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The Schroeder–Bernstein index for Banach spaces

by

Elói Medina Galego (São Paulo)

Abstract. In relation to some Banach spaces recently constructed by W. T. Gowers and B. Maurey, we introduce the notion of Schroeder–Bernstein index $\mathrm{SBi}(X)$ for every Banach space X. This index is related to complemented subspaces of X which contain some complemented copy of X. Then we establish the existence of a Banach space E which is not isomorphic to E^n for every $n \in \mathbb{N}$, $n \geq 2$, but has a complemented subspace isomorphic to E^2. In particular, $\mathrm{SBi}(E)$ is uncountable. The construction of E follows the approach given in 1996 by W. T. Gowers to obtain the first solution to the Schroeder–Bernstein Problem for Banach spaces.

1. Introduction. Let X and Y be Banach spaces. We write $X \overset{c}{\hookrightarrow} Y$ if X is isomorphic to a complemented subspace of Y, $X \sim Y$ if X is isomorphic to Y, and $X \not\sim Y$ when X is not isomorphic to Y. If $n \in \mathbb{N} = \{1, 2, \ldots\}$, then X^n denotes the sum of n copies of X. The first infinite cardinal number will be denoted by $\aleph_0$ and the first uncountable cardinal by $\aleph_1$.

In 1996 W. T. Gowers [6] presented the first solution to the Schroeder–Bernstein Problem for Banach spaces, that is, he constructed Banach spaces X_1 and X_2 such that

$$X_1 \overset{c}{\hookrightarrow} X_2, \quad X_2 \overset{c}{\hookrightarrow} X_1, \quad X_1 \not\sim X_2. \tag{1.1}$$

Afterwards, in 1997, for each $p \in \mathbb{N}$, $p \geq 2$, W. T. Gowers and B. Maurey [8, p. 563] defined a finite sequence of Banach spaces $X_1, \ldots, X_p$ such that for all $m, n \leq p$, we have

$$X_m \overset{c}{\hookrightarrow} X_n, \quad X_m \not\sim X_n \quad \text{for } m \neq n. \tag{1.2}$$

Such sequences show that the structure of the complemented subspaces of a Banach space X which contain some complemented copy of X may be complicated. To examine this structure more closely it is natural to introduce the following definition:

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Definition 1.1. Let κ be a cardinal number, $\kappa \geq 2$. We say that a Banach space X has the *κ-Schroeder–Bernstein Property* (κ-SBP) if for each family $(X_\xi)_{\xi\in\kappa}$ of Banach spaces of cardinality κ which satisfies

(a) $X \sim X_1$;
(b) $X_\xi \overset{c}{\hookrightarrow} X_\gamma$, $\forall \xi, \gamma \in \kappa$;
(c) $X_\xi \not\sim X_\gamma$, $\forall \xi, \gamma \in \kappa$ with $\xi \neq 1$, $\gamma \neq 1$ and $\xi \neq \gamma$,

we have $X_1 \sim X_\xi$ for some $\xi \in \kappa$, $\xi \neq 1$.

The *Schroeder–Bernstein index* $\mathrm{SBi}(X)$ of X is defined by

$$\mathrm{SBi}(X) = \inf\{\kappa : X \text{ has the } \kappa\text{-SBP}\}.$$

Observe that this index is clearly well defined, and $\mathrm{SBi}(X) = 2$ if and only if X has the Schroeder–Bernstein Property (see [1]).

It follows directly from the definition that the Banach space X_1 in (1.1) satisfies $\mathrm{SBi}(X_1) > 2$; see also [2]–[4] for more examples of such spaces. Furthermore, the space X_1 in (1.2) satisfies $\mathrm{SBi}(X_1) > p$.

