Hardy spaces $H^1$
for Schrödinger operators with certain potentials

by

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Abstract. Let $\{K_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-L = \Delta - V$ with $V \geq 0$. We say that $f$ belongs to $H^1_L$ if $\|\sup_{t>0} |K_t f(x)|\|_{L^1(dx)} < \infty$. We state conditions on $V$ and $K_t$ which allow us to give an atomic characterization of the space $H^1_L$.

1. Introduction. Let $L f(x) = -\Delta f(x) + V(x) f(x)$ be a Schrödinger operator on $\mathbb{R}^d$, where $V \geq 0$, $V \neq 0$. We shall assume that $-L$ generates a semigroup $\{K_t\}_{t>0}$ of linear contractions on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. This is guaranteed if e.g. $V \in L^q_{\text{loc}}$ for some $q > d/2$.

We define the Hardy space $H^1_L$ related to the operator $L$ by

$$H^1_L = \left\{ f \in L^1(\mathbb{R}^d) : \|f\|_{H^1_L} = \left\| \sup_{t>0} |K_t f(x)| \right\|_{L^1(dx)} < \infty \right\}.$$  

Let $Q = \{Q_j\}_{j=1}^\infty$ be a collection of closed cubes with parallel sides whose interiors are disjoint such that $\mathbb{R}^d = \bigcup_{j=0}^\infty Q_j$. For a cube $Q$ let $d(Q)$ denote its diameter. We shall always assume that there exist constants $C_0, \beta > 0$ such that for $Q_{j_1}, Q_{j_2} \in Q$ if $Q_{j_1}^{**} \cap Q_{j_2}^{**} \neq \emptyset$, then $C_0^{-1} d(Q_{j_1}) \leq d(Q_{j_2}) \leq C_0 d(Q_{j_1})$, where $Q^*$ is the cube with the same center as $Q$ such that $d(Q^*) = (1 + \beta)d(Q)$.

In order to state our results we need the following notion of the local atomic Hardy space $H^1_Q$ associated with the collection $Q$. We say that a function $a$ is an $H^1_Q$-atom if there is a cube $Q \in Q$ such that $a$ is a classical $(1, \infty)$-atom having support contained in $Q^{**}$, or $a(x) = \left|Q\right|^{-1}1_Q(x)$, where, for a set $A \subset \mathbb{R}^d$, $1_A$ denotes the indicator function of $A$. Then $H^1_Q$ is defined

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\[(1.2) \quad H_Q^1 = \left\{ f \in L^1 : f(x) = \sum_s \lambda_s a_s(x), \sum_s |\lambda_s| < \infty \right\},\]

where \(a_s\) are \(H_Q^1\)-atoms. We set

\[(1.3) \quad \|f\|_{H_Q^1} = \inf \sum |\lambda_s|,\]

where the infimum is taken over all possible representations of \(f\) as in (1.2).

In [DZ2] the authors proved that if \(V\) satisfies the reverse Hölder inequality with an exponent \(q > d/2, d \geq 3\), that is,

\[(1.4) \quad \left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq C \frac{1}{|B|} \int_B V(y) \, dy \quad \text{for every ball } B,
\]

then the elements of \(H_L^1\) admit atomic decompositions of this type (cf. Section 8 of the present article).

The main goal of the present paper is to use ideas from [DZ1] and [DZ2] to see what the theory looks like when there are no reverse Hölder estimates for \(V\). We formulate here two conditions on \(V\), \(K_t\), and \(Q\) that guarantee that \(H_L^1\) is local, that is, the norms \(\cdot \|_{H_Q^1}\) and \(\cdot \|_{H_L^1}\) are equivalent (see Theorem 2.2). We shall verify that these conditions hold not only for \(V\) satisfying the reverse Hölder inequality but also for the following naturally occurring potentials:

\[(1.5) \quad V(x) = 1_{\mathbb{R}^d_+}(x), \quad \mathbb{R}^d_+ = \{(x_1, x_2, \ldots, x_d) : x_1 > 0\}, \quad d \geq 1,
\]

\[(1.6) \quad V(x) = \exp(x_1), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad d \geq 1,
\]

\[(1.7) \quad V(x) = \gamma |x|^{-2}, \quad \gamma > 0, \quad d \geq 3,
\]

and properly defined families \(Q\) (cf. Theorems 2.4, 2.6, and 2.8). The potentials (1.5) and (1.6) do not satisfy the doubling condition, so they do not belong to any reverse Hölder class. Obviously for \(q \geq d/2\) and \(V(x) = \gamma |x|^{-2}\) the condition (1.4) does not hold.

For results concerning Hardy spaces related to Schrödinger operators with potentials from reverse Hölder classes we refer the reader to [DZ1]–[DZ4].

At the end of the paper for \(0 < \alpha < 1\) and \(V\) being a nonnegative polynomial we consider the operator \((-\Delta)^\alpha + V\). We prove atomic decompositions of the elements of \(H_L^1(-\Delta)^\alpha + V\).

