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Hardy spaces H^1 for Schrödinger operators with certain potentials

by

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Abstract. Let $\{K_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-L = \Delta - V$ with $V \geq 0$. We say that f belongs to H_L^1 if $\|\sup_{t>0} |K_t f(x)|\,\|_{L^1(dx)} < \infty$. We state conditions on V and K_t which allow us to give an atomic characterization of the space H_L^1.

1. Introduction. Let $Lf(x) = -\Delta f(x) + V(x)f(x)$ be a Schrödinger operator on $\mathbb{R}^d$, where $V \geq 0$, $V \not\equiv 0$. We shall assume that $-L$ generates a semigroup $\{K_t\}_{t>0}$ of linear contractions on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. This is guaranteed if e.g. $V \in L^q_{\mathrm{loc}}$ for some $q > d/2$.

We define the Hardy space H_L^1 related to the operator L by

$$H_L^1 = \Big\{ f \in L^1(\mathbb{R}^d) : \|f\|_{H_L^1} = \Big\| \sup_{t>0} |K_t f(x)| \Big\|_{L^1(dx)} < \infty \Big\}. \tag{1.1}$$

Let $\mathcal{Q} = \{Q_j\}_{j=1}^{\infty}$ be a collection of closed cubes with parallel sides whose interiors are disjoint such that $\mathbb{R}^d = \bigcup_{j=0}^{\infty} Q_j$. For a cube Q let $d(Q)$ denote its diameter. We shall always assume that there exist constants $C_0, \beta > 0$ such that for $Q_{j_1}, Q_{j_2} \in \mathcal{Q}$ if $Q_{j_1}^{****} \cap Q_{j_2}^{****} \neq \emptyset$, then $C_0^{-1} d(Q_{j_1}) \leq d(Q_{j_2}) \leq C_0 d(Q_{j_1})$, where Q^* is the cube with the same center as Q such that $d(Q^*) = (1+\beta)d(Q)$.

In order to state our results we need the following notion of the local atomic Hardy space $H_{\mathcal{Q}}^1$ associated with the collection $\mathcal{Q}$. We say that a function a is an $H_{\mathcal{Q}}^1$-*atom* if there is a cube $Q \in \mathcal{Q}$ such that a is a classical $(1,\infty)$-atom having support contained in Q^{**}, or $a(x) = |Q|^{-1}\mathbf{1}_Q(x)$, where, for a set $A \subset \mathbb{R}^d$, $\mathbf{1}_A$ denotes the indicator function of A. Then $H_{\mathcal{Q}}^1$ is defined

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$$H^1_{\mathcal{Q}} = \Big\{ f \in L^1 : \ f(x) = \sum_s \lambda_s a_s(x), \ \sum_s |\lambda_s| < \infty \Big\}, \tag{1.2}$$

where a_s are $H^1_{\mathcal{Q}}$-atoms. We set

$$\|f\|_{H^1_{\mathcal{Q}}} = \inf \sum |\lambda_s|, \tag{1.3}$$

where the infimum is taken over all possible representations of f as in (1.2).

In [DZ2] the authors proved that if V satisfies the reverse Hölder inequality with an exponent $q > d/2$, $d \geq 3$, that is,

$$\Big(\frac{1}{|B|} \int_B V(y)^q \, dy \Big)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy \quad \text{for every ball } B, \tag{1.4}$$

then the elements of H^1_L admit atomic decompositions of this type (cf. Section 8 of the present article).

The main goal of the present paper is to use ideas from [DZ1] and [DZ2] to see what the theory looks like when there are no reverse Hölder estimates for V. We formulate here two conditions on V, K_t, and $\mathcal{Q}$ that guarantee that H^1_L is local, that is, the norms $\|\cdot\|_{H^1_{\mathcal{Q}}}$ and $\|\cdot\|_{H^1_L}$ are equivalent (see Theorem 2.2). We shall verify that these conditions hold not only for V satisfying the reverse Hölder inequality but also for the following naturally occurring potentials:

$$V(x) = \mathbf{1}_{\mathbb{R}^d_+}(x), \quad \mathbb{R}^d_+ = \{(x_1, x_2, \ldots, x_d) : \ x_1 > 0\}, \ d \geq 1, \tag{1.5}$$

$$V(x) = \exp(x_1), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ d \geq 1, \tag{1.6}$$

$$V(x) = \gamma |x|^{-2}, \quad \gamma > 0, \ d \geq 3, \tag{1.7}$$

and properly defined families $\mathcal{Q}$ (cf. Theorems 2.4, 2.6, and 2.8). The potentials (1.5) and (1.6) do not satisfy the doubling condition, so they do not belong to any reverse Hölder class. Obviously for $q \geq d/2$ and $V(x) = \gamma|x|^{-2}$ the condition (1.4) does not hold.

For results concerning Hardy spaces related to Schrödinger operators with potentials from reverse Hölder classes we refer the reader to [DZ1]–[DZ4].

At the end of the paper for $0 < \alpha < 1$ and V being a nonnegative polynomial we consider the operator $(-\Delta)^\alpha + V$. We prove atomic decompositions of the elements of $H^1_{(-\Delta)^\alpha + V}$.

