcal{F}_{m-1})|$, $i = 1, \dots, M$.

The last result of Part I is Theorem 3. To see that the dyadic martingale square function for the dyadic martingale $\{E(\Lambda_{\varepsilon,k}^i \mid \mathcal{F}_{m-1})\}_{m=1}^\infty$ can be bounded by $C\|A_\gamma^\delta f\|_\infty$ a.e., it is helpful to notice that for a fixed $Q \in \mathcal{Q}^i$ if $Q_{m-1} \supseteq 3Q$ and $\ell(\widehat{P}^{-1}(Q_{m-1})) = 2^3 \ell(Q)$, then $\fint_{Q_{m-1}} \lambda_Q^i = 0$ so $d_j \chi_{Q_{j-1}} = 0$ if $j > m$. When $j = m$, we have

$$\|d_m \chi_{Q_{m-1}}\|_\infty = \sup_{Q_m \subseteq Q_{m-1}} \Big| \fint_{Q_m} \lambda_Q^i - \fint_{Q_{m-1}} \lambda_Q^i \Big|$$

if $\theta_x \in Q_{m-1}$. The estimate for this term in the dyadic square function only involves $\nabla u(r\theta_y)$ for $r\theta_y \in T_Q$ and the most extreme case will be that Q lies in the 2^nth section of Q_{m-1}, Q_m, that is farthest from θ_x. Then at most the region in $\|d_m \chi_{Q_{m-1}}\|_\infty$ will involve 5 cubes beyond the cube Q such that $\theta_x \in \operatorname{supp} T_Q$, i.e. at a distance $6\sqrt{n-1}(1/r_0)\ell(\widehat{P}^{-1}(Q))$, so one needs $\gamma \geq \beta(1 + 12\sqrt{n-1}/r_0)$.

THEOREM 3. *For $\Lambda_k^i(\theta_x)$ the martingale limit function of the dyadic martingale*

$$\{E(\Lambda_{1/2^{m+1},k}^i \mid \mathcal{F}_{m-1})\}_{m=1}^\infty,$$

with $\Lambda_{\varepsilon,k}^i$ defined above, the dyadic martingale square function satisfies $S\Lambda_k^i(\theta_x) \leq C A_\gamma^\delta f(\theta_x)$ with C independent of θ_x and f.

Proof. Notice that it is obvious that $\{E(\Lambda_{1/2^{m+1},k}^i \mid \mathcal{F}_{m-1})\}_{m=1}^\infty$ is a dyadic martingale. The rest of the proof will actually show that each dyadic square function S_m is uniformly bounded, with constant independent of m. This implies that the family of martingale functions is uniformly integrable and so has an L^1 limit function $\Lambda_{\varepsilon,k}^i$. Actually the uniform bound on S_m is

enough to prove Theorem 1. Now with $\sum'_Q$ as in the previous proof,

$$
S^2 \Lambda^i(\theta_x) = \sum_{Q_j \in \mathcal{F}^i} \Big\| \oint_{Q_j} \sum \lambda_Q - \oint_{Q_{j-1}} \sum \lambda_Q \Big\|_\infty^2
$$

$$
= \sum_{Q_j \in \mathcal{F}^i} \Big\| \sum_Q {}' \Big(\oint_{Q_j} \lambda_Q - \oint_{Q_{j-1}} \lambda_Q \Big) \Big\|_\infty^2
$$

$$
\leq \sum_{Q_j \in \mathcal{F}^i} \Big\| \sum_Q {}' \oint_{Q_j} \oint_{Q_{j-1}} (\lambda_Q(\theta_w) - \lambda_Q(\theta_z))\, d\theta_z\, d\theta_w \Big\|_\infty^2
$$

$$
\leq \sum_{Q_j \in \mathcal{F}^i} \Big(\sum_Q {}' \oint_{Q_j} \oint_{Q_{j-1}} \|\lambda_Q\|_{\mathrm{Lip}\,\alpha}\, \ell(Q_{j-1})^\alpha\, d\theta_z\, d\theta_w \Big)^2
$$

$$
= c \sum_{Q_j \in \mathcal{F}^i} \ell(Q_{j-1})^{2\alpha} \Big(\sum \|\lambda_Q\|_{\mathrm{Lip}\,\alpha} \frac{\ell(Q)^\beta}{\ell(Q)^\beta} \Big)^2
$$

$$
\leq C \sum_{Q_j \in \mathcal{F}^i} \ell(Q_{j-1})^{2\alpha} \Big(\sum_Q {}' \frac{1}{\ell(\widehat{P}^{-1}(Q))^{2\beta}} \Big) \Big(\sum \|\lambda_Q\|_{\mathrm{Lip}\,\alpha}^2 \ell(Q)^{2\beta} \Big)
$$

$$
= C \sum_{Q_j \in \mathcal{F}^i} \ell(Q_{j-1})^{2\alpha} \cdot \ell(Q_{j-1})^{-2\beta} \sum_Q {}' \|\lambda_Q\|_{\mathrm{Lip}\,\alpha, Q_{j-1}}^2 \ell(Q)^{2\beta}
$$

$$
\leq C \sum_{Q \in \mathcal{Q}} \|\lambda_Q\|_{\mathrm{Lip}\,\alpha}^2 \ell(Q)^{2\beta} \cdot \sum_{\ell(Q_{j-1}) \leq 2^3 \ell(Q)} \ell(Q_{j-1})^{2(\alpha-\beta)}.
$$

Here T_Q is in the sum of λ_Q that make up Λ^i, and θ_x is near T_Q as described above. This means

$$
S^2 \Lambda^i(\theta_x) \leq C \sum_Q \|\lambda_Q\|_{\mathrm{Lip}\,\alpha}^2 \ell(Q)^{2\alpha}
$$

$$
\leq C \sum_Q \int_{T_Q} |\nabla u|^2 \cdot (1-r)^{2-n} \leq C A_\gamma^{\delta_0} f(\theta_x). \quad \blacksquare
$$