The aim of this paper is to provide a Banach space E with $\mathrm{SBi}(E) > \aleph_0$. In fact, we will show how to construct a Banach space E with the properties announced in the abstract. Then Lemma 3.5 implies that $\mathrm{SBi}(E)$ is uncountable, because defining $X_n = E^n$ for $n \in \mathbb{N}$, we have

$$X_m \overset{c}{\hookrightarrow} X_n, \quad X_m \not\sim X_n \quad \text{for } m \neq n.$$

The construction of E is inspired by the papers [5] and [6]. In the first one, Gowers defined Banach spaces X_1 and X_2 such that $X_1 \overset{c}{\hookrightarrow} X_2$ and $X_1 \overset{c}{\hookrightarrow} X_2$, but $X_1^m \not\sim X_2^n$, for all $m, n \in \mathbb{N}$ (see [3]). In the second, he exhibited the first Banach space X satisfying $X \sim X^3$ and $X \not\sim X^2$.

2. Preliminaries. As in [5] (see also [3]), we begin by fixing two totally incomparable Banach spaces X and Y from the class of sequence spaces constructed in [7]. We recall that the *support* of a vector $x = (x_n)_{n\in\mathbb{N}}$ in a sequence space, written $\mathrm{supp}(x)$, is $\{n \in \mathbb{N} : x_n \neq 0\}$. We will write $x < y$ to mean $i < j$ for every $i \in \mathrm{supp}(x)$ and $j \in \mathrm{supp}(y)$. If $x_1 < \cdots < x_m$, we will say that the vectors $x_1, \ldots, x_m$ are *successive*.

We know that X and Y contain normalized sequences $x_1 < x_2 < \cdots$ and $y_1 < y_2 < \cdots$ respectively such that if $T : X \to X$ and $L : Y \to Y$ are any bounded linear operators then there exist λ and μ such that $T(x_n) - \lambda x_n \to 0$ and $L(y_n) - \mu y_n \to 0$ as $n \to \infty$ (see [7, Lemma 22]). Moreover, we can prove in a way similar to the proof of [7, Lemma 23] that if $U : X \to Y$ and $V : Y \to X$ are any bounded linear operators, then $U(x_n) \to 0$ and $V(x_n) \to 0$ as $n \to \infty$.

Let $(x_n^*)_{n\in\mathbb{N}}$ and $(y_n^*)_{n\in\mathbb{N}}$ be sequences of support functionals for $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ respectively.

We also fix two sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of isometric copies of X and Y respectively. Let $Z_n = (X_n \oplus Y_n)_\infty$ be the sum of the Banach spaces X_n and Y_n, with the supremum norm. We denote by $\overline{P}_n$ and $\overline{Q}_n$ the canonical projections of Z_n onto X_n and Y_n respectively.

Let V denote the vector space of all sequences $v = (z_1, z_2, z_3, \ldots)$ such that $z_n \in Z_n$ for all $n \in \mathbb{N}$ and $z_n \neq 0$ for only finitely many n. We identify each Z_n with the subspace of V given by $\{(0, \ldots, z_n, 0, \ldots) : z_n \in Z_n\}$.

By x_{ij} and y_{ij}, for $i, j \in \mathbb{N}$, we denote the elements in V given respectively by

$$z_l = \begin{cases} (x_i, 0) & \text{if } l = j, \\ 0 & \text{if } l \neq j; \end{cases} \qquad z_l = \begin{cases} (0, y_i) & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases}$$

The functionals x^*_{ij}, y^*_{ij} and w^*_{ij} are defined by

$$x^*_{ij}(v) = x^*_i(\overline{P}_j(z_j)), \quad y^*_{ij}(v) = y^*_i(\overline{Q}_j(z_j)),$$
$$w^*_{ij}(v) = x^*_i(\overline{P}_j(z_j)) - y^*_i(\overline{Q}_j(z_j)),$$

for $v \in V$, $v = (z_1, z_2, z_3, \ldots)$, $i, j \in \mathbb{N}$.