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2. Statements of the results. Denote by $P_t(x)$ the convolution kernels of the heat semigroup $\{P_t\}_{t>0}$ on $\mathbb{R}^d$ generated by $\Delta$ and by $K_t(x,y)$ the integral kernels of the semigroup $\{K_t\}_{t>0}$ generated by the Schrödinger operator $-L = \Delta - V$, $V \geq 0$. Obviously,

$$0 \leq K_t(x,y) \leq P_t(x-y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/4t).$$

For $V \geq 0$ and a collection $Q$ of cubes as described above we consider the following two conditions:

(D) there exist constants $C, \varepsilon > 0$ such that

$$\sup_{y \in Q} \int K_{2s^2(Q)2}^{2s^2(Q)2}(x,y) \, dx \leq Cs^{-1-\varepsilon} \quad \text{for } Q \in Q, \ s \in \mathbb{N},$$

(K) there exist constants $C, \delta > 0$ such that

$$\int_0^{2t} (1_{Q^{\delta}} - V) * P_s(x) \, ds \leq C \left( \frac{t}{d(Q)^2} \right)^\delta \quad \text{for } x \in \mathbb{R}^d, \ Q \in Q, \ t \leq d(Q)^2.$$
Theorem 2.8. For $V(x) = \gamma |x|^{-2}$ on $\mathbb{R}^d$, $d \geq 3$, $\gamma > 0$, let $Q$ be the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$ that consists of dyadic cubes. Then conditions (D) and (K) hold.

3. Auxiliary lemmas. To prove the theorems stated in Section 2, we need a sequence of auxiliary results.

For $l > 0$ let $h_l$ denote the local Hardy space (cf. [Go]) with the norm $\|f\|_{h_l}$ defined by

$$\|f\|_{h_l} = \| \sup_{t \leq l^2} |P_t f(x)| \|_{L^1(dx)}.$$

The following theorem is a consequence of results of Goldberg [Go].

Theorem 3.2. There exists a constant $C > 0$ such that for every $l > 0$ we have

$$C^{-1} \|f\|_{H^l_{Q_l}} \leq \|f\|_{h_l} \leq C \|f\|_{H^l_{Q_l}},$$

where $Q_l$ is a partition of $\mathbb{R}^d$ into cubes of side-length $l$. Moreover, if $f \in h_l$ with $\text{supp } f \subset Q^*$ for some $Q \in Q_l$, then

$$f = \sum \lambda_s a_s, \quad \sum |\lambda_s| \leq C \|f\|_{h_l},$$

with $a_s$ being $H^l_{Q_l}$-atoms such that $\text{supp } a_s \subset Q^*$.

For a collection $Q$ of cubes let $\{\phi_Q\}_{Q \in Q}$ be a family of $C^\infty$ functions on $\mathbb{R}^d$ such that $\text{supp } \phi_Q \subset Q^*$, $0 \leq \phi_Q \leq 1$, $|\partial^\alpha \phi_Q| \leq C_{\alpha} d(Q)^{-|\alpha|}$, and $\sum_Q \phi_Q(x) = 1$ for all $x \in \mathbb{R}^d$.

Lemma 3.3. There exists a constant $C > 0$ such that for every $Q \in Q$ we have

$$\|\phi_Q f\|_{h_{d(Q)}} \leq C \| \sup_{t \leq d(Q)^2} |P_t (\phi_Q f)| \|_{L^1(Q^*)}.$$

Proof. There exist constants $C, c_1 > 0$ such that if $x \in (Q^*)^c$, $y \in Q^*$, and $t \leq d(Q)^2$, then $P_t (x - y) \leq C d(Q)^{-d} \exp(-c_1 |x - y|^2/d(Q)^2)$, where $y_Q$ denotes the center of $Q$. Hence

$$|P_t * (\phi_Q f)(x)| \leq C d(Q)^{-d} \|\phi_Q f\|_{L^1} \exp(-c_1 |x - y|^2/d(Q)^2).$$

Now the lemma is a consequence of (3.5) and Theorem 3.2. □

For $Q \in Q$ we set

$$Q'(Q) = \{ Q' \in Q : Q^{***} \cap (Q')^{***} \neq \emptyset \};$$

$$Q''(Q) = \{ Q'' \in Q : Q^{***} \cap (Q'')^{***} = \emptyset \}.$$

The lemma below is quite similar to those in our earlier papers (cf. [DZ1, Lemma 5.7], [DZ2, Lemma 3.11]).
Lemma 3.7. There exists a constant $C > 0$ such that for every $Q \in \mathcal{Q}$ and every $f \in L^1(\mathbb{R}^d)$ we have
\[
\left\| \sup_{t > 0} \left| K_t \left( \phi_Q \sum_{Q' \in \mathcal{Q}'(Q)} \phi_{Q'} f \right) - \phi_Q \left( K_t \sum_{Q' \in \mathcal{Q}'(Q)} \phi_{Q'} f \right) \right| \right\|_{L^1(Q^{**})} \leq C \sum_{Q' \in \mathcal{Q}'(Q)} \| \phi_{Q'} f \|_{L^1}.
\]

Proof. Let $g = \sum_{Q' \in \mathcal{Q}'(Q)} \phi_{Q'} f$. Then
\[
\sup_{t > 0} \left| K_t (\phi_Q g)(x) - \phi_Q(x) K_t g(x) \right| = \sup_{t > 0} \left| \int (\phi_Q(y) - \phi_Q(x)) K_t(x, y) g(y) \, dy \right|
\]
\[
\leq C \sup_{t > 0} \left| \int \frac{|x - y|}{d(Q)} K_t(x, y) g(y) \, dy \right| \leq C \frac{d(Q)}{|x - y|^{d-1}} \, dy.
\]
Integrating with respect to $x$ over $Q^{**}$ we obtain the lemma. ■