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2. Statements of the results. Denote by $P_t(x)$ the convolution kernels of the heat semigroup $\{P_t\}_{t>0}$ on $\mathbb{R}^d$ generated by Δ and by $K_t(x,y)$ the integral kernels of the semigroup $\{K_t\}_{t>0}$ generated by the Schrödinger operator $-L = \Delta - V$, $V \ge 0$. Obviously,

$$0 \le K_t(x,y) \le P_t(x-y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/4t). \tag{2.1}$$

For $V \ge 0$ and a collection $\mathcal{Q}$ of cubes as described above we consider the following two conditions:

(D) there exist constants $C, \varepsilon > 0$ such that

$$\sup_{y\in Q^*} \int K_{2^s d(Q)^2}(x,y)\, dx \le C s^{-1-\varepsilon} \qquad \text{for } Q \in \mathcal{Q},\ s \in \mathbb{N},$$

(K) there exist constants $C, \delta > 0$ such that

$$\int_0^{2t} (\mathbf{1}_{Q^{***}} V) * P_s(x)\, ds \le C\left(\frac{t}{d(Q)^2}\right)^{\delta} \qquad \text{for } x \in \mathbb{R}^d,\ Q \in \mathcal{Q},\ t \le d(Q)^2.$$

THEOREM 2.2. *Assume that for $V \ge 0$ and a collection $\mathcal{Q}$ of cubes conditions* (D) *and* (K) *hold. Then there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathcal{Q}}} \le \|f\|_{H^1_L} \le C\|f\|_{H^1_{\mathcal{Q}}}. \tag{2.3}$$

For $\ell > 0$ denote by $\widetilde{\mathcal{Q}}_\ell$ a partition of $\mathbb{R}^{d-1}$ into cubes whose sides have length ℓ.

The theorems below combined with Theorem 2.2 give atomic characterizations of the Hardy spaces related to Schrödinger operators with the potentials we are interested in.

THEOREM 2.4. *For the potential $V(x) = \mathbf{1}_{\mathbb{R}^d_+}(x)$ on $\mathbb{R}^d$, $d \ge 1$, the collection*

$$\begin{aligned} \mathcal{Q} = \{&[k,k+1] \times \widetilde{Q} : k = -1,0,1,2,\ldots,\ \widetilde{Q} \in \widetilde{\mathcal{Q}}_1\} \\ &\cup \{[-2^{k+1}, -2^k] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_{2^k},\ k = 0,1,2,\ldots\} \end{aligned} \tag{2.5}$$

satisfies (D) *and* (K).

THEOREM 2.6. *Let $V(x) = \exp(x_1)$ on $\mathbb{R}^d$, $d \ge 1$. Then the family of cubes*

$$\begin{aligned} \mathcal{Q} = \{&[-2^{j+1}, -2^j] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_{2^j},\ j = 0,1,2,\ldots\} \\ &\cup \{[-1,1] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_2\} \\ &\cup \{[r_j, r_{j+1}] \times \widetilde{Q}_j : r_1 = 1,\ r_{j+1} = r_j + \exp(-r_j/2), \\ &\qquad \widetilde{Q}_j \in \widetilde{\mathcal{Q}}_{\exp(-r_j/2)}\} \end{aligned} \tag{2.7}$$

satisfies (D) *and* (K).

THEOREM 2.8. *For* $V(x) = \gamma|x|^{-2}$ *on* $\mathbb{R}^d$, $d \geq 3$, $\gamma > 0$, *let* $\mathcal{Q}$ *be the Whitney decomposition of* $\mathbb{R}^d \setminus \{0\}$ *that consists of dyadic cubes. Then conditions* (D) *and* (K) *hold.*

3. Auxiliary lemmas. To prove the theorems stated in Section 2, we need a sequence of auxiliary results.

For $l > 0$ let $\mathbf{h}^1_l$ denote the local Hardy space (cf. [Go]) with the norm $\|f\|_{\mathbf{h}^1_l}$ defined by

$$\|f\|_{\mathbf{h}^1_l} = \Big\| \sup_{t \leq l^2} |P_t f(x)| \Big\|_{L^1(dx)}. \tag{3.1}$$

The following theorem is a consequence of results of Goldberg [Go].

THEOREM 3.2. *There exists a constant* $C > 0$ *such that for every* $l > 0$ *we have*

$$C^{-1}\|f\|_{H^1_{\mathcal{Q}_l}} \leq \|f\|_{\mathbf{h}^1_l} \leq C\|f\|_{H^1_{\mathcal{Q}_l}},$$

where $\mathcal{Q}_l$ *is a partition of* $\mathbb{R}^d$ *into cubes of side-length* l. *Moreover*, *if* $f \in \mathbf{h}^1_l$ *with* $\operatorname{supp} f \subset Q^*$ *for some* $Q \in \mathcal{Q}_l$, *then*

$$f = \sum \lambda_s a_s, \qquad \sum |\lambda_s| \leq C\|f\|_{\mathbf{h}^1_l},$$

with a_s *being* $H^1_{\mathcal{Q}_l}$*-atoms such that* $\operatorname{supp} a_s \subset Q^{**}$.

For a collection $\mathcal{Q}$ of cubes let $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a family of C^∞ functions on $\mathbb{R}^d$ such that $\operatorname{supp}\phi_Q \subset Q^*$, $0 \leq \phi_Q \leq 1$, $|\partial^\alpha \phi_Q| \leq C_\alpha d(Q)^{-|\alpha|}$, and $\sum_Q \phi_Q(x) = 1$ for all $x \in \mathbb{R}^d$.