It is useful to extend the class of operators L_0 for the application of Theorem 4. Note that Theorem 1 is also valid for any strictly elliptic constant coefficient operator. Basically the same proof works for solutions to these operator equations. If $L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \big(a_{ij} \frac{\partial}{\partial x_j} \big)$ in rectangular coordinates, then in spherical coordinates

$$
L = \Big(\sum_{i,j=1}^n a_{ij} \theta_i \theta_j \Big) \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \Big(\sum_{j<n,\, i=1}^n a_{ij} \theta_i \frac{\partial}{\partial \theta_j} + \sum_{i<n,\, j=1}^n a_{ij} \theta_j \frac{\partial}{\partial \theta_i} \Big)
$$

$$
+ \frac{1}{r} \Big(\sum_{j<n,\, i=1}^n a_{ij} \theta_i \frac{\partial}{\partial r \partial \theta_j} + \sum_{i<n,\, j=1}^n a_{ij} \theta_j \frac{\partial^2}{\partial r \partial \theta_i} \Big)
$$

$$- \frac{2}{r} \Big(\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j \Big) \Big(\sum_{l=1}^{n-1} \theta_l \frac{\partial^2}{\partial r \partial \theta_l} \Big) + \frac{1}{r^2} \Big(\sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \Big)$$

$$- \frac{1}{r^2} \Big(\sum_{j<n,i=1}^{n} a_{ij}\theta_i \Big(\sum_{l=1}^{n-1} \theta_l \frac{\partial^2}{\partial \theta_l \partial \theta_j} \Big) + \sum_{i<n,j=1}^{n} a_{ij}\theta_j \Big(\sum_{l=1}^{n-1} \theta_l \frac{\partial^2}{\partial \theta_l \partial \theta_i} \Big) \Big)$$

$$+ \frac{1}{r^2} \Big(\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j \Big) \Big(\sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \theta_k\theta_l \frac{\partial^2}{\partial \theta_k \partial \theta_l} \Big) + \frac{1}{r} \Big(\sum_{i=1}^{n} a_{ii} - \sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j \Big) \frac{\partial}{\partial r}$$

$$+ \frac{3}{r^2} \Big(\sum_{i,j=1}^{n} a_{ij}\theta_i\theta_j \Big) \Big(\sum_{l=1}^{n-1} \theta_l \frac{\partial}{\partial \theta_l} \Big) - \frac{1}{r^2} \Big(\sum_{i=1}^{n} a_{ii} \Big) \Big(\sum_{l=1}^{n-1} \theta_l \frac{\partial}{\partial \theta_\ell} \Big).$$

The fact that the a_{ij} are constant and satisfy $\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} \xi_i a_{ij}\xi_j \leq \lambda^{-1}|\xi|^2$ means that one can substitute for $\partial^2 u/\partial r^2$ as in the proof of Theorem 1. The formula is much longer; however the same kinds of estimations will work here to give the exponential square result, using direct estimation on the drift terms and integration by parts along with replacement by martingales on the principal order terms. To estimate the boundary terms the device of averaging replaces using Lemma 2 ([CS1]).

II. In this part of the paper $S_\alpha u(\theta) = (\int_{\Gamma_\alpha(\theta)} |\nabla u(y)|^2 \delta(y)^{2-n}\, dy)^{1/2}$ is used to devoted the Lusin area integral of $u(y)$ over the cone $\Gamma_\alpha(\theta) = \Gamma_\alpha^\delta(\theta)$, a Euclidean cone as defined in the Appendix. The following result will be established:

THEOREM 4. *For L_0, L_1 strictly elliptic divergence form operators with coefficients satisfying the condition* (Cc) *below, suppose the solutions to $L_0 u_0 = 0$ in $B_1(0)$ satisfy the exponential square theorem. Then solutions u to $L_1 u_1 = 0$ in $B_1(0)$ also satisfy the EST.*

DEFINITION (EST). For L a strictly elliptic divergence form operator on $B_1(0)$, a solution to $Lu = 0$ on $B_1(0)$ *satisfies the exponential square theorem* if $Su \in L^\infty(\partial B_1(0))$ implies that $u|_{\partial B} = f$ lies in the exponential square class.

(Cc) For all $q \in \partial B_1(0)$ and $0 < r < r_0$ there is a coefficient $\varepsilon(r)$ so that

$$\Big(\frac{1}{\omega_0(\Delta_r(q))} \int_{T_r(q)} \frac{a(y)^2}{\delta(y)^2} G_0(0;y)\, dy \Big) \leq \varepsilon(r)$$

with $\varepsilon(r) \to 0$ as $r \to 0$ and in fact $\varepsilon(2^j r) \leq 2^{\gamma j} r^\gamma$ where $\gamma < \alpha$, α is the coefficient of Hölder continuity at $\partial B_1(0)$ for solutions of $L_0 u = 0$ that vanish locally on $\partial B_1(0)$.

Here $\omega_0 = \omega_0^0$ is the elliptic measure associated with L_0; ω_0^x is this measure evaluated at the point x in $B_1(0)$; $G_0(x; y)$ is the Green function for L_0; $\Delta_r(q) = B_r(q) \cap \partial B_1(0)$ and $T_r(q) = B_r(q) \cap B_1(0)$ for $q \in \partial B_1(0)$; $\delta(y) = \text{dist}(y, \partial B_1(0))$ and $a(y) = \sup_{x \in B_{\delta(y)/2}(y)} |\varepsilon(x)|$, where $\varepsilon(x) = \sup_{i,j} |\varepsilon_{ij}(x)|$ and $\varepsilon_{ij}(x) = b_{ij}(x) - a_{ij}(x)$. Moreover, the operators

$$L_0 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right), \quad L_1 = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

satisfy $\lambda^{-1}|\xi|^2 \le \xi_i C_{ij}\xi_j \le \lambda|\xi|^2$ for $C_{ij} = a_{ij}$ and b_{ij} and for some $\lambda > 0$.

DEFINITION. $f \in \exp L^2(d\mu)$ if for any surface "cube" $Q \subseteq \partial B_1(0)$ there are constants $C_1, C_2 > 0$ independent of Q so that

$$\left(\frac{1}{\mu(Q)} \int_Q \exp\{C_1|f - f_{\mu(Q)}|^2\}\, d\mu(y)\right) \le C_2 < \infty,$$

where $f_{\mu(Q)} = \mu(Q)^{-1} \int_Q f(y)\, d\mu(y)$.

In this paper μ will be taken to be surface (Lebesgue) measure on $\partial B_1(0)$, and the elliptic measure associated to L_0 will be assumed to satisfy $d\omega_0 \in B^2(d\mu)$, i.e.

$$\left(\frac{1}{\sigma(Q)} \int_Q \left(\frac{d\omega_0}{d\sigma}(x)\right)^2 d\sigma(x)\right)^{1/2} \precsim \frac{\omega_0(Q)}{\sigma(Q)}.$$

The measure in Theorem 4 and in the EST will be surface measure. It is crucial that the results of [FKP] hold for the measure $d\omega_1$, given the Carleson condition (Cc).