Lemma 3.8. Assume that $\mathcal{Q}$ satisfies condition (D). Then there exists a constant $C > 0$ such that
\[
\sum_{Q \in \mathcal{Q}} \left\| 1_{Q^{**}} \sup_{t > 0} \left| K_t \left( \sum_{Q'' \in \mathcal{Q}''(Q)} \phi_{Q''} f \right) \right| \right\|_{L^1} \leq C \| f \|_{L^1}.
\]

Proof. Denote the left-hand side by $S$. Then
\[
S \leq \sum_{Q \in \mathcal{Q}} \sum_{Q'' \in \mathcal{Q}'(Q)} \left\| 1_{Q^{**}} \sup_{t > 0} (K_t |\phi_{Q''} f|) \right\|_{L^1}
\]
\[
\leq \sum_{Q'' \in \mathcal{Q}} \sum_{Q \in \mathcal{Q}''(Q''')} \left\| 1_{Q^{**}} \sup_{t > 0} (K_t |\phi_{Q''} f|) \right\|_{L^1}
\]
\[
\leq C \sum_{Q'' \in \mathcal{Q}} \left\| \sup_{t > 0} (K_t |\phi_{Q''} f|) \right\|_{L^1((Q'')^{**})}
\]
\[
\leq C \sum_{Q'' \in \mathcal{Q}} \left\| \sup_{0 < t \leq d(Q'')^2} (K_t |\phi_{Q''} f|) \right\|_{L^1((Q'')^{**})}
\]
\[
+ C \sum_{Q'' \in \mathcal{Q}} \sum_{j = 0}^{\infty} \left\| \sup_{2^j d(Q'')^2 \leq t \leq 2^{j+1} d(Q'')^2} (K_t |\phi_{Q''} f|) \right\|_{L^1((Q'')^{**})}.
\]

Note that for $s_j = 2^j d(Q'')^2 \leq t \leq 2^{j+1} d(Q'')^2 = s_{j+1}$ we have
\[
K_t(x, y) = \int K_{t-2^{j-1} d(Q'')^2} (x, z) K_{2^{j-1} d(Q'')^2} (z, y) \, dz
\]
\[
\leq \int P_{s_j}^\infty (x, z) K_{2^{j-1} d(Q'')^2} (z, y) \, dz,
\]
where, by (2.1),
\[
P_{s_j}^\infty (x, z) = \sup_{s_j/2 \leq s \leq 2s_j} P_s (x - z) \leq C_1 s_j^{-d/2} \exp(-c_1 |x - z|^2/s_j).
\]
Applying (D), we obtain
\[ S \leq C \sum_{Q'' \in \mathcal{Q}} \| \phi_{Q''} f \|_{L^1} + C \sum_{Q'' \in \mathcal{Q}} \sum_{j=0}^{\infty} j^{-1-\varepsilon} \| \phi_{Q''} f \|_{L^1} \leq C \| f \|_{L^1}. \]

**Lemma 3.10.**
\[ \int_{\mathbb{R}^d} \int_0^\infty V(x)(K_s f)(x) \, ds \, dx \leq \| f \|_{L^1}. \]

**Proof.** This result seems to be well known. We give a proof for completeness. The perturbation formula asserts that
\[ P_t = K_t + \int_0^t P_{t-s} V K_s \, ds. \]
Therefore, by (2.1), if \( f \geq 0 \) then
\[ \int_0^t P_{t-s} V K_s f(y) \, ds \leq P_t f(y). \]
Integrating with respect to \( y \) and applying the Fubini theorem we get
\[ \int_0^t \int V(x) K_s f(x) \, dx \, ds \leq \| f \|_{L^1}. \]
Letting \( t \to \infty \) we obtain the lemma.

The following lemma is a generalization of Lemma 3.9 of [DZ2] (see also [DZ1, Lemma 5.1]).

**Lemma 3.11.** Assume that \( Q \) satisfies (K). Then there exists a constant \( C > 0 \) such that
\[ \sup_{0 < t \leq d(Q)^2} \| (P_t - K_t)(\phi_Q f) \|_{L^1} \leq C \| \phi_Q f \|_{L^1}. \]

**Proof.** By (2.1) and (3.5) it suffices to estimate the quantity
\[ \sup_{0 < t \leq d(Q)^2} \| (P_t - K_t)(\phi_Q f) \|_{L^1(Q^{**})}. \]
The perturbation formula implies
\[ (P_t - K_t)(\phi_Q f)(x) = \int_0^t P_{t-s} V'' K_s(\phi_Q f)(x) \, ds \]
\[ + \int_0^t P_{t-s}(1_Q^{**} V) K_s(\phi_Q f)(x) \, ds, \]
where \( V = 1_Q^{**} V + V''. \)
For \( y \in (Q^*)^c, x \in Q^*, \) and \( 0 < s < t \leq d(Q)^2, \) we have \( P_{t-s}(x-y) \leq Cd(Q)^{-d}\exp(-c|x-y|^2/d(Q)^2). \) Hence

\[
\left| \int_0^t P_{t-s}V''K_s(\phi_Qf)(x) \, ds \right| = \left| \int_0^t P_{t-s}(x-y)V''(y)K_s(\phi_Qf)(y) \, ds \, dy \right|
\]

\[
\leq C \int_0^t d(Q)^{-d}\exp\left(\frac{-c|x-y|^2}{d(Q)^2}\right)V''(y)K_s(\|\phi_Qf\|)(y) \, ds \, dy
\]

\[
\leq Cd(Q)^{-d}\int_0^\infty V''(y)K_s(\|\phi_Qf\|)(y) \, ds \, dy \leq Cd(Q)^{-d}\|\phi_Qf\|_1.
\]