Let us fix three disjoint infinite subsets $\mathbb{N}_1, \mathbb{N}_2$ and $\mathbb{N}_3$ of $\mathbb{N}$. Now we introduce a convenient norm in V. If $v \in V$, $v = (z_1, z_2, z_3, \ldots)$, then $\|v\|$ is the maximum of the following three numbers:

$$\begin{cases} \sup\{\|z_i\| : i \in \mathbb{N}\}; \\ \sup\Big\{\Big(\sum_{j=1}^{\infty}(|x^*_{ij}(v)|^4 + |y^*_{ij}(v)|^4)\Big)^{1/4} : i \in \mathbb{N}_2\Big\}; \\ \sup\Big\{\Big(\sum_{j=1}^{\infty}|w^*_{ij}(v)|^2\Big)^{1/2} : i \in \mathbb{N}_3\Big\}. \end{cases}$$

Let Z be the completion of $(V, \|\ldots\|)$.

Note that Z_n is closed in Z, because $\|z_n\| \leq \|(0, \ldots, z_n, 0, \ldots)\| \leq 2^{1/2}\|z_n\|$ for all $z_n \in Z_n$. Moreover, Z is equal to the closed subspace generated by $(Z_n)_{n\in\mathbb{N}}$ in Z, and $\|\sum_{i=1}^{n} z_i\| \leq \|\sum_{i=1}^{n+m} z_i\|$ for every $z_i \in Z_i$ and $m, n \in \mathbb{N}$. Therefore [10, Theorem 15.5] implies that $(Z_n)_{n\in\mathbb{N}}$ is a Schauder decomposition of Z. We will write

$$Z = Z_1 \oplus Z_2 \oplus Z_3 \oplus \cdots.$$

It is easy to see that this sum is symmetric, that is, the norm of an element $(z_n)_{n\in\mathbb{N}}$ of this sum is not affected by changing the order of the terms or by multiplying some of them by -1. Thus

$$Z \sim Z_1 \oplus Z_3 \oplus Z_5 \oplus \cdots, \qquad Z \sim Z_2 \oplus Z_4 \oplus Z_6 \oplus \cdots$$

and

$$Z \sim (Z_1 \oplus Z_3 \oplus Z_5 \oplus \cdots) \oplus (Z_2 \oplus Z_4 \oplus Z_6 \oplus \cdots).$$

Consequently,

$$Z \sim Z^2. \tag{2.1}$$

For every $n \in \mathbb{N}$, we define

$$W_n = Z_{n+1} \oplus Z_{n+2} \oplus Z_{n+3} \oplus \cdots.$$

Let S denote the shift operator defined by $S(v) = (0, z_1, z_2, \ldots)$ for $v \in Z$, $v = (z_1, z_2, z_3, \ldots)$. It is clear from the definition of Z that S^n is an isometry from Z onto W_n for every $n \in \mathbb{N}$. So, for every $n \in \mathbb{N}$, we have

$$Z \sim W_n. \tag{2.2}$$

Now we put $E = X \oplus Z$. Hence E contains a complemented subspace isomorphic to its square E^2. Indeed, by (2.1) and (2.2) we get

$$E^2 \sim X^2 \oplus Z^2 \sim X^2 \oplus W_2 \sim X_1 \oplus X_2 \oplus W_2 \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} E.$$

To obtain more information about the finite sums of E, E^p, for every $p \in \mathbb{N}$, $p \geq 2$, it will be useful to define the Banach spaces

$$G_p = X_1 \oplus X_2 \oplus \cdots \oplus X_{p-1} \oplus W_{p-1}.$$

We observe that $E = X \oplus Z \sim X_1 \oplus W_1 = G_2$ and again by (2.1) and (2.2),

$$E^p \sim X^p \oplus Z^p \sim X^p \oplus W_p \sim G_{p+1}.$$

Hence to prove that $E \not\sim E^p$ for every $p \in \mathbb{N}$, $p \geq 2$, it suffices to show that

$$G_2 \not\sim G_{p+1}. \tag{2.3}$$

We will verify (2.3) by proving that $Y \oplus G_2 \not\sim Y \oplus G_{p+1}$, that is,

$$Z \not\sim G_p \tag{2.4}$$

for every $p \in \mathbb{N}$, $p \geq 2$ (see Theorem 3.4).

For this purpose we need some more definitions. For any $j \in \mathbb{N}$ we let P_j and Q_j be the canonical projections of Z onto X_j and Y_j respectively.

