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The Lebesgue constants for the Franklin orthogonal system

by

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Abstract. To each set of knots $t_i = i/2n$ for $i = 0, \ldots, 2\nu$ and $t_i = (i-\nu)/n$ for $i = 2\nu+1, \ldots, n+\nu$, with $1 \le \nu \le n$, there corresponds the space $\mathcal{S}_{\nu,n}$ of all piecewise linear and continuous functions on $I = [0,1]$ with knots t_i and the orthogonal projection $P_{\nu,n}$ of $L^2(I)$ onto $\mathcal{S}_{\nu,n}$. The main result is

$$\lim_{(n-\nu)\wedge\nu\to\infty} \|P_{\nu,n}\|_1 = \sup_{\nu,n\,:\,1\le\nu\le n} \|P_{\nu,n}\|_1 = 2 + (2-\sqrt{3})^2.$$

This shows that the Lebesgue constant for the Franklin orthogonal system is $2+(2-\sqrt{3})^2$.

1. Introduction. Fourier expansions with respect to a complete orthonormal system in $L^2[0,1]$ can be used to approximate functions e.g. in $L^p[0,1], 1 \le p < \infty$, or in $C[0,1]$ in case $p = \infty$. Suppose that $(g_n : n \ge 0)$ is such an orthonormal system. We introduce the partial sums

$$G_n(f) = \sum_{k=0}^{n} (f, g_k) g_k$$

and the subspaces

$$\mathcal{G}_n = \operatorname{span}\{g_k : k = 0, \ldots, n\}.$$

Clearly, each $G_n : L^p[0,1] \to \mathcal{G}_n$ is a projection (it is orthogonal in case $p = 2$) and its L^1 norm is equal to

$$L_n = \operatorname*{ess\,sup}_{[0,1]} \int_0^1 |K_n(\cdot, s)|\, ds,$$

where the Dirichlet kernel K_n is given by

$$K_n(t,s) = \sum_{k=0}^{n} f_k(t) f_k(s), \qquad t, s \in [0,1].$$

The *Lebesgue constant* is now defined as

$$L = \sup_{n\ge0} L_n.$$

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For finite p, by Hölder's inequality, the L^p norm of G_n satisfies

$$\|G_n\|_p \le L_n \le L,$$

where L may be infinite. For the best approximation of $f \in L^p[0,1]$ by $\mathcal{G}_n$ we introduce the customary notation

$$E_{n,p}(f) = \inf_{g\in\mathcal{G}_n} \|f-g\|_p.$$

It is elementary to see that

$$E_{n,p}(f) \le \|f - G_n(f)\|_p \le (1+L_n)E_{n,p}(f) \le (1+L)E_{n,p}(f).$$

It follows from the results of A. M. Olevskiĭ [9] or S. V. Bochkarev [2] that for any uniformly bounded orthonormal system we have $L=\infty$. In such cases the asymptotic behaviour of L_n may be important. In case $L<\infty$ the numerical value of L is important.

In particular, we may consider a family of orthonormal complete systems on $[0,1]$ indexed by sequences of simple knots $\mathcal{T} = (t_n : n \ge 0)$ dense in $[0,1]$ with $t_0 = 0$, $t_1 = 1$ and $t_n \in (0,1)$ for $n > 1$. For $n \ge 1$, the space of piecewise linear and continuous functions on $[0,1]$ with knots at $t_0,\ldots,t_n$ is denoted by $S_n(\mathcal{T})$. Now the *general orthonormal Franklin system* (f_n) is defined as follows: $f_0 = 1$, $f_1(t) = \sqrt{3}(2t-1)$ and for $n \ge 2$ the function f_n is uniquely determined by the following conditions: $f_n \in S_n(\mathcal{T})$, $f_n \perp S_{n-1}(\mathcal{T})$, $\|f_n\|_2 = 1$ and $f_n(t_n) > 0$. It follows from the results of Z. Ciesielski [3] that for the corresponding Lebesgue constant we have $L = L(\mathcal{T}) \le 3$. Moreover, it was shown in K. Oskolkov [10] and P. Oswald [11] that the upper bound 3 is the best possible, i.e. $\sup_{\mathcal{T}} L(\mathcal{T}) = 3$. It is also immediate that $L(\mathcal{T}) \ge 5/3$ with strict inequality for each $\mathcal{T}$ with simple knots, and with equality for $\mathcal{T}$ with all knots of multiplicity 2, or if the set of interior knots of $\mathcal{T}$ is empty.