In order to prove that the Carleson condition with vanishing trace is sufficient to guarantee that the exponential square theorem will hold for solutions of one operator whenever it holds for solutions of a given operator, one can obtain a pointwise comparison of area integrals. This is done by using arguments similar to those in [FKP]; the solution u_1 behaves like a perturbation of u_0 when the coefficients satisfy (Cc).

One reason for proving Theorem 1 in Part I was to guarantee that Theorem 4 is not vacuous. Theorem 1 shows that for $L_0 = \Delta$, the EST is true; so Theorem 4 then shows that EST will also hold for solutions to $L_1 u_1 = 0$ when L_1 is a mild perturbation of Δ (or of any strictly elliptic constant coefficient operator). This result extends the class of elliptic operators for which the EST is known to be valid. It is a start towards proving an EST for solutions to more general elliptic operators in the upper half-plane. In [CS1] a result for a restricted class of elliptic operators was established for the upper half-plane. The coefficients were assumed to match the Laplacian in the vertical direction, i.e. $a_{nj} = \delta_{nj}$, $j = 1, \ldots, n$. Theorem 4 removes

this restriction, replacing it by the broader Carleson condition. R. Bañuelos and C. Moore [BM] note that an exponential square result for any operator L on $\mathbb{R}^{n+1}_+$ whose pull-back by a Lipschitz map is Δ would open the way for an easy proof of good-λ inequalities for the nontangential maximal function and the Lusin area integral of a harmonic function with sharp constants. Theorem 4 does not accomplish this much, but it is a step in that direction.

There are other facts that can be deduced from the proof of Theorem 4. The comparison of area integrals obtained in the Main Lemma below may be of interest in itself. As a major step in proving the Main Lemma the key estimate

$$\left(\fint_{R_j^*} |F(y)|^2\, dy \right)^{1/2} \precsim \varepsilon(\ell(R_j)) M_{\omega_0} S_\beta(u_1)(q_0)$$

is shown to be valid for any fixed dilation R_j^* of a Whitney type region $R_j \subseteq B_1(0)$. Here $F(y) = u_1(y) - u_0(y)$, $\ell(R_j) \simeq \operatorname{diam} R_j^* \simeq \operatorname{dist}(R_j^*, \partial B_1(0))$, q_0 is any point on $\partial B_1(0)$ that lies in the radial projection of R_j onto $\partial B_1(0)$, $\varepsilon(\ell(R_j))$ is the Carleson coefficient and

$$M_{\omega_0} g(q) = \sup_{\Delta_r(q)} \frac{1}{\omega_0(\Delta_r(q))} \int_{\Delta_r(q)} |g(\widehat{q})|\, d\omega_0(\widehat{q})$$

is the Hardy–Littlewood maximal function with respect to the elliptic measure ω_0 of L_0.

It is worth noting that the above key estimate for $(\fint_{R_j^*} |F(y)|^2\, dy)^{1/2}$ can be used to prove the good-λ inequality in [FKP, Lemma 2.16]. With suitable modifications this condition and proof also give the good-λ inequality in [CS2, Lemma 7].

Let

$$M(q_0) = \max(M_{\omega_0} S_\beta u_1(q_0), [M_{\omega_0} S_\beta u_1(q_0)]^{1/2}).$$

Theorem 4 will follow from the

MAIN LEMMA. *Suppose $L_0 u_0 = 0 = L_1 u_1$ in $B_1(0)$, where L_0 and L_1 are as in Theorem 4. If $u_1|_{\partial B_1(0)} = f = u_0|_{\partial B_1(0)}$, where $f \in \operatorname{BMO}(\partial B_1, d\omega_0)$, then there is a constant $C = C(\lambda, n, \|f\|_{\operatorname{BMO}}, \alpha, \beta)$ so that*

$$S_\alpha u_0(q) \le C M(q) \quad \text{for } d\omega_0 \text{ a.e. } q \in \partial B_1(0).$$

Before proving the Main Lemma, it is easy to see that if $S_\beta u_1 \in L^\infty(\partial B_1, d\omega_0)$ then

$$M_{\omega_0} S_\beta u_1(q) \le \|S_\beta u_1\|_{\infty, \omega_0} \quad \text{for } d\omega_0 \text{ a.e. } q.$$

The Main Lemma says that if $S_\beta u_1 \in L^\infty(\partial B_1, d\omega_1)$ then $S_\alpha u_0 \in L^\infty(\partial B_1, d\omega_0)$ and by the hypothesis of Theorem 4, u_0 satisfies EST, so $f \in \exp L^2(\partial B_1)$. The reason that the assumption $f \in \operatorname{BMO}(\partial B_1, d\omega_0)$ is

not needed for Theorem 4 is the following: the fact that the coefficients of L_0 and L_1 satisfy (Cc) means that the B^2 result of [FKP] is valid so $\omega_0 \in B^2(\sigma)$ $\Rightarrow \omega_1 \in B^2(\sigma)$. This fact is enough to guarantee that

$$S_\beta u_1 \in L^\infty(\partial B_1, d\omega_1) \Rightarrow f \in \mathrm{BMO}(d\omega_0),$$

because one can use the following theorem of C. Kenig ([K, Chapter 1, Theorem 1.5.18], quoted below for clarity) to see

$$S_\beta u_1 \in L^\infty(\partial B_1, d\omega_1) \Rightarrow f \in \mathrm{BMO}(d\omega_1).$$

Then $d\omega_1 \in B^2(d\omega_0) \Rightarrow d\omega_0 \in B^q(d\omega_1)$ for some $q > 1$ so this is sufficient to have $f \in \mathrm{BMO}(d\omega_0)$.