Let T be a bounded linear operator from Z to G_p. For any $i, j \in \mathbb{N}$ let $T_{ij} : X_i \to X_j$ be the restriction of $P_j T$ to X_i. Similarly, let $U_{ij} : X_i \to Y_j$, $L_{ij} : Y_i \to Y_j$ and $V_{ij} : Y_i \to X_j$ be the restrictions of $Q_j T$ to X_i, $Q_j T$ to Y_i and $P_j T$ to Y_i respectively.

As we have already said, we can associate scalars λ_{ji} with each T_{ij} and μ_{ji} with each L_{ij}, so that

$$\begin{aligned} T_{ij}(x_{ni}) - \lambda_{ji} x_{nj} &\to 0, & L_{ij}(y_{ni}) - \mu_{ji} y_{nj} &\to 0, \\ U_{ij}(x_{ni}) &\to 0, & V_{ij}(y_{ni}) &\to 0, \end{aligned}$$

as $n \to \infty$, for all $i, j \in \mathbb{N}$. In particular, we have defined two infinite matrices

$$\Lambda = (\lambda_{ij})_{i,j=1}^{\infty} \quad \text{and} \quad M = (\mu_{ij})_{i,j=1}^{\infty}.$$

$\mathcal{S}$ stands for the space of all sequences of scalars. Let $a = (a_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$, $x \in X$ and $y \in Y$. We define

$$a \ominus x = (a_1x, 0, a_2x, 0, a_3x, 0, \ldots), \qquad a \ominus y = (0, a_1y, 0, a_2y, 0, a_3y, 0, \ldots).$$

Let c_{00} be the space of sequences of scalars all but finitely many of which are zero. Set $\Lambda a = (\sum_{i=1}^{\infty} \lambda_{ni}a_i)_{n\in\mathbb{N}}$ for $a = (a_n)_{n\in\mathbb{N}}$ in c_{00}.

Denote by $\mathcal{Z}$ the vector space of sequences $u = (z_1, z_2, z_3, \ldots)$ such that $z_j \in Z_j$ for all $j \in \mathbb{N}$, and define $\mathcal{P}_j(z_1, z_2, z_3, \ldots) = \overline{P}_j(z_j)$ and $\mathcal{Q}_j(z_1, z_2, z_3, \ldots) = \overline{Q}_j(z_j)$, for $(z_1, z_2, z_3, \ldots) \in \mathcal{Z}$ and $j \in \mathbb{N}$.

Finally, we will consider Z as a vector subspace of $\mathcal{Z}$. In particular, $P_j(u) = \mathcal{P}_j(u)$ and $Q_j(u) = \mathcal{Q}_j(u)$ for all $u \in Z$ and $j \in \mathbb{N}$.

3. The results. We are now prepared to obtain results similar to those in [5] for the spaces Z and G_p.

LEMMA 3.1. *Let T be a bounded linear operator from Z to G_p, $p \geq 2$, and $a \in c_{00}$. Then for all $j \in \mathbb{N}$, as $n \to \infty$,*

(a) $P_j(T(a \ominus x_n)) - \mathcal{P}_j(\Lambda a \ominus x_n) \to 0$.
(b) $Q_j(T(a \ominus x_n)) \to 0$.
(c) $Q_j(T(a \ominus y_n)) - \mathcal{Q}_j(\Lambda a \ominus y_n) \to 0$.
(d) $P_j(T(a \ominus x_n)) \to 0$.

Proof. By symmetry it suffices to prove (a) and (b).

(a) We have

$$P_j(T(a \ominus x_n)) = P_j\Big(\sum_{i=1}^{\infty} a_i T(x_{ni})\Big) = \sum_{i=1}^{\infty} a_i T_{ij}(x_{ni}).$$

So by the definition of λ_{ji},

$$P_j(T(a \ominus x_n)) - \Big(\sum_{i=1}^{\infty} a_i\lambda_{ji}\Big)x_{nj} \to 0 \quad \text{as } n \to \infty.$$

But the second term in the above expression is $\mathcal{P}_j(\Lambda a \ominus x_n)$.