We prove that for the classical Franklin orthonormal system corresponding to the dyadic knots in $[0,1]$ we have $L = 2 + (2-\sqrt{3})^2$. This result was suggested by numerical results presented in Z. Ciesielski and E. Niedźwiecka [7]. In one case the problem was settled in Z. Ciesielski [5]: If there are $n+1$ equally spaced knots in $[0,1]$ and P_n denotes the (orthogonal) projection onto the corresponding space of continuous piecewise linear functions, then $\|P_n\|_1 < 2$ and $\|P_n\|_1 \to 2$ as $n\to\infty$. It was also shown in P. Bechler [1] that for the Franklin–Strömberg wavelet (Franklin system on $\mathbb{R}$) the Lebesgue constant is $2 + (2-\sqrt{3})^2$. It turns out that this result can also be obtained from our result on $[0,1]$.

Finally, we mention that for higher order splines, of order k, according to A. Yu. Shadrin [12], the general spline orthogonal systems corresponding to $\mathcal{T}$ have finite Lebesgue constants $L(\mathcal{T},k)$ bounded uniformly in $\mathcal{T}$, i.e. for each $k \ge 1$,

$$L_k = \sup_{\mathcal{T}} L(\mathcal{T},k) < \infty.$$

2. Preliminaries. For $n \geq 1$ let $\pi = (t_i : i = -1, 0, \ldots, n, n+1)$ be a given sequence of knots in $I = [0,1]$: $t_i < t_{i+1}$ for $i = 0, \ldots, n-1$ and $t_{-1} = t_0 = 0$, $t_n = t_{n+1} = 1$. Set $I_i = (t_{i-1}, t_i)$ and $\delta_i = |I_i| = t_i - t_{i-1}$. The *diameter* of the partition π is $|\pi| = \sup\{\delta_i : i = 1, \ldots, n\}$. Let $\mathcal{S}_\pi$ be the space of continuous, real-valued functions defined on I and linear on each interval I_i, $i = 1, \ldots, n$.

The space $\mathcal{S}_\pi$ has a natural basis N_i, $i = 0, \ldots, n$, where

$$N_i(t) = \begin{cases} (t - t_{i-1})/\delta_i & \text{for } t_{i-1} \leq t \leq t_i, \\ (t_{i+1} - t)/\delta_{i+1} & \text{for } t_i \leq t \leq t_{i+1}, \\ 0 & \text{for } t \leq t_{i-1} \text{ or } t \geq t_{i+1}. \end{cases} \tag{2.1}$$

Note that $\operatorname{supp} N_i = [t_{i-1}, t_{i+1}]$; therefore, for any reals $(a_0, \ldots, a_n)$, the sum

$$\sum_{i=0}^{n} a_i N_i(t)$$

is in $\mathcal{S}_\pi$ and for each $t \in I$ it contains at most two non-zero terms. Moreover,

$$\sum_{i=0}^{n} N_i(t) = 1 \quad \text{for } t \in I.$$

Set $\nu_i = (1, N_i)$ and $b_{i,j} = (N_i, N_j)$, where $(f,g) = \int_I f(t)g(t)\,dt$. Observe that for $i, k = 0, \ldots, n$ we get $\nu_i = (\delta_i + \delta_{i+1})/2$ and

$$b_{i,k} = \begin{cases} 0 & \text{for } |i-k| > 1, \\ \delta_i/6 & \text{for } k = i-1, \\ (\delta_i + \delta_{i+1})/3 & \text{for } k = i, \\ \delta_{i+1}/6 & \text{for } k = i+1. \end{cases} \tag{2.2}$$

We use the notation $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$. The Gram matrix $\mathbf{B} = [b_{i,j} : i,j = 0, \ldots, n]$ is a three-diagonal band matrix. Its inverse $\mathbf{A} = \mathbf{B}^{-1} = [a_{i,j} : i,j = 0, \ldots, n]$ has a number of important properties (cf. [8]):

$$a_{i,j} = a_{j,i}, \qquad a_{i,j} = (-1)^{i+j}|a_{i,j}|, \tag{2.3}$$

$$2|a_{i-1,j}| \leq |a_{i,j}| \quad \text{for } i \leq j, \tag{2.4}$$

$$|a_{i,j}| \geq 2|a_{i+1,j}| \quad \text{for } j \leq i, \tag{2.5}$$

$$|a_{i,j}| \leq \frac{2}{2^{|i-j|}} \cdot \frac{1}{\nu_i \vee \nu_j}, \tag{2.6}$$

$$\sum_i \nu_i |a_{i,j}| = 3. \tag{2.7}$$

Now, let

$$K_\pi(t,s) = \sum_{i,j=0}^{n} a_{i,j} N_i(t) N_j(s) \quad \text{for } t, s \in I. \tag{2.8}$$