THEOREM (C. Kenig). *Let $Lu = 0$ in $B_1(0)$. Then*

$$u(x) = \int\limits_{\partial B_1(0)} k(x; q) f(q)\, d\omega(q)$$

with $f \in \mathrm{BMO}(\partial B_1, d\omega)$ if and only if

$$\frac{1}{\omega(\Delta_r(q))} \int\limits_{T_r(q)} G(0; y) |\nabla u(y)|^2\, dy \leq C.$$

If $S_\beta u \in L^\infty(d\omega)$ then the basic estimate, for y^* the projection of y onto S^{n-1},

$$G(0; y) \simeq \frac{\omega(\Delta_{\delta(y)}(y^*))}{\delta(y)^{n-2}}$$

shows that

$$\frac{1}{\omega(\Delta_r)} \int\limits_{T_r} G(0; y)|\nabla u(y)|^2\, dy \leq \frac{1}{\omega(\Delta_r)} \int\limits_{T_r} \frac{\omega(\Delta_{\delta(y)}(y^*))}{\delta(y)^{n-2}} |\nabla u(y)|^2\, dy$$

$$\underset{\text{Fubini}}{\precsim} \frac{1}{\omega(\Delta_r)} \int\limits_{\Delta_r} \int\limits_{\Gamma_\beta^{\delta_0}(q)} |\nabla u(y)|^2 \delta(y)^{2-n}\, dy\, d\omega(q)$$

$$= \frac{1}{\omega(\Delta_r)} \int\limits_{\Delta_r} S_\beta u(q)\, d\omega(q) \leq \|S_\beta u\|_\infty.$$

So $S_\beta u_1 \in L^\infty(\partial B_1, d\omega_1) \Rightarrow f \in \mathrm{BMO}(\partial B_1, d\omega_1)$.

To prove the Main Lemma, an argument similar to that of [FKP] in their proof of Lemma 2.9 can be employed. Here, however, all estimates will be kept local, both for $\fint_{R_j} |\delta(y)\nabla F(y)|^2\, dy$ and for $(\fint_{R_j} |F(y)|^2\, dy)^{1/2}$ until the very end of the proof. Then instead of taking sups, the local estimates will be added to show $S_\alpha F(q) \precsim M(q)$ for a.e. $q \in \partial B_1(0)$.

It is easy to see this implies $S_\alpha u_0(q) \leq c'M(q)$ also. Start with the inequality

$$\fint_{R_j} |\delta(y)\nabla F(y)|^2\, dy \le \frac{2}{\ell(R_j)} \int_{\ell(R_j)}^{3\ell(R_j)/2} \left(\fint_{R_j} |\delta\nabla F|^2 \right) dx$$

$$\lesssim \frac{\delta_j^2}{|R_j|} \left(\frac{1}{\delta_j} \int_{N(\partial R_j)} |F(y)|\, |A\nabla F(y)\cdot \vec{n}| + \frac{1}{\delta_j} \int_{N(\partial R_j)} |F|\, |\varepsilon|\, |\nabla u_1 \cdot \vec{n}| \right.$$

$$\left. + \int_{R_j^*} |\vec{\nabla} F|\, |\varepsilon \vec{\nabla} u_1| \right) \le C(\lambda, n) \left(\fint_{R_j^*} |F|^2 \right)^{1/2} \left(\fint_{R_j^*} |\delta\nabla F|^2 \right)^{1/2}$$

$$+ \varepsilon_j \left(\fint_{R_j^*} |F|^2 \right)^{1/2} \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right) + \varepsilon_j \left(\fint_{R_j^*} |\delta\nabla F|^2 \right)^{1/2} \cdot \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right)^{1/2}$$

as in [FKP, p. 87], with $\varepsilon_j = \varepsilon(\ell(R_j))$ replacing ε_0.

Now the key estimate $(\fint_{R_j^*} |F(y)|^2\, dy)^{1/2} \le \varepsilon_j M_{\omega_0} S_\beta u_1(q_0)$ along with $|\nabla F| \lesssim |\nabla u_1| + |\nabla u_0|$ gives

$$\int_{R_j} |\nabla F(y)|^2 \delta(y)^{2-n} dy \lesssim \varepsilon_j M_{\omega_0} S_\beta u_1(q_0) \cdot \left(\fint_{R_j^*} \delta(y)^2 |\nabla F(y)|^2 \right)^{1/2}$$

$$+ \varepsilon_j^2 M_{\omega_0} S_\beta u_1(q_0) \cdot \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right)^{1/2}$$

$$+ \varepsilon_j \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right)^{1/2} \left(\fint_{R_j^*} |\delta\nabla u_0|^2 + \fint_{R_j^*} |\delta\nabla u_1|^2 \right)^{1/2}.$$

The terms involving $\fint_{R_j^*} |\delta\nabla u_0|^2$ can be handled by using

$$\fint_{R_j^*} |\delta\nabla u_0|^2 \lesssim \int_{R_j^*} |\nabla u_0(y)|^2 \delta(y)^{2-n} \lesssim \int_{R_j^*} \frac{1}{\omega_0(\Delta_j^*)} \cdot \frac{\omega_0(\Delta_j^*)}{\delta(x)^{n-2}} |\nabla u_0(y)|^2$$

$$\lesssim \frac{1}{\omega_0(\Delta_j^*)} \int_{R_j^*} G_0(0;y) |\nabla u_0(y)|^2\, dy \le C\|f\|_{\mathrm{BMO}}$$

according to Kenig's theorem, because $R_j^* \subseteq T_{r_j}(q_0)$ and ω_0 is a doubling measure, so $\omega_0(\Delta_{r_j}) \sim \omega_0(\Delta_j^*)$, where $\Delta_j^* = \mathrm{proj}|_{\partial B_1} R_j^*$, $r_j = \mathrm{diam}\, R_j$.

Thus $\fint_{R_j^*} |\delta\nabla u_0|^2 \le C = C(\|f\|_{\mathrm{BMO}})$ and this gives

$$\int_{R_j} |\nabla F(y)|^2 \delta(y)^{2-n}\, dy \lesssim \varepsilon_j M_{\omega_0} S_\beta u_1(q_0)$$

$$+ \varepsilon_j M_{\omega_0} S_\beta u_1(q) \cdot \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right)^{1/2} + \varepsilon_j \left(\fint_{R_j} |\delta\nabla u_1|^2 \right)^{1/2}$$

$$+ \varepsilon_j^2 M_{\omega_0} S_\beta u_1(q) \cdot \left(\fint_{R_j} |\delta\nabla u_1|^2 \right)^{1/2} + \varepsilon_j \left(\fint_{R_j^*} |\delta\nabla u_1|^2 \right).$$

Summing over j with $R_j \cap \Gamma_\alpha(q_0) \neq \emptyset$ on both sides gives

$$S_\alpha F(q_0)^2 \lesssim C(M_{\omega_0} S_\beta u_1(q_0) + M_{\omega_0}^2 S_\beta u_1(q_0))$$

using $\sum \varepsilon_j \leq \varepsilon_0$ and $(\sum \varepsilon_j^2)^{1/2} \leq \varepsilon_0$, Cauchy–Schwarz and the fact that $S_\beta u_1(q_0) \lesssim M_{\omega_0} S_\beta u_1(q_0)$. Taking square roots gives the required inequality.