(b) It suffices to note that $Q_j(T(a \ominus x_n)) = \sum_{i=1}^{\infty} a_i U_{ij}(x_{ni})$. ∎

LEMMA 3.2. *If $(v_i)_{i=1}^{q}$ are successive elements of Z, then*

$$\Big\|\sum_{i=1}^{q} v_i\Big\| \leq \Big(\sum_{i=1}^{q} \|v_i\|^2\Big)^{1/2}.$$

Proof. This follows from the definition of the norm in Z. ∎

In order to prove the next lemma, we recall that c_0, ℓ_∞, ℓ_2 and ℓ_4 denote the classical Banach sequence spaces, and $\{e_n : n \in \mathbb{N}\}$ are their unit vectors.

Lemma 3.3. *If T is an isomorphism from Z onto G_p, $p \geq 2$, then Λ is the matrix of an isomorphism from c_0 onto c_0, while M is the matrix of an isomorphism between c_0 and its subspace $c_0^{(p-1)}$ generated by $\{e_n : n \geq p\}$.*

Proof. Let $C \in \mathbb{R}$ be such that $\max\{\|T\|, \|T^{-1}\|\} \leq C$. First we show that Λ represents a bounded linear operator from c_0 to ℓ_∞. If this were not the case, then there would exist $a \in c_{00}$ such that $\|a\|_\infty = 1$ and $\|\Lambda a\|_\infty \geq 4C$. Set $b = \Lambda a$ and let $j \in \mathbb{N}$ be such that $|b_j| \geq 3C$. By Lemma 3.1, $P_j(T(a \ominus x_n)) - \mathcal{P}_j(b \ominus x_n) \to 0$ as $n \to \infty$. Since $\mathcal{P}_j(b \ominus x_n) = b_j x_n$ we can pick $n \in \mathbb{N}_1$ such that $\|P_j T(a \ominus x_n)\| \geq 2C$, therefore $\|T(a \ominus x_n)\| \geq 2C$. However, since $n \in \mathbb{N}_1$, we have $\|a \ominus x_n\| = \sup\{\|a_i x_n\| : i \in \mathbb{N}\} = \|a\|_\infty = 1$, which is a contradiction.

A similar argument shows that Λ is the matrix of a bounded linear operator from ℓ_4 to ℓ_4. Indeed, otherwise we could find $a \in c_{00}$ such that $\|a\|_4 = 1$, $\|b\|_4 \geq 4C$, where $b = \Lambda a$. Again, Lemma 3.1 implies that $P_j(T(a \ominus x_n)) - \mathcal{P}_j(b \ominus x_n) \to 0$ as $n \to \infty$. Choose $q \in \mathbb{N}$ such that $(\sum_{l=1}^{q} |b_l|^4)^{1/4} \geq 3C$.

Now, by Lemma 3.1, we can choose $n \in \mathbb{N}_2$ such that

$$|x_n^*(P_j(T(a \ominus x_n)) - b_j x_n)| \leq C/2^j \ , \quad \forall j \leq q.$$

By the definition of the norm in G_p we have

$$\|T(a \ominus x_n)\| \geq \Big(\sum_{j=1}^{q} |x_{nj}^*(T(a \ominus x_n))|^4\Big)^{1/4} \geq \Big(\sum_{j=1}^{q} |b_j|^4\Big)^{1/4} - C \geq 2C.$$

Since $n \in \mathbb{N}_2$, we have $\|a \ominus x_n\| \leq \|a\|_4 = 1$, again giving a contradicition.

This shows in particular that the image of any element in c_0 under Λ is in c_0. Hence Λ represents a bounded linear operator from c_0 to c_0.

Note that the preceding arguments also show that $\Lambda(T^{-1})$ (the matrix obtained from T^{-1}) represents a bounded linear operator from c_0 to c_0.

We will complete the proof by showing that both $\Lambda(T^{-1})\Lambda(T)$ and $\Lambda(T)\Lambda(T^{-1})$ are the identity on c_0.