It follows that

$$a_{i,j} = K_\pi(t_i, t_j) \quad \text{for } i, j = 0, \ldots, n. \tag{2.9}$$

The operator P_π defined by

$$P_\pi f(t) = \int_I K_\pi(t, s) f(s)\, ds \tag{2.10}$$

is the orthogonal projection of $L^2(I)$ onto $\mathcal{S}_\pi$. It can be seen that for its L^1 norm we have the formula

$$\|P_\pi\|_1 = \sup_{0\le k\le n} \int_I |K_\pi(t_k, s)|\, ds \tag{2.11}$$

and, since $\operatorname{sgn} a_{i,k} = -\operatorname{sgn} a_{i-1,k}$,

$$\int_I |K_\pi(t_k, s)|\, ds = \frac{1}{2}\sum_{i=1}^{n} \delta_i \,\frac{|a_{i-1,k}|^2 + |a_{i,k}|^2}{|a_{i-1,k}| + |a_{i,k}|}. \tag{2.12}$$

Formula (2.12) can be rewritten as a mean value with respect to the weights

$$p_{i,k} = \frac{\delta_i}{6}\,(|a_{i,k}| + |a_{i-1,k}|) \quad \text{for } i = 1, \ldots, n, \tag{2.13}$$

which by (2.7) satisfy

$$\sum_{i=1}^{n} p_{i,k} = 1. \tag{2.14}$$

Now, if $\phi(t) = 3(1+t^2)/(1+t)^2$, then

$$\int_I |K_\pi(t_k, s)|\, ds = \sum_{i=1}^{n} p_{i,k}\, \phi\bigg(\frac{|a_{i,k}|}{|a_{i-1,k}|}\bigg). \tag{2.15}$$

Since the rational function $\phi(t)$ is important for the rest of the paper, its properties are collected in

LEMMA 2.1. *The function* $\phi(t) = 3(1+t^2)/(1+t)^2$ *has the following properties*:

$$\phi(t) = \phi(1/t) \quad \textit{for } t > 0,$$

ϕ *is increasing on* $[1, \infty)$ *and*

$$\phi(1) = 3/2 < \phi(2) = 5/3 < \phi(2+\sqrt{3}) = 2 < \phi(\infty) = 3.$$

Moreover,

$$2u + \sqrt{3}(u-1) < \phi(2 - \sqrt{3}\,u) < 2u \quad \textit{for } \sqrt{3}/2 < u < 1, \tag{2.16}$$

and for $u < 1$ *and close to* 1 *we have*

$$\phi(2 - \sqrt{3}\,u) = 2u + o(1). \tag{2.17}$$

Since $3 > \phi(t) \ge 5/3$ for $t \in [2, \infty)$ it follows from (2.15) by (2.4) and (2.5) that

$$5/3 \le \|P_\pi\|_1 < 3, \tag{2.18}$$

and it is known that the constants $5/3$ and 3 are the best possible (see [10] and [11]). The constant $5/3$ is attained for knots of multiplicity 2 each or for $n = 1$, and 3 can be approached by knots distributed geometrically.

Our aim is to prove that the Lebesgue constant for the classical orthogonal Franklin system is $2 + (2 - \sqrt{3})^2$. For this, we need delicate numerical estimates for (2.15) in the case of dyadic knots.