To prove

$$\left(\fint_{R_j} |F(y)|^2 \, dy \right)^{1/2} \lesssim \varepsilon(\ell(R_j)) M_{\omega_0} S u_1(q_0), \qquad q_0 \in \mathrm{proj}(R_j),$$

the stopping time argument that [FKP] use to prove that $\widetilde{N}(F)(q) \leq \varepsilon_0 M_{\omega_0} S u_1(q)$ a.e. $d\sigma$ actually gives this estimate. The notation is that of [FKP].

Dividing the potential $F(x) = \int_{B_1(0)} \nabla_y G(x; y) \cdot ([\varepsilon_{ij}(y)] \nabla u)$ into the same regions as in [FKP],

$$\int_{B_{\delta(x)/4}(x)} + \int_{\Omega_0} + \sum_{j=1}^{N} \int_{\Omega_j \cap \Gamma(q_0)} + \sum_{j=1}^{N} \int_{\Omega_j \cap \Gamma(q_0)^c} + \int_{B_1(0) \setminus B_{1/2}(q_0)}$$

one has only to notice that in

$$\int_{B_{\delta(x)/4}(x)} \nabla_y G(x; y) \cdot [\varepsilon_{ij}(y)] \nabla u(y),$$

the estimate on $\varepsilon(y)$ actually gives $\varepsilon(\delta(x))$ because $y \in B_{\delta(x)/4}(x)$. So one has

$$\left(\fint_{R_j} |F_1(x)|^2 dx \right)^{1/2} \leq \varepsilon(\delta(x)) \left(\int_{R_j^*} |\nabla u_1(y)|^2 \delta(y)^{2-n} \right)^{1/2}.$$

For $\int_{\Omega_0}$ the stopping time argument uses the Carleson condition on the term

$$\sum_{I \in I_k} \int_{I^+} \frac{a(y)^2}{\delta(y)^2} G(0; y) \, dy,$$

where I is a "dyadic" cube in Δ_0, I_k is the collection of cubes in Δ_0 with side length $\sim 2^{-k} \delta(x)$, and I^+ is the upper half of the Carleson box associated with I. So

$$\int_{I^+} \frac{a(y)^2}{\delta(y)^2} G(0; y) \, dy \leq \varepsilon(\ell(I)) \omega_0(I) \leq \varepsilon(\delta(x)) \omega_0(I)$$

because of the definition of $\Omega_0 = B_{\delta(x)/2}(q_0) \cap B_1(0)$. So one has the estimate

$$\left| \int_{\Omega_0} \nabla G(x; y) \cdot [\varepsilon_{ij}] \nabla u(y) \right| \leq \varepsilon(\delta(x)) M_{\omega_0} S u_1(q_0).$$

For the integrals over $\Omega_j \cap \Gamma(q_0)$, condition (Cc) is used on

$$\left(\int\limits_{\Omega_j \cap \Gamma(q_0)} \frac{a(y)^2}{\delta(y)^2} G_0(0;y) \right)^{1/2} \leq \varepsilon(2^j \delta(x)) \omega_0(\Delta_j)^{1/2}$$

to give

$$\left| \int\limits_{\Omega_j \cap \Gamma(q_0)} \nabla G_0(x,y)[\varepsilon_{ij}(y)] \nabla u_1(y)\, dy \right| \lesssim 2^{-j\alpha} \varepsilon(2^j \delta(x)) \cdot S u_1(q_0)$$

with every other part of the argument the same as in [FKP]. Now since $\varepsilon(2^j \delta(x)) \leq 2^{\gamma j}(\delta(x))$, we have

$$\left| \int\limits_{\Omega_j \cap \Gamma(q_0)} \nabla G_0 \cdot [\varepsilon_{ij}] \nabla u_1 \right| \leq 2^{-(\alpha-\gamma)j} \delta(x)^\gamma \cdot S u_1(q_0)$$

and as $\gamma < \alpha$, one can sum over j to get

$$\left| \sum_{j=1}^{N} \int\limits_{\Omega_j \cap \Gamma(q_0)} \nabla G_0 \cdot [\varepsilon_{ij}] \nabla u_1 \right| \lesssim \delta(x)^\gamma S u_1(q_0).$$

Similar estimates on the Green function $G_0(x;y)$ along with the stopping time argument imply that

$$\left| \sum_{j=1}^{N} \int\limits_{\Omega_j \cap \Gamma(q_0)^c} \nabla G_0 \cdot [\varepsilon_{ij}] \nabla u_1 \right| \lesssim \varepsilon(\delta(x)) M_{\omega_0} S u_1(q_0).$$

Finally, the fact that

$$\left| \int\limits_{B_1(0) \setminus B_{1/2}(q_0)} \nabla G_0 \cdot [\varepsilon_{ij}] \nabla u_1 \, dy \right| \leq C \int\limits_{\partial B_1 \setminus \Delta(q_0, 1/2)} S u_1(q)\, d\omega^x(q)$$

shows that this part of the potential is bounded by $C \delta(x)^\alpha M_{\omega_0} S u_1(q_0)$ by using the estimate

$$\frac{d\omega^x}{d\omega^0}(q) \lesssim \frac{2^{-\alpha j}}{\omega^0(\Delta_j)} \quad \text{if } q \in \Delta_j \setminus \Delta_{j-1}, \quad \Delta_j = \{q : |q - q_0| \lesssim 2^j \delta(x)\}.$$

For N such that $2^N \delta(x) = 1/2$ we have

$$\int\limits_{\partial B_1 \setminus \Delta(q_0, 1/2)} S u_1(q)\, d\omega^x(q) = \sum_{j=N-1}^{M} \int\limits_{R_j} S u_1(q)\, d\omega^x(q)$$

$$\leq \sum_{j=N}^{\infty} \frac{2^{-j\alpha}}{\omega^0(\Delta_j)} \int\limits_{\Delta_j} S u_1\, d\omega^0 \leq C \cdot 2^{-N\alpha} M_{\omega_0} S u_1(q_0) \lesssim \delta(x)^\alpha M_{\omega_0} S u_1(q_0).$$

This finishes the proof of the Main Lemma. ∎

Finally, we prove a version of the good-λ inequality (see [FKP, Lemma 2.16]) as a result of independent interest.