LEMMA 3.3. *There exists a constant* $C > 0$ *such that for every* $Q \in \mathcal{Q}$ *we have*

$$\|\phi_Q f\|_{\mathbf{h}^1_{d(Q)}} \leq C\Big\| \sup_{t \leq d(Q)^2} |P_t(\phi_Q f)| \Big\|_{L^1(Q^{**})}. \tag{3.4}$$

Proof. There exist constants $C, c_1 > 0$ such that if $x \in (Q^{**})^{\mathrm{c}}$, $y \in Q^*$, and $t \leq d(Q)^2$, then $P_t(x-y) \leq Cd(Q)^{-d}\exp(-c_1|x-y_Q|^2/d(Q)^2)$, where y_Q denotes the center of Q. Hence

$$|P_t * (\phi_Q f)(x)| \leq Cd(Q)^{-d}\|\phi_Q f\|_{L^1}\exp(-c_1|x-y_Q|^2/d(Q)^2). \tag{3.5}$$

Now the lemma is a consequence of (3.5) and Theorem 3.2. ∎

For $Q \in \mathcal{Q}$ we set

$$\begin{aligned} \mathcal{Q}'(Q) &= \{Q' \in \mathcal{Q} : Q^{***} \cap (Q')^{***} \neq \emptyset\}, \\ \mathcal{Q}''(Q) &= \{Q'' \in \mathcal{Q} : Q^{***} \cap (Q'')^{***} = \emptyset\}. \end{aligned} \tag{3.6}$$

The lemma below is quite similar to those in our earlier papers (cf. [DZ1, Lemma 5.7], [DZ2, Lemma 3.11]).

LEMMA 3.7. *There exists a constant $C>0$ such that for every $Q\in\mathcal{Q}$ and every $f\in L^1(\mathbb{R}^d)$ we have*

$$\Big\|\sup_{t>0}\Big|K_t\Big(\phi_Q\sum_{Q'\in\mathcal{Q}'(Q)}\phi_{Q'}f\Big)-\phi_Q\Big(K_t\sum_{Q'\in\mathcal{Q}'(Q)}\phi_{Q'}f\Big)\Big|\Big\|_{L^1(Q^{**})}\leq C\sum_{Q'\in\mathcal{Q}'(Q)}\|\phi_{Q'}f\|_{L^1}.$$

Proof. Let $g=\sum_{Q'\in\mathcal{Q}'(Q)}\phi_{Q'}f$. Then

$$\sup_{t>0}|K_t(\phi_Q g)(x)-\phi_Q(x)K_tg(x)|=\sup_{t>0}\Big|\int(\phi_Q(y)-\phi_Q(x))K_t(x,y)g(y)\,dy\Big|$$
$$\leq C\sup_{t>0}\int\frac{|x-y|}{d(Q)}K_t(x,y)|g(y)|\,dy\leq\frac{C}{d(Q)}\int\frac{|g(y)|}{|x-y|^{d-1}}\,dy.$$

Integrating with respect to x over Q^{**} we obtain the lemma. ∎

LEMMA 3.8. *Assume that $\mathcal{Q}$ satisfies condition* (D). *Then there exists a constant $C>0$ such that*

$$\sum_{Q\in\mathcal{Q}}\Big\|\mathbf{1}_{Q^{***}}\sup_{t>0}\Big|K_t\Big(\sum_{Q''\in\mathcal{Q}''(Q)}\phi_{Q''}f\Big)\Big|\Big\|_{L^1}\leq C\|f\|_{L^1}.$$

Proof. Denote the left-hand side by S. Then

$$\begin{aligned}S&\leq\sum_{Q\in\mathcal{Q}}\sum_{Q''\in\mathcal{Q}''(Q)}\|\mathbf{1}_{Q^{***}}\sup_{t>0}(K_t|\phi_{Q''}f|)\|_{L^1}\\
&\leq\sum_{Q''\in\mathcal{Q}}\sum_{Q\in\mathcal{Q}''(Q'')}\|\mathbf{1}_{Q^{***}}\sup_{t>0}(K_t|\phi_{Q''}f|)\|_{L^1}\\
&\leq C\sum_{Q''\in\mathcal{Q}}\|\sup_{t>0}(K_t|\phi_{Q''}f|)\|_{L^1((Q'')^{**c})}\\
&\leq C\sum_{Q''\in\mathcal{Q}}\|\sup_{0<t\leq d(Q'')^2}(K_t|\phi_{Q''}f|)\|_{L^1((Q'')^{**c})}\\
&\quad+C\sum_{Q''\in\mathcal{Q}}\sum_{j=0}^{\infty}\|\sup_{2^jd(Q'')^2\leq t\leq 2^{j+1}d(Q'')^2}(K_t|\phi_{Q''}f|)\|_{L^1((Q'')^{**c})}.\end{aligned}$$

Note that for $s_j=2^jd(Q'')^2\leq t\leq 2^{j+1}d(Q'')^2=s_{j+1}$ we have

$$\begin{aligned}K_t(x,y)&=\int K_{t-2^{j-1}d(Q'')^2}(x,z)K_{2^{j-1}d(Q'')^2}(z,y)\,dz\\
&\leq\int P^{\max}_{s_j}(x,z)K_{2^{j-1}d(Q'')^2}(z,y)\,dz,\end{aligned}$$

where, by (2.1),

$$P^{\max}_{s_j}(x,z)=\sup_{s_j/2\leq s\leq 2s_j}P_s(x-z)\leq C_1s_j^{-d/2}\exp(-c_1|x-z|^2/s_j).\tag{3.9}$$

Applying (D), we obtain

$$S \le C \sum_{Q''\in\mathcal{Q}} \|\phi_{Q''} f\|_{L^1} + C \sum_{Q''\in\mathcal{Q}} \sum_{j=0}^{\infty} j^{-1-\varepsilon} \|\phi_{Q''} f\|_{L^1} \le C\|f\|_{L^1}. \ \blacksquare$$

Lemma 3.10.

$$\int_{\mathbb{R}^d} \int_0^\infty V(x)(K_s|f|)(x)\, ds\, dx \le \|f\|_{L^1}.$$

Proof. This result seems to be well known. We give a proof for completeness. The perturbation formula asserts that

$$P_t = K_t + \int_0^t P_{t-s} V K_s\, ds.$$

Therefore, by (2.1), if $f \ge 0$ then

$$\int_0^t P_{t-s} V K_s f(y)\, ds \le P_t f(y).$$

Integrating with respect to y and applying the Fubini theorem we get

$$\int_0^t \int V(x) K_s f(x)\, dx\, ds \le \|f\|_{L^1}.$$

Letting $t \to \infty$ we obtain the lemma. ∎

The following lemma is a generalization of Lemma 3.9 of [DZ2] (see also [DZ1, Lemma 5.1]).