Given any $n \in \mathbb{N}_1$ and $a \in c_0$, $\|a\| = 1$, it follows from the definition of the norm in Z that $\Lambda a \ominus x_n \in G_p$. So, by Lemma 3.1 and the continuity of T and Λ, we can write

$$T(a \ominus x_n) = \Lambda a \ominus x_n + v_n \tag{3.1}$$

with

$$v_n \in G_p, \quad P_j(v_n) \to 0 \quad \text{as } n \to \infty,\ \forall j \in \mathbb{N}. \tag{3.2}$$

Setting $\Lambda' = \Lambda(T^{-1})$, again by the analogue of Lemma 3.1 for T^{-1}, we can write

$$T^{-1}(\Lambda a \ominus x_n) = \Lambda'(\Lambda a) \ominus x_n + u_n \tag{3.3}$$

with

$$u_n \in Z, \quad P_j(u_n) \to 0 \quad \text{as } n \to \infty,\ \forall j \in \mathbb{N}. \tag{3.4}$$

Therefore, by using (3.1) and (3.3), we deduce that

$$a \ominus x_n = \Lambda'(\Lambda a) \ominus x_n + u_n + T^{-1}(v_n). \tag{3.5}$$

The bounds of $\|\Lambda\|_\infty$, $\|\Lambda'\|_\infty$ and $\|T\|$ yield $\|v_n\| \le 5C$ and $\|u_n\| \le 30C^2$.

Suppose that there exists $a \in c_0$ with $\|a\|_\infty = 1$ such that $\Lambda'(\Lambda a) = c \neq a$, so $(a-c) \ominus x_n = u_n + T^{-1}(v_n)$. By using (3.2) and (3.4) we can find an infinite subset $\mathcal{N}$ of $\mathbb{N}_1$, a sequence $(u'_n)_{n\in\mathbb{N}}$ of successive elements in Z and a sequence $(v'_n)_{n\in\mathbb{N}}$ of successive elements in G_p such that $\|u'_n - u_n\| \le 100^{-n}$ and $\|v'_n - v_n\| \le 100^{-n}$ for all $n \in \mathcal{N}$.

Now, we pick $n_1, \ldots, n_t$ from $\mathcal{N}$. Then by (3.5),

$$\sum_{i=1}^{t} (a-c) \ominus x_{n_i} = \sum_{i=1}^{t} u'_{n_i} + T^{-1}\Big(\sum_{i=1}^{t} v'_{n_i}\Big) + u(t), \tag{3.6}$$

where

$$u(t) \in G_p, \quad \|u(t)\| \le (1+C) \sum_{i=1}^{t} 100^{-n_i}. \tag{3.7}$$

But for a suitable $j \in \mathbb{N}$ we have

$$\Big\|\sum_{i=1}^{t} (a-c) \ominus x_{n_i}\Big\| = \|(a-c)\|_\infty \Big\|\sum_{i=1}^{t} x_{n_i j}\Big\|. \tag{3.8}$$

However, since the vectors $x_{n_1}, \ldots, x_{n_t}$ are successive and normalized in X, it follows from the definition of the norm in X [7, p. 863] that

$$\Big\|\sum_{i=1}^{t} x_{n_i}\Big\| \ge \frac{t}{\log_2(t+1)}. \tag{3.9}$$

On the other hand, by Lemma 3.2,

$$\Big\|\sum_{i=1}^{t} u'_{n_i}\Big\| \le \Big(\sum_{i=1}^{t} \|u'_{n_i}\|^2\Big)^{1/2} \le 30t^{1/2}C^2 + \Big(\sum_{i=1}^{t} 100^{-2n_i}\Big)^{1/2}. \tag{3.10}$$

We also have, by Lemma 3.2,

$$\begin{aligned} \Big\|T^{-1}\Big(\sum_{i=1}^{t} v'_{n_i}\Big)\Big\| &\le C\Big(\sum_{i=1}^{t} \|v'_{n_i}\|^2\Big)^{1/2} \\ &\le 5t^{1/2}\, C^2 + C\Big(\sum_{i=1}^{t} 100^{-2n_i}\Big)^{1/2}. \end{aligned} \tag{3.11}$$

Hence, by (3.7), (3.10) and (3.11), the right hand side of (3.6) has norm at most $35t^{1/2}C^2 + (1+C)(\sum_{i=1}^{t} 100^{-n_i} + (\sum_{i=1}^{t} 100^{-2n_i})^{1/2})$. This contradicts (3.9) and (3.6) when t is large enough.