Let
$$E = \{q \in \Delta_r : S_\alpha F(q) > \beta_0 \lambda, \; F^*_{\beta'}(q) M_{\omega_0} S_{\beta'} u_1(q) \leq (\gamma\lambda)^2,$$
$$\widetilde{N}_{\beta'}(\delta \nabla F)(q) \cdot M_{\omega_0} S_{\beta'} u_1(q) \leq (\gamma\lambda)^2 \}.$$

Then for Δ_r a Whitney cube in the decomposition of $\{q : S_\alpha F(q) > \lambda\}$ one has
$$\lambda^2 \omega_0(E \cap \Delta_r) \lesssim \int\limits_{E \cap \Delta_r} \int\limits_{\Gamma_\alpha^{\tau r}(q)} |\nabla F(y)|^2 \delta(y)^{2-n} \, dy \, d\omega_0(q)$$

since $S_\alpha F(q) > \lambda \Rightarrow S_\alpha^{\tau r} F(q) > \lambda/2$ on $E_r = E \cap \Delta_r$ for any $0 < \tau < 1$, $r = \delta(x)$, if γ is chosen to be small enough (see [DJK]).

As in [FKP], with $A = [a_{ij}]$ the coefficient matrix for L_0,
$$\int\limits_{E \cap \Delta_r} (S_\alpha^{\tau r} F(q))^2 \, d\omega_0(q) \lesssim \int\limits_{\Omega} \left(\frac{1}{2} L_0 F^2 - F L_0 F \right) G_0(0; y) \, dy$$

$$\lesssim \int\limits_{\partial\Omega} F(y) G_0(0; y) A \nabla F \cdot \vec{n} - \frac{1}{2} \int\limits_{\Omega} A \nabla F^2 \cdot \nabla G_0$$

$$- \int\limits_{\partial\Omega} F G_0 [\varepsilon_{ij}] \nabla u_1(y) \cdot \vec{n} + \int\limits_{\Omega} \nabla(F G_0) \cdot [\varepsilon_{ij}] \nabla u_1(y)$$

$$= \int\limits_{\partial\Omega} F(y) G_0(0; y) A \nabla F(y) \cdot \vec{n} - \int\limits_{\partial\Omega} \frac{1}{2} F^2 A \nabla G_0 \cdot \vec{n}$$

$$+ \int\limits_{\Omega} \frac{1}{2} F(y)^2 L_0 G_0 - \int\limits_{\partial\Omega} F G_0 [\varepsilon_{ij}] \nabla u_1 \cdot \vec{n}$$

$$+ \int\limits_{\Omega} \nabla F \cdot [\varepsilon_{ij}] \nabla u_1(y) G_0(y) + \int\limits_{\Omega} F(y) \nabla G_0(y) \cdot [\varepsilon_{ij}] \nabla u_1(y).$$

Now
$$\Omega = \bigcup_{q \in E \cap \Delta_r} \Gamma_\alpha^{\tau r}(q)$$

so $0 \notin \Omega$ and $\int_\Omega F(y)^2 \, L_0 G_0 = 0$.

This leaves three boundary integrals and two integrals over Ω to be estimated. To handle the boundary integrals, which technically may not exist since ∇F and ∇G_0 exist as $L^2_{\text{loc}}(B_1(0))$ functions, one can use averaging over cones as in [FKP]: Take $\alpha < \beta < \beta'$; then
$$\int\limits_{\Delta_r \cap E} (S_\alpha^{\tau r} F(q))^2 \, d\omega_0(q) = \int\limits_{E_r} \int\limits_{\Gamma_\alpha^{\tau r}(q)} |\nabla F(y)|^2 \delta(y)^{2-n} \, dy \, d\omega_0(q)$$

$$\lesssim \int\limits_{E_r} \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \int\limits_{\Gamma_\eta^{\tau r}(q)} |\nabla F(y)|^2 \, \delta(y)^{2-n} \, dy \, d\eta \, d\omega_0(q)$$

$$\simeq \frac{1}{\beta-\alpha}\int_\alpha^\beta \int_{E_r}\int_{\Gamma_\eta^{\tau r}(q)} |\nabla F(y)|^2 \delta(y)^{2-n}\, dy\, d\omega_0(q)\, d\eta$$

$$\lesssim \frac{1}{\beta-\alpha}\int_\alpha^\beta \int_{\Omega_\eta} (A\nabla F(y))\cdot \nabla F(y)\delta(y)^{2-n}\omega_0(\Delta_{\delta(y)}(y^*))\, dy\, d\eta,$$

where

$$\Omega_\eta = \bigcup_{q\in E_r}\Gamma_\eta^{\tau r}(q), \qquad E_r = \Delta_r \cap E.$$

Now averaging over the top of Ω_η as well means that for $I(y) \geq 0$ each boundary integral can be replaced by

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \int_{\partial\Omega_\eta} I(y)\, dy\, d\eta \lesssim \int_{N(\partial\Omega_\eta)}\frac{1}{\delta(y)}I(y)\, dy \lesssim \int_{\Omega_\beta}\frac{1}{\delta(I)}I(y)\, dy,$$

where $N(\partial\Omega_\eta)$ is a solid neighborhood of $\partial\Omega_\eta^{\tau r}\subseteq \Omega_\beta^{\tau r}=\Omega_\beta$.

Obviously

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \int_{\Omega_\eta} I(y)\, dy\, d\eta \leq \int_{\Omega_\beta} I(y)\, dy.$$

So one needs to estimate the following five integrals:

(1)
$$\int_{\Omega_\beta}\frac{1}{\delta(y)}|F(y)|\,|G_0(0;y)|\,|\nabla F(y)|,$$

(2)
$$\int_{\Omega_\beta}\frac{1}{\delta(y)}|F(y)|^2|\nabla G_0(0;y)|,$$

(3)
$$\int_{\Omega_\beta}\frac{1}{\delta(y)}|F(y)|\,|G_0(0;y)|\,|[\varepsilon_{ij}]\nabla u_1(y)|,$$

(4)
$$\int_{\Omega_\beta}|G_0(0;y)|\,|\nabla F(y)|\varepsilon(y)|\nabla u_1(y)|,$$

(5)
$$\int_{\Omega_\beta}|F(y)|\,|\nabla G_0(0;y)|\varepsilon(y)|\nabla u_1(y)|.$$