Lemma 3.11. *Assume that $\mathcal{Q}$ satisfies* (K). *Then there exists a constant $C > 0$ such that*

$$\Big\| \sup_{0<t\le d(Q)^2} |(P_t - K_t)(\phi_Q f)| \Big\|_{L^1} \le C\|\phi_Q f\|_{L^1}. \tag{3.12}$$

Proof. By (2.1) and (3.5) it suffices to estimate the quantity $\big\|\sup_{0<t\le d(Q)^2} |(P_t - K_t)(\phi_Q f)|\big\|_{L^1(Q^{**})}$. The perturbation formula implies

$$\begin{aligned}(P_t - K_t)(\phi_Q f)(x) = \int_0^t & P_{t-s} V'' K_s(\phi_Q f)(x)\, ds \\ &+ \int_0^t P_{t-s}((\mathbf{1}_{Q^{***}} V) K_s(\phi_Q f))(x)\, ds,\end{aligned}$$

where $V = \mathbf{1}_{Q^{***}} V + V''$.

For $y \in (Q^{***})^{\mathrm{c}}$, $x \in Q^{**}$, and $0 < s < t \le d(Q)^2$, we have $P_{t-s}(x-y) \le Cd(Q)^{-d}\exp(-c|x-y|^2/d(Q)^2)$. Hence

$$\Big|\int_0^t P_{t-s}V''K_s(\phi_Q f)(x)\,ds\Big| = \Big|\iint_0^t P_{t-s}(x-y)V''(y)K_s(\phi_Q f)(y)\,ds\,dy\Big|$$
$$\le C\iint_0^t d(Q)^{-d}\exp\bigg(\frac{-c|x-y|^2}{d(Q)^2}\bigg)V''(y)K_s(|\phi_Q f|)(y)\,ds\,dy$$
$$\le Cd(Q)^{-d}\int\int_0^\infty V''(y)K_s(|\phi_Q f|)(y)\,ds\,dy \le Cd(Q)^{-d}\|\phi_Q f\|_{L^1}.$$

In the last inequality we have used Lemma 3.10. Thus

$$\Big\|\sup_{0<t\le d(Q)^2}\Big|\int_0^t P_{t-s}V''K_s(\phi_Q f)(x)\,ds\Big|\Big\|_{L^1(Q^{**})} \le C\|\phi_Q f\|_{L^1}.$$

We now turn to estimating the integral that contains $\mathbf{1}_{Q^{***}}V$:

$$\Big|\int_0^t P_{t-s}\mathbf{1}_{Q^{***}}VK_s(\phi_Q f)(x)\,ds\Big| \le \int_0^t P_{t-s}(\mathbf{1}_{Q^{***}}V)P_s(|\phi_Q f|)(x)\,ds$$
$$= \int_0^{t/2} + \int_{t/2}^t = I_t(x) + J_t(x).$$

For $t_j = 2^{-j}d(Q)^2 \le t \le 2^{-j+1}d(Q)^2 = 2t_j$ we have

$$I_j^*(x) = \sup_{t_j\le t\le 2t_j} I_t(x) \le \int_0^{2t_j}\int P_{t_j}^{\max}(x,z)V(z)\mathbf{1}_{Q^{***}}(z)P_s(|\phi_Q f|)(z)\,dz\,ds$$

(see (3.9)). Hence, applying (K) and (3.9), we conclude that

$$\|\sup_{0<t\le d(Q)^2} I_t(x)\|_{L^1} \le \sum_{j\ge 1}\|I_j^*\|_{L^1}$$
$$\le \sum_{j\ge 1}\iiint_0^{t_j} P_{t_j}^{\max}(x,z)\mathbf{1}_{Q^{***}}(z)V(z)P_s(|\phi_Q f|)(z)\,ds\,dz\,dx$$
$$\le C\sum_{j\ge 1}\iiint_0^{t_j}(\mathbf{1}_{Q^{***}}V)(z)P_s(z-y)(|\phi_Q(y)f(y)|)\,ds\,dy\,dz$$
$$\le C\sum_{j\ge 1}2^{-j\delta}\|\phi_Q f\|_{L^1}.$$

The L^1-norm of $J^*(x) = \sup_{0<t\le d(Q)^2} J_t(x)$ can be estimated in a similar way. ■