A similar proof shows that $\Lambda\Lambda'$ is the identity on c_0, and finally a similar argument to the whole of this proof shows that M is an isomorphism from c_0 onto $c_0^{(p-1)}$. ∎

THEOREM 3.4. *Z is isomorphic to G_p for no $p \in \mathbb{N}$, $p \geq 2$.*

Proof. Suppose that T is an isomorphism from Z onto G_p. Then Lemma 3.3 implies that $\Lambda : c_0 \to c_0$ is Fredholm with index 0, and M is Fredholm with index $p-1$. We know that $\Lambda - M$ cannot be strictly singular (see [9, Proposition 2.c.10]). In particular, we can find $\delta > 0$ and a sequence $a_1 < a_2 < \cdots$ of elements in c_{00} of norm one such that $\|(\Lambda - M)(a_n)\| \geq \delta$ for every $n \in \mathbb{N}$.

Let $C \in \mathbb{R}$ be such that $\max\{\|T\|, \|T^{-1}\|\} \leq C$ and choose $q > (4C/\delta)^2$, $q \in \mathbb{N}$.

Since $(\Lambda a_n)_i \to 0$ and $(Ma_n)_i \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$, we can obtain integers $0 = n_0 < n_1 < \cdots < n_q$ and q elements of $\{a_n : n \in \mathbb{N}\}$ such that after relabelling these elements as $a_1, \ldots, a_q$, we have

$$\sup_{i \notin [n_{l-1}+1, n_l]} |(\Lambda(a_l))_i| < \frac{\delta}{8q}, \qquad l = 1, \ldots, q, \tag{3.12}$$

and

$$\sup_{i \notin [n_{l-1}+1, n_l]} |(M(a_l))_i| < \frac{\delta}{8q}, \qquad l = 1, \ldots, q. \tag{3.13}$$

Let j_l with $n_{l-1} + 1 \leq j_l < n_l$ be such that

$$|((\Lambda - M)(a_l))_{j_l}| \geq \delta, \qquad l = 1, \ldots, q. \tag{3.14}$$

According to Lemma 3.1, we know that

$$\Big\|T\Big(\sum_{n=1}^{q}(a_n \ominus x_k + a_n \ominus y_k)\Big)\Big\| = \Big\|\sum_{n=1}^{q}(\Lambda a_n \ominus x_k + Ma_n \ominus y_k) + u_k\Big\|, \tag{3.15}$$

where

$$u_k \in \mathcal{Z}, \quad \mathcal{P}_j(u_k) \to 0, \quad \mathcal{Q}_j(u_k) \to 0 \quad \text{as } k \to \infty,\ \forall j \in \mathbb{N}. \tag{3.16}$$

Fix m with $1 \leq m \leq q$ and $k \in \mathbb{N}$ sufficiently large such that $\|\mathcal{P}_{j_m}(u_k)\| < \delta/8$ and $\|\mathcal{Q}_{j_m}(u_k)\| < \delta/8$. Then

$$\begin{aligned}
&\Big|w^*_{kj_m}\Big(\sum_{n=1}^{q}(\Lambda a_n \ominus x_k + Ma_n \ominus y_k) + u_k\Big)\Big| \\
&\quad = |((\Lambda - M)(a_1))_{j_m} + \cdots + ((\Lambda - M)(a_m))_{j_m} + \cdots + ((\Lambda - M)(a_q))_{j_m} \\
&\qquad + x^*_k(\mathcal{P}_{j_m}(u_k)) - y^*_k(\mathcal{Q}_{j_m}(u_k))| \\
&\quad \geq \delta - 2(q-1)\,\frac{\delta}{8q} - \frac{\delta}{4} \geq \frac{\delta}{2}.
\end{aligned}$$