In the last inequality we have used Lemma 3.10. Thus

\[
\left\| \sup_{0 < s \leq d(Q)^2} \int_0^t P_{t-s}V''K_s(\phi_Qf)(x) \, ds \right\|_{L^1(Q^*)} \leq C\|\phi_Qf\|_1.
\]

We now turn to estimating the integral that contains \( 1_{Q^*}V: \)

\[
\left| \int_0^t P_{t-s}1_{Q^*}V K_s(\phi_Qf)(x) \, ds \right| \leq \int_0^t P_{t-s}(1_{Q^*}V) P_s(\|\phi_Qf\|)(x) \, ds
\]

\[
= \int_0^{t/2} + \int_{t/2}^t = I_t(x) + J_t(x).
\]

For \( t_j = 2^{-j}d(Q)^2 \leq t \leq 2^{-j+1}d(Q)^2 = 2t_j \) we have

\[
I_j^*(x) = \sup_{t_j \leq t \leq 2t_j} I_t(x) \leq \int_0^{2t_j} P_{t_j}^{\text{max}}(x,z)V(z)1_{Q^*}(z) P_s(\|\phi_Qf\|)(z) \, dz \, ds
\]

(see (3.9)). Hence, applying (K) and (3.9), we conclude that

\[
\left\| \sup_{0 < t \leq d(Q)^2} I_t(x) \right\|_1 \leq \sum_{j \geq 1} \|I_j^*\|_1
\]

\[
\leq \sum_{j \geq 1} \int_0^{t_j} P_{t_j}^{\text{max}}(x,z)1_{Q^*}(z)V(z) P_s(\|\phi_Qf\|)(z) \, dz \, dx
\]

\[
\leq C \sum_{j \geq 1} \int_0^{t_j} (1_{Q^*}V)(z)P_s(z-y)(\|\phi_Q(y)f(y)\|) \, dy \, dz
\]

\[
\leq C \sum_{j \geq 1} 2^{-j\delta}\|\phi_Qf\|_1.
\]

The \( L^1 \)-norm of \( J^*(x) = \sup_{0 < t \leq d(Q)^2} J_t(x) \) can be estimated in a similar way. \( \blacksquare \)
4. Proof of Theorem 2.2. We start by proving the first inequality in (2.3). From Lemmas 3.3 and 3.11 we deduce that

\[
\sum_{Q \in \mathcal{Q}} \| \phi_Q f(x) \|_{H_{d(Q)}^1} \leq C \sum_{Q \in \mathcal{Q}} \| 1_{Q^{**}} \|_{L^1} \sup_{t \leq d(Q)^2} \| P_t \phi_Q f \|_{L^1} \\
\leq C \sum_{Q \in \mathcal{Q}} \| 1_{Q^{**}} \|_{L^1} \sup_{t \leq d(Q)^2} \| (P_t - K_t) (\phi_Q f) \|_{L^1} \\
+ C \sum_{Q \in \mathcal{Q}} \| 1_{Q^{**}} \|_{L^1} \sup_{t \leq d(Q)^2} \| K_t (\phi_Q f) \|_{L^1} \\
\leq C \| f \|_{L^1} + C \sum_{Q \in \mathcal{Q}} \| 1_{Q^{**}} \|_{L^1} \sup_{t \leq d(Q)^2} \| K_t (\phi_Q f) \|_{L^1}
\]

Note that

\[
K_t (\phi_Q f)(x) = K_t \left( \phi_Q \left( \sum_{Q' \in Q''(Q)} \phi_{Q'} f \right) \right)(x) \\
- \phi_Q(x) K_t \left( \sum_{Q' \in Q''(Q)} (\phi_{Q'} f) \right)(x) \\
- \phi_Q(x) K_t \left( \sum_{Q'' \in Q''(Q)} (\phi_{Q''} f) \right)(x) + \phi_Q(x) K_t f(x).
\]

Lemmas 3.7 and 3.8 combined with (4.2) lead to

\[
\sum_{Q \in \mathcal{Q}} \| 1_{Q^{**}} \|_{L^1} \sup_{t \leq d(Q)^2} \| K_t (\phi_Q f) \|_{L^1} \leq C \| f \|_{H_{d(Q)}^1}.
\]

Hence, applying (4.1), (4.3), and Theorem 3.2, we obtain

\[
\phi_Q(x) f(x) = \sum_s \lambda_s(Q) a_s(Q)
\]

with

\[
\sum_{Q \in \mathcal{Q}} \sum_s |\lambda_s(Q)| \leq C \| f \|_{H_{d(Q)}^1},
\]

where \(a_s(Q)\) are \(H_{d(Q)}^1\)-atoms having supports contained in \(Q^{**}\). The first inequality in (2.3) follows from (4.4) and (4.5).