4. Proof of Theorem 2.2. We start by proving the first inequality in (2.3). From Lemmas 3.3 and 3.11 we deduce that

$$\begin{aligned} \sum_{Q\in\mathcal{Q}} \|\phi_Q f(x)\|_{\mathbf{h}^1_{d(Q)}} &\le C\sum_{Q\in\mathcal{Q}} \Big\| \mathbf{1}_{Q^{**}} \sup_{t\le d(Q)^2} |P_t\phi_Q f|\Big\|_{L^1} \\ &\le C\sum_{Q\in\mathcal{Q}} \Big\| \mathbf{1}_{Q^{**}} \sup_{t\le d(Q)^2} |(P_t-K_t)(\phi_Q f)|\Big\|_{L^1} \\ &\quad + C\sum_{Q\in\mathcal{Q}} \Big\| \mathbf{1}_{Q^{**}} \sup_{t\le d(Q)^2} |K_t(\phi_Q f)|\Big\|_{L^1} \\ &\le C\|f\|_{L^1} + C\sum_{Q\in\mathcal{Q}} \Big\| \mathbf{1}_{Q^{**}} \sup_{t\le d(Q)^2} |K_t(\phi_Q f)|\Big\|_{L^1} \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned} K_t(\phi_Q f)(x) &= K_t\Big(\phi_Q\Big(\sum_{Q'\in\mathcal{Q}'(Q)} \phi_{Q'} f\Big)\Big)(x) \\ &\quad - \phi_Q(x) K_t\Big(\sum_{Q'\in\mathcal{Q}'(Q)} (\phi_{Q'} f)\Big)(x) \\ &\quad - \phi_Q(x) K_t\Big(\sum_{Q''\in\mathcal{Q}''(Q)} (\phi_{Q''} f)\Big)(x) + \phi_Q(x) K_t f(x). \end{aligned} \tag{4.2}$$

Lemmas 3.7 and 3.8 combined with (4.2) lead to

$$\sum_{Q\in\mathcal{Q}} \Big\| \mathbf{1}_{Q^{**}} \sup_{t\le d(Q)^2} |K_t(\phi_Q f)|\Big\|_{L^1} \le C\|f\|_{H^1_L}. \tag{4.3}$$

Hence, applying (4.1), (4.3), and Theorem 3.2, we obtain

$$\phi_Q(x) f(x) = \sum_s \lambda_s(Q) a_s(Q) \tag{4.4}$$

with

$$\sum_{Q\in\mathcal{Q}} \sum_s |\lambda_s(Q)| \le C\|f\|_{H^1_L}, \tag{4.5}$$

where $a_s(Q)$ are $H^1_{\mathcal{Q}_{d(Q)}}$-atoms having supports contained in Q^{**}. The first inequality in (2.3) follows from (4.4) and (4.5).

We now turn to the second inequality in (2.3). Let a be an $H^1_{\mathcal{Q}}$-atom with $\operatorname{supp} a \subset Q^{**}$. There exists an integer $m \ge 0$ independent of Q such that

$$\inf\{d(Q')^2 : Q' \in \mathcal{Q}'(Q)\} \ge 2^{-m} d(Q)^2.$$

Then, by Lemma 3.11,

$$\Big\| \sup_{t\le 2^{-m}d(Q)^2} \Big|(P_t-K_t)\sum_{Q'\in\mathcal{Q}'(Q)} \phi_{Q'} a\Big|\Big\|_{L^1} \le C\|a\|_{L^1}. \tag{4.6}$$

Thus

$$\text{(4.7)}\quad \Big\|\sup_{t\le 2^{-m}d(Q)^2}|K_t(\phi_{Q'}a)|\Big\|_{L^1}\le C\|a\|_{L^1}+\Big\|\sup_{t\le 2^{-m}d(Q)^2}|P_t(\phi_{Q'}a)|\Big\|_{L^1}.$$

By standard arguments, the last summand on the right-hand side of (4.7) is controlled by the H^1_Q-norm of a. Therefore it suffices to estimate $\big\|\sup_{t>2^{-m}d(Q)^2}|K_ta|\big\|_{L^1}$. Observe that

$$\text{(4.8)}\qquad \Big\|\sup_{t>2^{-m}d(Q)^2}|K_ta|\Big\|_{L^1}\le\sum_{j\ge -m}\Big\|\sup_{2^jd(Q)^2\le t\le 2^{j+1}d(Q)^2}|K_ta|\Big\|_{L^1}$$

and, by (3.9),

$$\text{(4.9)}\quad \Big\|\sup_{2^jd(Q)^2\le t\le 2^{j+1}d(Q)^2}|K_ta|\Big\|_{L^1}$$
$$\le\Big\|\sup_{2^{j-1}d(Q)^2\le t\le 3\cdot 2^{j-1}d(Q)^2}K_t|K_{2^{j-1}d(Q)^2}a|\Big\|_{L^1}\le C\big\|K_{2^{j-1}d(Q)^2}|a|\big\|_{L^1},$$

which, combined with (2.1) and (D), allows us to sum up the expression on the right-hand side of (4.8). This completes the proof of Theorem 2.2. ∎

5. Proof of Theorem 2.4. Let $\mathcal{Q}$ be the collection of cubes described in (2.5). Obviously $\mathcal{Q}$ satisfies (K). Therefore it remains to show that (D) holds.

Let $k_t^{\{\gamma\}}(x,y)$ be the integral kernels of the semigroup generated by $-L^{\{\gamma\}}=\Delta-\gamma\mathbf{1}_{\mathbb{R}^d_+}$, $\gamma>0$. The Feynman–Kac formula implies

$$B_t^{\{\gamma\}}(y)=\int k_t^{\{\gamma\}}(x,y)\,dx=E^{y_1}\exp\bigg(-\frac{\gamma}{2}\int_0^{2t}\mathbf{1}_{[0,\infty)}(W_s)\,ds\bigg),$$

where W_s is one-dimensional Brownian motion with infinitesimal generator $\frac{1}{2}\frac{d^2}{dx^2}$, and $y=(y_1,\widetilde{y})\in\mathbb{R}\times\mathbb{R}^{d-1}=\mathbb{R}^d$. Applying e.g. [BS, formula 1.4.3, p. 156], we get

$$\text{(5.1)}\quad B_t^{\{\gamma\}}(y)$$
$$=\begin{cases}\operatorname{Erf}\Big(\dfrac{-y_1}{\sqrt{4t}}\Big)+\dfrac{e^{-\gamma t}}{\pi}\displaystyle\int_0^{2t}\frac{\exp(\gamma s/2-y_1^2/2s)}{\sqrt{s(2t-s)}}\,ds & \text{for } y_1\le 0,\\[2ex] e^{-\gamma t}\operatorname{Erf}\Big(\dfrac{y_1}{\sqrt{4t}}\Big)+\dfrac{1}{\pi}\displaystyle\int_0^{2t}\frac{\exp(-\gamma s/2-y_1^2/2s)}{\sqrt{s(2t-s)}}\,ds & \text{for } y_1\ge 0,\end{cases}$$

where $\operatorname{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}\,dv$. An immediate consequence of (5.1) is

$$\text{(5.2)}\quad \int k_t^{\{1\}}(x,y)\,dx=B_t^{\{1\}}(y)\le\begin{cases}C\min\{1,t^{-1/2}(1+|y_1|)\} & \text{for } y_1\le 0,\\ C\min\{1,t^{-1/2}\} & \text{for } y_1\ge 0.\end{cases}$$