For (1),

$$\int_{\Omega_\beta}\frac{1}{\delta(y)}|F(y)|\,|G_0(0;y)|\,|\nabla F(y)|\, dy = \sum_{I\in\mathcal{D}(\Delta_r)}\int_{I^+\cap\Omega_\beta}\frac{1}{\delta(y)}|F|\,|G_0|\,|\nabla F|$$

$$\lesssim \sum_{I\in\mathcal{D}(\Delta_r)}\frac{1}{\ell(I)}\frac{\omega_0(I)}{\ell(I)^{n-2}}\left(\int_{I^+\cap\Omega_\beta}|F(y)|^2\, dy\right)^{1/2}\left(\int_{I^+\cap\Omega_\beta}|\delta\nabla F|^2\right)^{1/2}\ell(I)^{n-1}$$

$$\lesssim \sum_{I \in \mathcal{D}(\Delta_r)} \frac{\ell(I)}{\ell(I)} \, \omega_0(I) \varepsilon(\ell(I)) \cdot M_{\omega_0} S_\beta u_1(q_{E \cap I}) \cdot \widetilde{N}_\beta(\delta \nabla F)(q_{E \cap I})$$

$$\lesssim \gamma^2 \lambda^2 \sum_k \sum_{\substack{\ell(I)=2^{-k}r \\ I \in \mathcal{D}(\Delta_r)}} \omega_0(I) \cdot \varepsilon(\ell(I)) \lesssim \gamma^2 \lambda^2 \omega_0(\Delta_r)$$

since diam $\Omega_\beta \sim r$ and by hypothesis

$$\sum_{\substack{k=0 \\ \ell(I)=2^{-k}r}}^{\infty} \varepsilon(\ell(I)) \le C_0.$$

Here $q_{E \cap I}$ can be any point in $E \cap I$ for $\beta' > \beta$ sufficiently large, and $\mathcal{D}(\Delta_r)$ is the collection of all dyadic subcubes of Δ_r.

The integral (2) can also be estimated:

$$\int_{\Omega_\beta} \frac{1}{\delta(y)} F(y)^2 |\nabla G_0(0;y)| \, dy = \sum_{I \in \mathcal{D}(\Delta_r)} \int_{I^+ \cap \Omega_\beta} \frac{1}{\delta} F^2 |\nabla G_0|$$

$$\lesssim \sum_{I \in \mathcal{D}(\Delta_r)} \frac{1}{\ell(I)} F^*(q_{E \cap I}) \frac{\omega_0(I)}{\ell(I)^{n-1}} \Big(\fint_{I^+ \cap \Omega_\beta} |F|^2 \Big)^{1/2} \cdot \ell(I)^{\frac{n}{2} \cdot 2}$$

$$\lesssim \sum_{I \subseteq \mathcal{D}(\Delta_r)} \omega_0(I) F^*_{\beta'}(q_{E \cap I}) \varepsilon(\ell(I)) M_{\omega_0} S_{\beta'}(q_{E \cap I})$$

$$\lesssim \gamma^2 \lambda^2 \sum_j \sum_{\ell(I)=2^{-j}r} \varepsilon(\ell(I)) \omega_0(I) = C \gamma^2 \lambda^2 \omega_0(\Delta_r)$$

as above.

For (3),

$$\int_{\Omega_\beta} \frac{1}{\delta(y)} |F(y)| \, |G_0(0;y)| \, |\varepsilon(y)| \, |\nabla u_1(y)| \, dy$$

$$= \sum_{I \in \mathcal{D}(\Delta_r)} \int_{I^+ \cap \Omega_\beta} \frac{1}{\delta(y)} |F| \, |G_0| \, |\varepsilon| \, |\nabla u_1|$$

$$\lesssim \sum_{I \in \mathcal{D}(\Delta_r)} \varepsilon(\ell(I)) \frac{\omega_0(I)}{\ell(I)^{n-1}} \Big(\int_{I^+ \cap \Omega_\beta} |F(y)|^2 \, dy \Big)^{1/2} \Big(\int_{I^+ \cap \Omega_\beta} |\nabla u_1(y)|^2 \Big)^{1/2}$$

$$\lesssim \sum_k \sum_{\substack{\ell(I)=2^{-k}r \\ I \cap \Omega_\beta \ne \emptyset}} \varepsilon(2^{-k}r) \omega_0(I) \gamma^2 \lambda^2 \le C \gamma^2 \lambda^2 \omega_0(\Delta_r).$$

For (4),

$$\int\limits_{\Omega_\beta} |G_0(0;y)|\,|\nabla F(y)|\,\varepsilon(y)\,|\nabla u_1(y)|$$

$$\leq \sum_{I\in\mathcal{D}(\Delta_r)} \Big(\fint_{I^+\cap\Omega_\beta} |\delta\nabla F(y)|^2 \Big)^{1/2} \Big(\fint_{I^+\cap\Omega_\beta} |\delta\nabla u_1(y)|^2 \Big)^{1/2}$$

$$\cdot \varepsilon(\ell(I)) \sup_{y\in I^+} |G_0(0;y)| \cdot \ell(I)^{n-2}$$

$$\lesssim \sum_{I\in\mathcal{D}(\Delta_r)} \gamma^2\lambda^2\varepsilon(\ell(I))\omega_0(I) \leq C\gamma^2\lambda^2\omega_0(\Delta_r) \quad \text{as before.}$$

Finally, we estimate the fifth integral:

$$\int\limits_{\Omega_\beta} |F(y)|\,|\nabla G_0(0;y)|\varepsilon(y)|\nabla u_1(y)|$$

$$= \sum_{I\in\mathcal{D}(\Delta_r)} \int\limits_{I^+\cap\Omega_\beta} |F(y)|\,|\nabla G_0(0;y)|\,|\varepsilon(y)|\,|\nabla u_1(y)|$$

$$\lesssim \sum_{I\in\mathcal{D}(\Delta_r)} F_{\beta'}^*(q_{E\cap I}) \cdot \varepsilon(\ell(I)) \Big(\fint_{I^+\cap\Omega_\beta} |\delta\nabla G_0|^2 \Big)^{1/2} \Big(\fint_{I^+\cap\Omega_\beta} |\delta\nabla u_1|^2 \Big)^{1/2} \ell(I)^{n-2}$$

$$\lesssim \sum_{I\in\mathcal{D}(\Delta_r)} F_{\beta'}^*(q_{E\cap I}) \cdot S_{\beta'}u_1(q_{E\cap I}) \cdot \varepsilon(\ell(I)) \sup_{y\in I^+} |G_0(0;y)|\ell(I)^{n-2}$$

$$\lesssim \gamma^2\lambda^2 \sum_j \sum_{\ell(I)=2^{-j}r} \varepsilon(\ell(I))\omega_0(I) \lesssim C\gamma^2\lambda^2\omega_0(\Delta_r).$$

So $\lambda^2\omega_0(E_r) \leq C\gamma^2\lambda^2\omega_0(\Delta_r)$. Dividing by λ^2 and summing over the Whitney cubes Δ_r gives $\omega_0(E) \leq C\gamma^2\omega_0(SF > \lambda)$.