We now turn to the second inequality in (2.3). Let \(a\) be an \(H_{d(Q)}^1\)-atom with \(\text{supp} \ a \subset \Gamma(Q^{**})\). There exists an integer \(m \geq 0\) independent of \(Q\) such that

\[
\inf \{ d(Q')^2 : Q' \in Q'(Q) \} \geq 2^{-m} d(Q)^2.
\]

Then, by Lemma 3.11,

\[
\left\| \sup_{t \leq 2^{-m} d(Q)^2} \left( P_t - K_t \right) \sum_{Q' \in Q'(Q)} \phi_{Q'} a \right\|_{L^1} \leq C \| a \|_{L^1}.
\]
Thus
\begin{equation}
\left\| \sup_{t \leq 2^{-m}d(Q)^2} |K_t(\phi_Q a)| \right\|_{L^1} \leq C\|a\|_{L^1} + \left\| \sup_{t \leq 2^{-m}d(Q)^2} |P_t(\phi_Q a)| \right\|_{L^1}.
\end{equation}
By standard arguments, the last summand on the right-hand side of (4.7) is controlled by the $H^1_Q$-norm of $a$. Therefore it suffices to estimate $\left\| \sup_{t > 2^{-m}d(Q)^2} |K_t a| \right\|_{L^1}$. Observe that
\begin{equation}
\left\| \sup_{t > 2^{-m}d(Q)^2} |K_t a| \right\|_{L^1} \leq \sum_{j \geq -m} \left\| \sup_{2^j d(Q)^2 \leq t \leq 2^{j+1}d(Q)^2} |K_t a| \right\|_{L^1}
\end{equation}
and, by (3.9),
\begin{equation}
\left\| \sup_{2^j d(Q)^2 \leq t \leq 2^{j+1}d(Q)^2} |K_t a| \right\|_{L^1} \leq \left\| \sup_{2^{j-1}d(Q)^2 \leq t \leq 3 \cdot 2^{j-1}d(Q)^2} K_t|K_{2^{j-1}d(Q)^2} a| \right\|_{L^1} \leq C\|K_{2^{j-1}d(Q)^2} a\|_{L^1},
\end{equation}
which, combined with (2.1) and (D), allows us to sum up the expression on the right-hand side of (4.8). This completes the proof of Theorem 2.2. ■

5. Proof of Theorem 2.4. Let $Q$ be the collection of cubes described in (2.5). Obviously $Q$ satisfies (K). Therefore it remains to show that (D) holds.

Let $k_t^{\{\gamma\}}(x, y)$ be the integral kernels of the semigroup generated by $-L^{\{\gamma\}} = \Delta - \gamma 1_{\mathbb{R}_+^d}$, $\gamma > 0$. The Feynman–Kac formula implies
\begin{equation}
B_t^{\{\gamma\}}(y) = \int k_t^{\{\gamma\}}(x, y) \, dx = E^{y_1} \exp \left( -\frac{\gamma}{2} \int_0^1 1_{[0, \infty)}(W_s) \, ds \right),
\end{equation}
where $W_s$ is one-dimensional Brownian motion with infinitesimal generator $\frac{1}{2} \frac{d^2}{dx^2}$, and $y = (y_1, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^d$. Applying e.g. [BS, formula 1.4.3, p. 156], we get
\begin{equation}
B_t^{\{\gamma\}}(y) = \begin{cases} 
\text{Erf}\left( \frac{-y_1}{\sqrt{4t}} \right) + \frac{e^{-\gamma t}}{\pi} \int_0^{2t} \frac{\exp(\gamma t/2 - y_1^2/2s)}{\sqrt{s(2t-s)}} \, ds & \text{for } y_1 \leq 0,
\end{cases}
\end{equation}
\begin{equation}
\begin{aligned}
e^{-\gamma t} \text{Erf}\left( \frac{y_1}{\sqrt{4t}} \right) + \frac{1}{\pi} \int_0^{2t} \frac{\exp(-\gamma t/2 - y_1^2/2s)}{\sqrt{s(2t-s)}} \, ds & \text{for } y_1 \geq 0,
\end{aligned}
\end{equation}
where Erf$(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} \, dv$. An immediate consequence of (5.1) is
\begin{equation}
k_t^{\{1\}}(x, y) \, dx = B_t^{\{1\}}(y) \leq \begin{cases} 
C \min\{1, t^{-1/2}(1 + |y_1|)\} & \text{for } y_1 \leq 0,
\end{cases}
\end{equation}
\begin{equation}
C \min\{1, t^{-1/2}\} & \text{for } y_1 \geq 0.
\end{cases}
\end{equation}
Now (D) follows from (5.2). ■
6. Proof of Theorem 2.6. Let \( K_t(x, y) \) be the integral kernels of the semigroup generated by \(-L = \Delta - V(x)\), where \( V(x) = \exp(x_1) \), and let \( Q \) be the corresponding collection of cubes (see (2.1)). It is not difficult to check that \( Q \) satisfies (K). What is left is to prove (D). We shall consider two cases.