Now (D) follows from (5.2). ∎

6. Proof of Theorem 2.6. Let $K_t(x,y)$ be the integral kernels of the semigroup generated by $-L = \Delta - V(x)$, where $V(x) = \exp(x_1)$, and let $\mathcal{Q}$ be the corresponding collection of cubes (see (2.7)). It is not difficult to check that $\mathcal{Q}$ satisfies (K). What is left is to prove (D). We shall consider two cases.

Case 1: $Q = [-2^{j+1}, -2^j] \times \widetilde{Q}$, $\widetilde{Q} \in \widetilde{\mathcal{Q}}_{2^j}$, $j = 0, 1, 2, \ldots$, or $Q = [-1,1] \times \widetilde{Q}$, $\widetilde{Q} \in \widetilde{\mathcal{Q}}_2$. Since $V \ge \mathbf{1}_{\mathbb{R}^d_+}$, we have $K_t(x,y) \le k_t^{\{1\}}(x,y)$. Hence, applying (5.2), we obtain

$$\int K_t(x,y)\,dx \le Ct^{-1/2}(1+|y_1|) \quad \text{for } y = (y_1, \widetilde{y}),\ y_1 \le 1, \tag{6.1}$$

and, consequently, (D) holds.

Case 2: $Q = [r_j, r_{j+1}] \times \widetilde{Q}_j$, $r_1 = 1$, $r_{j+1} = r_j + \exp(-r_j/2)$, $\widetilde{Q}_j \in \widetilde{\mathcal{Q}}_{\exp(-r_j/2)}$. Let $k_t^{[j]}(x,y)$ denote the integral kernels of the semigroup generated by the Schrödinger operator $\Delta - e^{r_{j+1}}\mathbf{1}_{\{x=(x_1,\widetilde{x})\,:\,x_1 > r_{j+1}\}}$. Obviously, $K_t(x,y) \le k_t^{[j]}(x,y)$. Moreover, (5.1) implies

$$\begin{aligned}\int K_t(x,y)\,dx &\le \int k_t^{[j]}(x,y)\,dx \\ &= \int k_t^{\{e^{r_{j+1}}\}}(x - r_{j+1}\mathbf{e}_1, y - r_{j+1}\mathbf{e}_1)\,dx \le Ct^{-1/2}e^{-r_j/2}\end{aligned}$$

for $y = (y_1, \widetilde{y})$, $|y_1 - r_{j+1}| \le 2e^{-r_j/2}$, and condition (D) is verified. ■

7. Proof of Theorem 2.8. The fact that (K) holds is obvious. In order to prove (D) we denote by $K_t^{\{\gamma\}}(x,y)$ the integral kernels of the semigroup generated by $-L^{\{\gamma\}} = \Delta - \gamma|x|^{-2}$. Then $K_t^{\{\gamma_2\}}(x,y) \le K_t^{\{\gamma_1\}}(x,y)$ for $0 < \gamma_1 \le \gamma_2$. Therefore it suffices to verify (D) for $\gamma > 0$ small. Theorem 2 of [MS] combined with (2.1) gives

$$K_1^{\{\gamma\}}(x,y) \le C\phi(x)\phi(y)e^{-|x-y|^2/5}, \tag{7.1}$$

with

$$\phi(x) = |x|^\sigma \quad \text{for } |x| < 1, \qquad \phi(x) = 1 \quad \text{for } |x| \ge 1, \tag{7.2}$$

where $\sigma > 0$ is an exponent that depends on γ. Since $L^{\{\gamma\}}$ is homogeneous of degree 2,

$$K_t^{\{\gamma\}}(x,y) = t^{-d/2}K_1^{\{\gamma\}}(t^{-1/2}x, t^{-1/2}y). \tag{7.3}$$

Now (D) follows from (7.1)–(7.3). ■

8. Remarks. In the present section we give two further examples of potentials and families of cubes for which conditions (D) and (K) hold.

• If $V(x) = \mathbf{1}_{[-1,1]}(x_1)$, $x = (x_1, \widetilde{x}) \in \mathbb{R}^d$, $d \geq 1$, and

$$\begin{aligned}\mathcal{Q} = \{[-2^{j+1}, -2^j] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_{2^j},\ j = 0, 1, \ldots\} \\ \cup \{[-1, 1] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_2\} \\ \cup \{[2^j, 2^{j+1}] \times \widetilde{Q} : \widetilde{Q} \in \widetilde{\mathcal{Q}}_{2^j},\ j = 0, 1, \ldots\},\end{aligned}$$

then conditions (D) and (K) hold. We omit the proof.