This is enough to prove

$$\int\limits_{\partial B_1} (SF(q))^p \, d\omega_0(q) \lesssim \int\limits_{\partial B_1} (\widetilde{N}(F)(q))^p \, d\omega_0(q)$$

$$+ \int\limits_{\partial B_1} (\widetilde{N}(\delta\nabla F)(q))^p \, d\omega_0(q) + \int\limits_{\partial B_1} f(q)^p \, d\omega_0(q)$$

using the maximal theorem, the estimates

$$F^*(q) \leq \widetilde{N}(F)(q) + u_0^*(q), \quad S_\beta u_1(q) \leq S_\beta F(q) + S_\beta u_0(q)$$

and standard arguments.

It can be noted here that the same kind of argument proves the good-λ inequality for parabolic functions also given the proper decay of the Carleson coefficient.

Appendix. We recall that the spherical cone is

$$\Gamma_\gamma^\delta(\theta_x) = \{(r,\theta) : \delta < r < 1, |\theta - \theta_x| < \gamma(1-r)\},$$

and the Euclidean cone is

$$\Gamma_\gamma^\delta(x) = \{y : |y - r^*\theta_x| < \gamma(1-r^*) \text{ where}$$
$$1 - r^* = \text{dist}(y, \mathbb{R}^{n-1}) \text{ and } \text{dist}(y, \text{center of } S^{n-1}) \geq \delta\}.$$

Here $\mathbb{R}^{n-1}$ is tangent to S^{n-1} at $\theta_x = x$.

We will show that for γ fixed there are apertures β, β' so that $\Gamma_{\beta'}^\delta(\theta_x) \subset \Gamma_\gamma^\delta(x)$ and $\Gamma_\gamma^\delta(x) \subset \Gamma_\beta^\delta(\theta_x)$.

CLAIM. $|r - r^*| < 1 - r \leq 1 - r^*$ for $y = r\theta$.

It is enough to show this in $\mathbb{R}^2$. Take $\theta_x = x = 0$ and S^1 centered at $(0,1)$.

Let $(x,y) = \vec{y}$ lie on a circle of radius $r = \varrho$ about $(0,1)$, where ϱ is large enough to have this circle intersect $\Gamma_\alpha^\delta(0)$. The line $y = \alpha x$ intersects the circle $x^2 + (y-1)^2 = \varrho^2$ at x such that $x^2 + (\alpha x - 1)^2 = \varrho^2$. Solving gives

$$x = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2 + 1)(1 - \varrho^2)}}{2(\alpha^2 + 1)}.$$

Choosing the point on the circle nearest 0 gives

$$\vec{y} = \left(\frac{\alpha - \sqrt{-1 + \varrho^2(\alpha^2 + 1)}}{(\alpha^2 + 1)}, \frac{\alpha^2 - \alpha\sqrt{-1 + \varrho^2(\alpha^2 + 1)}}{(\alpha^2 + 1)}\right) = (x, 1 - r^*)$$

so

$$|r - r^*| = \left|\varrho - \left(1 - \frac{\alpha^2 - \alpha\sqrt{-1 + p^2(\alpha^2 + 1)}}{(\alpha^2 + 1)}\right)\right|,$$

$$\frac{|r - r^*|}{1 - r} = \left|\frac{(\alpha^2 + 1)\varrho + \alpha^2 - \alpha\sqrt{-1 + \varrho^2(\alpha^2 + 1)} - (\alpha^2 + 1)}{(\alpha^2 + 1)(1 - \varrho)}\right|$$

$$= \left|\frac{(\varrho - 1)(\alpha^2 + 1) + \alpha^2 - \alpha\sqrt{\varrho^2(\alpha^2 + 1) - 1}}{(\alpha^2 + 1)(1 - \varrho)}\right|.$$

Using l'Hospital we obtain

$$\lim_{\varrho \uparrow 1} \frac{|r - r^*|}{1 - r} = \lim_{\varrho \to 1} \left|\frac{(\alpha^2 + 1) - \frac{\alpha\varrho(\alpha^2 + 1)}{\sqrt{\varrho^2(\alpha^2 + 1) - 1}}}{\alpha^2 + 1}\right| = 1 - \frac{\alpha}{\sqrt{\alpha^2 + 1} - 1} = 0.$$

So there is a height δ_0 so that $|r^* - r| < 1 - r$ if $r \geq \delta_0$.

The fact that $1 - r \leq 1 - r^*$ is obvious so if $y \in \Gamma_\gamma^\delta(\theta_x)$ then $|\theta_y - \theta_x| < \gamma(1-r)$, hence $|r\theta_y - r^*\theta_x| < r\gamma(1-r) + |r^*\theta_x - r\theta_x| = r\gamma(1-r) + |r^* - r||\theta_x|$ and so $|y - r^*\theta_x| < (\gamma + |\theta_x|)(1 - r^*)$, that is, $y \in \Gamma_{\beta'}^\delta(x)$ if $\beta' \geq 2\gamma + |\theta_x|$.

Conversely, if $y \in \Gamma^\delta_\beta(x)$ then $|y - r^*\theta_x| < \gamma(1 - r^*)$ and

$$|r\theta_y - r\theta_x| \le \gamma(1 - r^*) + |r\theta_x - r^*\theta_x|$$

so that

$$|\theta_y - \theta_x| \le \frac{\gamma}{r}\,(1 - r) + \frac{\gamma}{r}\,(r - r^*) + \frac{|\theta_x|}{r}\,|r - r^*|$$

and hence

$$|\theta_y - \theta_x| < \frac{2\gamma}{\delta_0}\,(2(1 - r)) + \frac{|\theta_x|}{\delta_0}\,(1 - r) \le \beta(1 - r)$$

if $\beta \ge (2\gamma + |\theta_x|)/\delta_0$.

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