So, by the definition of the norm in Z, for sufficiently large $k \in \mathbb{N}_3$ we obtain

$$\Big\| \sum_{n=1}^{q} (\Lambda a_n \ominus x_k + M a_n \ominus y_k) + u_k \Big\| \geq \frac{\delta}{2} \sqrt{q}. \tag{3.17}$$

On the other hand the action of each w_{kj}^* is zero on every vector of the form $a \ominus x_k + a \ominus y_k$ with $a \in c_{00}$. Thus, again by the definition of the norm in Z we have $\| \sum_{n=1}^{q} (a_n \ominus x_k + a_n \ominus y_k) \| = 1$. Therefore by (3.15) and (3.17) we conclude that $\|T\| \geq \delta\sqrt{q}/2$. Since we have chosen q to be greater than $(4C/\delta)^2$, we have shown that $\|T\| > C$. This contradiction completes the proof. ■

LEMMA 3.5. *Let X be a Banach space such that $X^2 \overset{c}{\hookrightarrow} X$. If $X^p \sim X^q$ for some $p, q \in \mathbb{N}$ with $p < q$, then $X \sim X^{q-p+1}$.*

Proof. Since $X^2 \overset{c}{\hookrightarrow} X$, it follows that $X^p \overset{c}{\hookrightarrow} X$. Therefore there exists a Banach space B such that $X \sim X^p \oplus B$. Thus

$$X^{q-p+1} \sim X^{q-p} \oplus X \sim X^{q-p} \oplus X^p \oplus B \sim X^q \oplus B \sim X^p \oplus B \sim X. \ \blacksquare$$

4. Remarks and problems. The existence of the Banach space E constructed in this paper leads in the natural way to some problems related to the structure of complemented subspaces in Banach spaces. We now mention some of them.

PROBLEM 4.1. Does there exist a Banach space X with $\mathrm{SBi}(X) = \aleph_1$?

Let X be a Banach space with an unconditional basis. An unsolved problem is whether $\mathrm{SBi}(X) = 2$ (see [8]). In fact, we do not even know the answer to

PROBLEM 4.2. Let X be a Banach space with an unconditional basis. Is it true that $\mathrm{SBi}(X) < \aleph_0$?

Now we state some natural problems concerning the Schroeder–Bernstein index. In [2] a Banach space X with $2 = \mathrm{SBi}(X^*) < \mathrm{SBi}(X)$ was exhibited, where X^* is the dual space of X. Nevertheless Problems 4.3 and 4.4 below are still open even in the case where $\mathrm{SBi}(X) = 2$ and $\mathrm{SBi}(Y) = 2$ (see [1]).

PROBLEM 4.3. Is it true that $\mathrm{SBi}(X^*) \leq \mathrm{SBi}(X)$ for every Banach space X?

PROBLEM 4.4. Is it true that $\mathrm{SBi}(X \oplus Y) \leq \sup\{\mathrm{SBi}(X), \mathrm{SBi}(Y)\}$ for any Banach spaces X and Y?

Finally, let X represent the complex Banach space constructed by W. T. Gowers and B. Maurey in [8, Section 4.3]. Then X is isomorphic to its subspaces of even codimension while not being isomorphic to those of odd codimension [8, Theorem 19]. In particular, $X \oplus \mathbb{C} \overset{c}{\hookrightarrow} X$ and X

is not isomorphic to $X \oplus \mathbb{C}$. Hence $\mathrm{SBi}(X) \geq 3$. Furthermore, if Y is any infinite-dimensional complemented subspace of X, then either $Y \sim X$ or $Y \sim X \oplus \mathbb{C}$ (see remarks after [8, Theorem 19]), thus $\mathrm{SBi}(X) = 3$.

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Department of Mathematics
IME, University of São Paulo
São Paulo 05315-970 Brazil
E-mail: eloi@ime.usp.br

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