Case 1: \( Q = [-2^j+1, -2^j] \times \widetilde{Q}_j, \widetilde{Q} \in \widetilde{Q}_2, j = 0, 1, 2, \ldots \), or \( Q = [-1, 1] \times \widetilde{Q}, \widetilde{Q} \in \widetilde{Q}_2 \). Since \( V \geq 1_\mathbb{R}^d \), we have \( K_t(x, y) \leq k_t^{(1)}(x, y) \). Hence, applying (5.2), we obtain

\[
(6.1) \quad \int K_t(x, y) \, dx \leq C t^{-1/2} (1 + |y_1|) \quad \text{for } y = (y_1, \widetilde{y}), \ y_1 \leq 1,
\]

and, consequently, (D) holds.

Case 2: \( Q = [r_j, r_{j+1}] \times \widetilde{Q}_j, r_1 = 1, r_{j+1} = r_j + \exp(-r_j/2), \widetilde{Q}_j \in \widetilde{Q}_{\exp(-r_j/2)} \). Let \( k_t^{(j)}(x, y) \) denote the integral kernels of the semigroup generated by the Schrödinger operator \( \Delta - e^{r_j+1} 1_{\{x = (x_1, \bar{x}) : x_1 > r_{j+1}\}} \). Obviously, \( K_t(x, y) \leq k_t^{(j)}(x, y) \). Moreover, (5.1) implies

\[
\int K_t(x, y) \, dx \leq \int k_t^{(j)}(x, y) \, dx = \int k_t^{(r_j+1)}(x - r_{j+1} e_1, y - r_{j+1} e_1) \, dx \leq C t^{-1/2} e^{-r_j/2}
\]

for \( y = (y_1, \widetilde{y}), |y_1 - r_{j+1}| \leq 2e^{-r_j/2} \), and condition (D) is verified. ■

7. Proof of Theorem 2.8. The fact that (K) holds is obvious. In order to prove (D) we denote by \( K_t^{(\gamma)}(x, y) \) the integral kernels of the semigroup generated by \(-L^{(\gamma)} = \Delta - \gamma |x|^{-2} \). Then \( K_t^{(\gamma_2)}(x, y) \leq K_t^{(\gamma_1)}(x, y) \) for \( 0 < \gamma_1 \leq \gamma_2 \). Therefore it suffices to verify (D) for \( \gamma > 0 \) small. Theorem 2 of [MS] combined with (2.1) gives

\[
(7.1) \quad K_t^{(\gamma)}(x, y) \leq C \phi(x) \phi(y) e^{-|x-y|^2/5},
\]

with

\[
(7.2) \quad \phi(x) = |x|^\sigma \quad \text{for } |x| < 1, \quad \phi(x) = 1 \quad \text{for } |x| \geq 1,
\]

where \( \sigma > 0 \) is an exponent that depends on \( \gamma \). Since \( L^{(\gamma)} \) is homogeneous of degree 2,

\[
(7.3) \quad K_t^{(\gamma)}(x, y) = t^{-d/2} K_t^{(1)}(t^{-1/2} x, t^{-1/2} y).
\]

Now (D) follows from (7.1)–(7.3). ■

8. Remarks. In the present section we give two further examples of potentials and families of cubes for which conditions (D) and (K) hold.
• If $V(x) = 1_{[-1,1]}(x_1)$, $x = (x_1, \tilde{x}) \in \mathbb{R}^d, d \geq 1$, and
  \[ Q = \{[-2^{j+1}, -2^j] \times \tilde{Q} : \tilde{Q} \in \tilde{Q}_{2^j}, j = 0, 1, \ldots\} \]
  \[ \cup \{[-1, 1] \times \tilde{Q} : \tilde{Q} \in \tilde{Q}_2\} \]
  \[ \cup \{[2^j, 2^{j+1}] \times \tilde{Q} : \tilde{Q} \in \tilde{Q}_{2^j}, j = 0, 1, \ldots\}, \]
  then conditions (D) and (K) hold. We omit the proof.

• One can check, using estimates derived e.g. in [K], [DZ1]–[DZ3], and [Sh], that for $V$ satisfying the reverse H"older inequality with an exponent $q > d/2, d \geq 3$, and for the family $Q$ of cubes defined as follows:
  \[ (8.1) \quad Q \in Q \Leftrightarrow Q \text{ is the maximal dyadic cube for which} \]
  \[ \frac{d(Q)^2}{|Q|} \int_Q V(y) \, dy \leq 1, \]
  conditions (D) and (K) are satisfied and, consequently, the norms $\| \cdot \|_{H^1_Q}$ and $\| \cdot \|_{H^1_{\tilde{Q}}}$ are equivalent.

We now show how to verify (D) in a slightly simpler way than it was done in [DZ1]–[DZ3]. Let $m(x) = d(Q)^{-1}$, where $Q$ is a cube from $Q$ such that $x \in Q$ (the function $m(x)$ is well defined for almost every $x$). By Lemma 1.4 of [Sh] there exist constants $C > 0$ and $0 < \theta < 1$ such that
  \[ (8.2) \quad C^{-1} m(y)(1 + |x - y|m(y))^{-\theta} \leq m(x) \leq C m(y)(1 + |x - y|m(y))^{\theta/(1-\theta)}. \]