• One can check, using estimates derived e.g. in [K], [DZ2]–[DZ3], and [Sh], that for V satisfying the reverse Hölder inequality with an exponent $q > d/2$, $d \geq 3$, and for the family $\mathcal{Q}$ of cubes defined as follows:

$$Q \in \mathcal{Q} \Leftrightarrow Q \text{ is the maximal dyadic cube for which } \frac{d(Q)^2}{|Q|} \int_Q V(y)\, dy \leq 1, \tag{8.1}$$

conditions (D) and (K) are satisfied and, consequently, the norms $\|\cdot\|_{H^1_{\mathcal{Q}}}$ and $\|\cdot\|_{H^1_L}$ are equivalent.

We now show how to verify (D) in a slightly simpler way than it was done in [DZ2]–[DZ3]. Let $m(x) = d(Q)^{-1}$, where Q is a cube from $\mathcal{Q}$ such that $x \in Q$ (the function $m(x)$ is well defined for almost every x). By Lemma 1.4 of [Sh] there exist constants $C > 0$ and $0 < \theta < 1$ such that

$$\begin{aligned}C^{-1} m(y)(1 + |x-y| m(y))^{-\theta} &\leq m(x) \\ &\leq C m(y)(1 + |x-y| m(y))^{\theta/(1-\theta)}.\end{aligned} \tag{8.2}$$

Then, by applying (2.1) and the Schwarz inequality, one gets

$$\begin{aligned}I = \Big(\int K_t(x,y)\, dx\Big)^2 &\leq 2\Big(\int_{|x-y|\leq R} K_t(x,y)\, dx\Big)^2 + \Big(\int_{|x-y|>R} P_t(x-y)\, dx\Big)^2 \\ &\leq CR^d \int_{|x-y|\leq R} K_t(x,y)^2\, dx + CtR^{-2}.\end{aligned}$$

Using (8.2) and the Fefferman–Phong inequality (see [Sh, Lemma 1.9]) we obtain

$$\begin{aligned}I &\leq CR^d m(y)^{-2}(1 + Rm(y))^{2\theta} \int_{|x-y|\leq R} m(x)^2 K_t(x,y)^2\, dx + CtR^{-2} \\ &\leq CR^d m(y)^{-2}(1 + Rm(y))^{2\theta} \langle LK_t(\cdot\,, y), K_t(\cdot\,, y)\rangle + CtR^{-2}.\end{aligned}$$

By (2.1) and the holomorphy of the semigroup $\{K_t\}$, we have

$$\langle LK_t(\cdot\,, y), K_t(\cdot\,, y)\rangle \leq Ct^{-1-d/2}.$$

Hence, putting $R = t^{(1+\varepsilon)/2} m(y)^{\varepsilon}$ with $\varepsilon > 0$ small enough, we get (D).

9. Fractional Schrödinger operators. Let $L=-(-\Delta)^\alpha+V$, where $0<\alpha<1$ and $V\ge 0$ is a polynomial. Then $-L$ generates a semigroup $\{K_t\}_{t>0}$ of linear operators with integral kernels $K_t(x,y)$ such that

$$0\le K_t(x,y)\le P_t^\alpha(x-y), \tag{9.1}$$

where $P_t^\alpha(x)$ are the convolution kernels of the symmetric stable semigroup $\{P_t^\alpha\}_{t>0}$ generated by $-(-\Delta)^\alpha$. Let $\mathcal{Q}$ be defined by the condition

$$Q\in\mathcal{Q} \;\Leftrightarrow\; Q \text{ is the maximal dyadic cube for which } \frac{d(Q)^{2\alpha}}{|Q|}\int_Q V(y)\,dy\le 1.$$

Set $d(x)=d(Q)$, where $Q\in\mathcal{Q}$ is such that $x\in Q$. Then there exist constants $C>0$ and $0<\theta<1$ such that

$$C^{-1}d(x)\left(1+\frac{|x-y|}{d(x)}\right)^{-\theta/(1-\theta)}\le d(y)\le Cd(x)\left(1+\frac{|x-y|}{d(x)}\right)^{\theta}. \tag{9.2}$$

The estimates in (9.2) could be proved for V satisfying (1.4) with $q=d/2\alpha$ (see [Sh, Lemma 1.4 and its proof]). It follows from (9.2) that $\mathcal{Q}$ forms a covering of $\mathbb{R}^d$ such that the diameters of any two neighboring cubes from $\mathcal{Q}$ are comparable.

We are now in a position to state the following two conditions that are valid for K_t and V:

(D^α) there exist constants $C,\varepsilon>0$ such that

$$\sup_{y\in Q^*}\int K_{2^s d(Q)^{2\alpha}}(x,y)\,dx\le Cs^{-1-\varepsilon}\quad \text{for } Q\in\mathcal{Q},\ s\in\mathbb{N},$$

(K^α) there exist constants $C,\delta>0$ such that

$$\int_0^{2t}(\mathbf{1}_{Q^{***}}V)*P_s^\alpha(x)\,ds\le C\left(\frac{t}{d(Q)^{2\alpha}}\right)^{\delta}\quad \text{for } x\in\mathbb{R}^d,\ Q\in\mathcal{Q},\ t\le d(Q)^{2\alpha}.$$

By using ideas similar to those of the proof of Theorem 2.2, one gets the following theorem.

THEOREM 9.3. *The Hardy space H^1_L defined by K_t is a local Hardy space associated with $\mathcal{Q}$, that is, the norms $\|\cdot\|_{H^1_L}$ and $\|\cdot\|_{H^1_{\mathcal{Q}}}$ are comparable.*

Sketch of the proof. It suffices to repeat the proof of Theorem 2.2 replacing the classical heat kernel by P_t^α. Condition (K^α) is valid for V satisfying (1.4) with $q=d/2\alpha$, and can be verified by the same method as in [DZ2]–[DZ3] (see also [Sh] for the idea of the proof). We omit the details. The only nontrivial fact we have to show is condition (D^α). Using arguments similar to those in Section 8 one can reduce the proof of (D^α) to the following variant of the uncertainty principle (cf. [F]).