3. The norms of orthogonal projections on $[0, 1]$

3.1. *The case of equally spaced knots.* In this section we assume that $n \geq 1$ and consider the particular knots $\pi_n = (t_i : i = -1, \ldots, n+1)$ with $t_i = i/n$ for $i = 0, \ldots, n$, i.e. $\delta_i = 1/n$ for $i = 1, \ldots, n$. For $\mathcal{S}_{\pi_n}$ and P_{π_n} we simply write $\mathcal{S}_n$ and P_n, respectively. Moreover, $\nu_0 = \nu_n = 1/2n$ and $\nu_i = 1/n$ for $0 < i < n$. In this case we have a fairly simple formula for the inverse matrix $\mathbf{A}$. To write the formula for its entries it is convenient to introduce two sequences of integers: $(A(n) : n = 0, 1, \ldots)$ and $(B(n) : n = 0, 1, \ldots)$ defined by the following recurrence relations:

$$\begin{cases} A(n+1) = 2A(n) + 3B(n) & \text{with } A(0) = 1, \\ B(n+1) = A(n) + 2B(n) & \text{with } B(0) = 0. \end{cases} \tag{3.1}$$

In particular, $A(1) = 2$, $A(2) = 7$, $A(3) = 26$, $B(1) = 1$ and $B(2) = 4$. Explicit formulas for $A(n)$ and $B(n)$ are sometimes helpful: if α is the positive solution to the equation $\cosh \alpha = 2$, then $A(n) = \cosh(n\alpha)$ and $\sqrt{3}\, B(n) = \sinh(n\alpha)$. Note that $\exp(\alpha) = 2 + \sqrt{3}$ and $\exp(-\alpha) = 2 - \sqrt{3}$; the notation $q = 2 + \sqrt{3}$ will occasionally be used later on as well.

PROPOSITION 3.1 ([4]). *If $n \geq 1$, then for $i, k = 0, \ldots, n$,*

$$a_{i,k} = \frac{2n}{B(n)} (-1)^{i+k} A(i \wedge k) A(n - i \vee k). \tag{3.2}$$

This result can be used to prove

COROLLARY 3.2 ([5]). *For each $n \geq 1$ we have $\|P_n\|_1 < 2$. Moreover,*

$$\lim_{n \to \infty} \|P_n\|_1 = 2. \tag{3.3}$$

For completeness, we present the proof of Corollary 3.2, given in [6] (cf. the proof of Theorem 5.2 in [6]): $\phi(t) = \phi(1/t)$, $\phi(2 + \sqrt{3}) = 2$ and $\phi(t) < 2$ for $2 - \sqrt{3} < t < 2 + \sqrt{3}$. Moreover, for $0 \leq k \leq n$ we have

$$\phi\left(\frac{|a_{i,k}|}{|a_{i-1,k}|}\right) = \begin{cases} \phi\left(\dfrac{A(i)}{A(i-1)}\right) & \text{for } 1 \leq i \leq k, \\[2ex] \phi\left(\dfrac{A(n-i+1)}{A(n-i)}\right) & \text{for } k < i \leq n\,. \end{cases} \tag{3.4}$$

But, according to (3.1),

$$2 \le \frac{A(m+1)}{A(m)} = 2 + \sqrt{3}\tanh(\alpha m) \quad \text{for } m \ge 0. \tag{3.5}$$

These inequalities and formula (2.15) imply the first part of the statement of Corollary 3.2. For the proof of (3.3) let us fix m and take k, n such that $m < k \le n - m$. Then

$$\begin{aligned} \|P_n\|_1 &\ge \sum_{i=m+1}^{k} p_{i,k}\,\phi(2+\sqrt{3}\tanh(\alpha(i-1))) \\ &\quad + \sum_{i=k+1}^{n-m} p_{i,k}\,\phi(2+\sqrt{3}\tanh(\alpha(n-i))) \\ &\ge \phi(2+\sqrt{3}\tanh(\alpha m)) \times \sum_{i=m+1}^{n-m} p_{i,k} \\ &\ge \phi(2+\sqrt{3}\tanh(\alpha m)) \times \Big(1 - \Big(\sum_{i=1}^{m} + \sum_{i=n-m+1}^{n}\Big) p_{i,k}\Big). \end{aligned}$$

Now, choosing $k = \lfloor n/2 \rfloor$ and using (2.13) and (3.2) we find that

$$\Big(\sum_{i=1}^{m} + \sum_{i=n-m+1}^{n}\Big) p_{i,k} \to 0 \quad \text{as } n \to \infty,$$

whence

$$\liminf_{n\to\infty} \|P_n\|_1 \ge \phi(2+\sqrt{3}\tanh(\alpha m)).$$

Now, letting $m \to \infty$ and applying the first part of Corollary 3.2 completes the proof.