Then, by applying (2.1) and the Schwarz inequality, one gets
  \[ I = \left( \int_{|x-y| \leq R} K_t(x,y) \, dx \right)^2 \leq 2 \left( \int_{|x-y| \leq R} K_t(x,y) \, dx \right)^2 \]
  \[ + \left( \int_{|x-y| > R} P_t(x-y) \, dx \right)^2 \]
  \[ \leq CR^d \int_{|x-y| \leq R} K_t(x,y)^2 \, dx + CtR^{-2}. \]

Using (8.2) and the Fefferman–Phong inequality (see [Sh, Lemma 1.9]) we obtain
  \[ I \leq CR^d m(y)^{-2}(1 + Rm(y))^{2\theta} \int_{|x-y| \leq R} m(x)^2K_t(x,y)^2 \, dx + CtR^{-2} \]
  \[ \leq CR^d m(y)^{-2}(1 + Rm(y))^{2\theta}(LK_t(\cdot,y), K_t(\cdot,y)) + CtR^{-2}. \]

By (2.1) and the holomorphy of the semigroup $\{K_t\}$, we have
  \[ \langle LK_t(\cdot,y), K_t(\cdot,y) \rangle \leq Ct^{-1-d/2}. \]

Hence, putting $R = t^{(1+\varepsilon)/2}m(y)^{\varepsilon}$ with $\varepsilon > 0$ small enough, we get (D).
9. Fractional Schrödinger operators. Let \( L = -(-\Delta)\alpha + V \), where \( 0 < \alpha < 1 \) and \( V \geq 0 \) is a polynomial. Then \(-L\) generates a semigroup \( \{K_t\}_{t>0} \) of linear operators with integral kernels \( K_t(x,y) \) such that

\[
0 \leq K_t(x,y) \leq P_t^\alpha(x-y),
\]

where \( P_t^\alpha(x) \) are the convolution kernels of the symmetric stable semigroup \( \{P_t^\alpha\}_{t>0} \) generated by \(-(-\Delta)\alpha\). Let \( Q \) be defined by the condition

\[
Q \in \mathcal{Q} \Leftrightarrow Q \text{ is the maximal dyadic cube for which }
\frac{d(Q)^{2\alpha}}{|Q|} \int_Q V(y) \, dy \leq 1.
\]

Set \( d(x) = d(Q) \), where \( Q \in \mathcal{Q} \) is such that \( x \in Q \). Then there exist constants \( C > 0 \) and \( 0 < \theta < 1 \) such that

\[
C^{-1} d(x) \left( 1 + \frac{|x-y|}{d(x)} \right)^{-\theta/(1-\theta)} \leq d(y) \leq C d(x) \left( 1 + \frac{|x-y|}{d(x)} \right)^{\theta}.
\]

The estimates in (9.2) could be proved for \( V \) satisfying (1.4) with \( q = d/2\alpha \) (see [Sh, Lemma 1.4 and its proof]). It follows from (9.2) that \( \mathcal{Q} \) forms a covering of \( \mathbb{R}^d \) such that the diameters of any two neighboring cubes from \( \mathcal{Q} \) are comparable.

We are now in a position to state the following two conditions that are valid for \( K_t \) and \( V \):

\begin{enumerate}
  \item[(D\(\alpha\))] there exist constants \( C, \varepsilon > 0 \) such that
  \[
  \sup_{y \in Q^*} \int_{Q^*} K_{2s}^\alpha d(Q)^{2\alpha} (x,y) \, dx \leq C s^{-1-\varepsilon} \quad \text{for } Q \in \mathcal{Q}, \ s \in \mathbb{N},
  \]
  \item[(K\(\alpha\))] there exist constants \( C, \delta > 0 \) such that
  \[
  \int_0^{2t} (1_Q \cdots V)^s P_s^\alpha(x) \, ds \leq C \left( \frac{t}{d(Q)^{2\alpha}} \right)^{\delta} \quad \text{for } x \in \mathbb{R}^d, \ Q \in \mathcal{Q}, \ t \leq d(Q)^{2\alpha}.
  \]
\end{enumerate}

By using ideas similar to those of the proof of Theorem 2.2, one gets the following theorem.

**Theorem 9.3.** The Hardy space \( H_{L^1}^1 \) defined by \( K_t \) is a local Hardy space associated with \( Q \), that is, the norms \( \| \cdot \|_{H_{L^1}^1} \) and \( \| \cdot \|_{H_{Q}^1} \) are comparable.

**Sketch of the proof.** It suffices to repeat the proof of Theorem 2.2 replacing the classical heat kernel by \( P_t^\alpha \). Condition (K\(\alpha\)) is valid for \( V \) satisfying (1.4) with \( q = d/2\alpha \), and can be verified by the same method as in [DZ2]–[DZ3] (see also [Sh] for the idea of the proof). We omit the details. The only nontrivial fact we have to show is condition (D\(\alpha\)). Using arguments similar to those in Section 8 one can reduce the proof of (D\(\alpha\)) to the following variant of the uncertainty principle (cf. [F]).
Theorem 9.4. Let $w(y) = d(x)^{-2\alpha}$. Then there exists a constant $C > 0$ such that
\begin{equation}
\int w(x)|f(x)|^2 \, dx \leq C\langle Lf, f \rangle.
\end{equation}

Proof. Write $\nabla^\alpha = (-\Delta)^{\alpha/2}$. Let $\phi_Q$ be a smooth resolution of identity associated with the collection $Q$ (see Section 3). For $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, and a real number $A > 0$ let $\psi_A^Q(x) = (Ad(Q)^{-1})^d \psi(Ad(Q)^{-1}x)$.