THEOREM 9.4. *Let $w(y)=d(x)^{-2\alpha}$. Then there exists a constant $C>0$ such that*

$$\int w(x)|f(x)|^2\,dx\le C\langle Lf,f\rangle. \tag{9.5}$$

Proof. Write $\nabla^\alpha=(-\Delta)^{\alpha/2}$. Let ϕ_Q be a smooth resolution of identity associated with the collection $\mathcal{Q}$ (see Section 3). For $\psi\in C_c^\infty$, $\psi\ge 0$, $\int\psi=1$, and a real number $A>0$ let $\psi_Q^A(x)=(Ad(Q)^{-1})^d\psi(Ad(Q)^{-1}x)$.

Obviously, $|\widehat{\psi}(\omega)-1|\le C|\omega|^\alpha$. Hence, from the Plancherel formula, we obtain

$$\begin{aligned}\int_{Q^*}|\psi_Q^A*(\phi_Qf)-\phi_Qf|^2&\le CA^{-2\alpha}d(Q)^{2\alpha}\int|\nabla^\alpha(\phi_Qf)|^2\\&\le CA^{-2\alpha}d(Q)^{2\alpha}\Big(\int\phi_Q^2|\nabla^\alpha f|^2+\int|[\phi_Q,\nabla^\alpha]f|^2\Big).\end{aligned}$$

Moreover, the A^∞ condition for V implies that there exist constants $C,\xi>0$ such that for $\Omega_\varepsilon=\Omega_\varepsilon(Q)=\{x\in Q^*:V(x)\le\varepsilon d(Q)^{-2\alpha}\}$ we have $|\Omega_\varepsilon|\le C\varepsilon^\xi|Q|$ independently of Q and ε. Therefore

$$\begin{aligned}\int|\phi_Q^A*(\phi_Qf)|^2&\le\|\phi_Q^A\|_{L^2}^2\|\phi_Qf\|_{L^1}^2\le(Ad(Q)^{-1})^d\Big(\int|\phi_Qf|\Big)^2\\&\le C(Ad(Q)^{-1})^d|\Omega_\varepsilon|\int_{\Omega_\varepsilon}|\phi_Qf|^2+C\varepsilon^{-1}d(Q)^{2\alpha}A^d\int V|\phi_Qf|^2\\&\le CA^d\varepsilon^\xi\int|\phi_Qf|^2+C\varepsilon^{-1}d(Q)^{2\alpha}A^d\int V|\phi_Qf|^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_{Q^*}|\phi_Qf|^2\le CA^{-2\alpha}d(Q)^{2\alpha}\int\phi_Q^2|\nabla^\alpha f|^2+CA^{-2\alpha}d(Q)^{2\alpha}\int|[\phi_Q,\nabla^\alpha]f|^2\\+CA^d\varepsilon^\xi\int|\phi_Qf|^2+C\varepsilon^{-1}d(Q)^{2\alpha}A^d\int V|\phi_Qf|^2.\end{aligned}$$

Summing up over $Q\in\mathcal{Q}$ we get

$$\begin{aligned}\int w|f|^2&\le\sum_{Q\in\mathcal{Q}}\int d(Q)^{-2\alpha}|\phi_Qf|^2\\&\le C_{A,\varepsilon}\langle Lf,f\rangle+CA^{-2\alpha}\sum_Q\int|[\phi_Q,\nabla^\alpha]f|^2\\&\le C_{A,\varepsilon}\langle Lf,f\rangle+CA^{-2\alpha}\int w|f|^2,\end{aligned}$$

provided $CA^d\varepsilon^\xi\le 1/2$. The last inequality has been deduced from the following lemma.

LEMMA 9.6. *The operator $Tf(x,Q)=[\phi_Q,\nabla^\alpha](w^{-1/2}f)(x)$ is bounded from $L^2(\mathbb{R}^d)$ into $l^2(L^2(\mathbb{R}^d))$.*

The theorem follows by fixing A sufficiently large and then taking $\varepsilon > 0$ small enough. ∎

Proof of Lemma 9.6. It suffices to prove that $T : L^1 \to l^1(L^1)$ and $T : L^\infty \to l^\infty(L^\infty)$ and then interpolate.

The first statement follows from

$$\sum_Q \int \frac{|\phi_Q(x) - \phi_Q(y)|}{|x-y|^{d+\alpha}} d(y)^\alpha \, dx \le C' \int\limits_{|x-y|\le Cd(y)} \frac{d(y)^\alpha}{d(Q)|x-y|^{d+\alpha-1}} \, dx + C' \int\limits_{|x-y|\ge Cd(y)} \frac{d(y)^\alpha}{|x-y|^{d+\alpha}} \, dx \le C'',$$

where C'' is a constant independent of y. The second statement is a consequence of

$$\sup_{x,Q} \int \frac{|\phi_Q(x) - \phi_Q(y)|}{|x-y|^{d+\alpha}} d(y)^\alpha \, dy \le C' \int\limits_{|x-y|\le Cd(x)} \frac{d(y)^\alpha}{d(Q)|x-y|^{d+\alpha-1}} \, dy + C' \int\limits_{|x-y|\ge Cd(x)} \frac{d(y)^\alpha}{|x-y|^{d+\alpha}} \, dy \le C'',$$

with a constant C'' independent of x and Q. In the above estimates we have used (9.2). The proof of the lemma is complete. ∎

Remark. Let us finally point out that Theorem 9.4 for V being a nonnegative polynomial could also be proved by using nilpotent Lie groups methods and maximal subelliptic estimates for accretive kernels proved by P. Głowacki (see [G]).

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