Obviously, $|\hat{\psi}(\omega) - 1| \leq C|\omega|^\alpha$. Hence, from the Plancherel formula, we obtain
\begin{equation}
\int |\psi_A^Q * (\phi_Q f) - \phi_Q f|^2 \leq CA^{-2\alpha}d(Q)^{2\alpha} \int |\nabla^\alpha (\phi_Q f)|^2
\end{equation}
\begin{equation}
\leq CA^{-2\alpha}d(Q)^{2\alpha} \left( \int \phi_Q^2 |\nabla^\alpha f|^2 + \int ||\phi_Q, \nabla^\alpha f||^2 \right).
\end{equation}

Moreover, the $A^\infty$ condition for $V$ implies that there exist constants $C, \xi > 0$ such that for $\Omega_\varepsilon = \Omega_\varepsilon^Q = \{x \in Q^* : V(x) \leq \varepsilon d(Q)^{-2\alpha}\}$ we have $|\Omega_\varepsilon| \leq C\varepsilon^\xi|Q|$ independently of $Q$ and $\varepsilon$. Therefore
\begin{equation}
\int |\phi_Q^A * (\phi_Q f)|^2 \leq \|\phi_Q^A\|^2_{L^2} \|\phi_Q f\|^2_{L^1} \leq (Ad(Q)^{-1})^d \left( \int |\phi_Q f|^2 \right)^2
\end{equation}
\begin{equation}
\leq C(Ad(Q)^{-1})^d|\Omega_\varepsilon| \int |\phi_Q f|^2 + C\varepsilon^{-1}d(Q)^{2\alpha}A^d \int V|\phi_Q f|^2
\end{equation}
\begin{equation}
\leq CA^d\varepsilon^\xi \int |\phi_Q f|^2 + C\varepsilon^{-1}d(Q)^{2\alpha}A^d \int V|\phi_Q f|^2.
\end{equation}

Hence
\begin{equation}
\int |\phi_Q f|^2 \leq CA^{-2\alpha}d(Q)^{2\alpha} \int \phi_Q^2 |\nabla^\alpha f|^2 + CA^{-2\alpha}d(Q)^{2\alpha} \int ||\phi_Q, \nabla^\alpha f||^2
\end{equation}
\begin{equation}
+ CA^d\varepsilon^\xi \int |\phi_Q f|^2 + C\varepsilon^{-1}d(Q)^{2\alpha}A^d \int V|\phi_Q f|^2.
\end{equation}

Summing up over $Q \in Q$ we get
\begin{equation}
\int w|f|^2 \leq \sum_{Q \in Q} d(Q)^{-2\alpha} |\phi_Q f|^2
\end{equation}
\begin{equation}
\leq CA,\varepsilon \langle Lf, f \rangle + CA^{-2\alpha} \sum_{Q} ||\phi_Q, \nabla^\alpha f||^2
\end{equation}
\begin{equation}
\leq CA,\varepsilon \langle Lf, f \rangle + CA^{-2\alpha} \int w|f|^2,
\end{equation}
provided $CA^d\varepsilon^\xi \leq 1/2$. The last inequality has been deduced from the following lemma.

Lemma 9.6. The operator $Tf(x, Q) = [\phi_Q, \nabla^\alpha](w^{-1/2}f)(x)$ is bounded from $L^2(\mathbb{R}^d)$ into $l^2(L^2(\mathbb{R}^d))$. 
The theorem follows by fixing $A$ sufficiently large and then taking $\varepsilon > 0$ small enough.

**Proof of Lemma 9.6.** It suffices to prove that $T : L^1 \to l^1(L^1)$ and $T : L^{\infty} \to l^{\infty}(L^{\infty})$ and then interpolate.

The first statement follows from

$$\sum_Q \int \frac{\left| \phi_Q(x) - \phi_Q(y) \right|}{|x-y|^{d+\alpha}} d(y)^\alpha dx \leq C' \int_{|x-y| \leq C d(y)} \frac{d(y)^\alpha}{d(Q)|x-y|^{d+\alpha-1}} dx$$

$$+ C'' \int_{|x-y| \geq C d(y)} \frac{d(y)^\alpha}{|x-y|^{d+\alpha}} dx \leq C'',$$

where $C''$ is a constant independent of $y$. The second statement is a consequence of

$$\sup_{x,Q} \left\{ \int \frac{\left| \phi_Q(x) - \phi_Q(y) \right|}{|x-y|^{d+\alpha}} d(y)^\alpha dy \leq C' \int_{|x-y| \leq C d(x)} \frac{d(y)^\alpha}{d(Q)|x-y|^{d+\alpha-1}} dy$$

$$+ C'' \int_{|x-y| \geq C d(x)} \frac{d(y)^\alpha}{|x-y|^{d+\alpha}} dy \leq C'' ,$$

with a constant $C''$ independent of $x$ and $Q$. In the above estimates we have used (9.2). The proof of the lemma is complete.

**Remark.** Let us finally point out that Theorem 9.4 for $V$ being a non-negative polynomial could also be proved by using nilpotent Lie groups methods and maximal subelliptic estimates for accretive kernels proved by P. Głowacki (see [G]).

**References**


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