3.2. *Partially equally spaced knots.* The set of knots $\pi_{\nu,n} = (t_i : i = -1, \ldots, n+1)$ is determined by the two integer parameters $n \ge 1$ and $1 \le \nu \le n$ and it is defined as follows: $t_{-1} = 0$, $t_{n+1} = 1$ and

$$t_i = \begin{cases} i/2n & \text{for } i = 0, \ldots, 2\nu, \\ (i-\nu)/n & \text{for } i = 2\nu+1, \ldots, N, \end{cases} \tag{3.6}$$

where $N = n+\nu$ and $1 \le \nu \le n$. These knots are obtained as follows: we take a uniform partition $\pi_n = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, and add to it middle points of the first ν intervals, i.e. the points $\{1/2n, 3/2n, \ldots, (2\nu-1)/2n\}$. Note that the case $n = \nu$ corresponds to the equally spaced knots π_{2n}. As this case was treated separately, it is assumed in what follows, if necessary, that $\nu < n$. We call knots of this kind *partially equally spaced.* For simplicity the orthogonal projection $P_{\pi_{\nu,n}}$ is denoted by $P_N = P_{\nu,n}$ and the space $\mathcal{S}_{\pi_{\nu,n}}$ by $\mathcal{S}_N = \mathcal{S}_{\nu,n}$. To any such partially equally spaced knots corresponds the inverse matrix $\mathbf{A} = \mathbf{A}_N = \mathbf{A}_{\nu,n}$ and in what follows the dependence on $\{\nu, n\}$ will be suppressed in notation.

Our main goal is to prove

THEOREM 3.3. *For each integer $n \geq 1$,*

$$\sup_{\nu\,:\,1\leq\nu\leq n} \|P_{\nu,n}\|_1 < D = 2 + (2-\sqrt{3})^2. \tag{3.7}$$

Moreover,

$$\lim_{(n-\nu)\wedge\nu\to\infty} \|P_{\nu,n}\|_1 = D. \tag{3.8}$$

Before starting the proof of Theorem 3.3 some auxiliary results need to be recalled or established. For the matrix $\mathbf{A}$ we have an explicit formula (see [4]):

PROPOSITION 3.4. *If $N = n + \nu$ with $1 \leq \nu \leq n$, then*

$$a_{i,k} = \frac{2n}{C(N)}(-1)^{i+k}\varepsilon_{i,k}, \tag{3.9}$$

where $C(N) = B(N) + B(N-2\nu)A(2\nu)$ and for $0 \leq i, k \leq N$,

$$\varepsilon_{i,k} = \begin{cases} 2A(i\wedge k)\cdot K & \text{if } i\vee k \leq 2\nu, \\ A(N - i\vee k)\cdot L & \text{if } i\wedge k > 2\nu, \\ 2A(i\wedge k)A(N-i\vee k) & \text{if } i\wedge k \leq 2\nu < i\vee k, \end{cases} \tag{3.10}$$

where $K = A(N - i\vee k) + 3B(N-2\nu)B(2\nu - i\vee k)$ and $L = A(i\wedge k) + A(2\nu)A(i\wedge k - 2\nu)$.

LEMMA 3.5. *Let*

$$\alpha(s,t) = A(s+t) + 3B(s)B(t), \qquad \beta(s,t) = \alpha(s,t+1)/\alpha(s,t).$$

Then

$$2\alpha(s,t) = 3A(s+t) - A(s-t), \tag{3.11}$$

$$\beta(s,t) \geq \beta(s,t+1) > 1 \quad \text{for } s \geq 1,\ t \geq 0, \tag{3.12}$$

$$1 < \beta(s,t) \leq \beta(s+1,t) \quad \text{for } s \geq 1,\ t \geq 0. \tag{3.13}$$

LEMMA 3.6. *Let $\phi(t) = 3(1+t^2)/(1+t)^2$ for $t > 0$. Then*

$$\beta(s,0) \nearrow \beta_0 = 2 + 2\sqrt{3} \quad \text{as } s \nearrow \infty, \tag{3.14}$$

$$\phi(\beta_0) = D + 2(2-\sqrt{3})^2, \tag{3.15}$$

$$\beta(s,1) \nearrow \beta_1 = \frac{7+8\sqrt{3}}{2+2\sqrt{3}} \quad \text{as } s \nearrow \infty, \tag{3.16}$$

$$\phi(\beta_1) = \frac{257+120\sqrt{3}}{127+60\sqrt{3}} < D. \tag{3.17}$$

LEMMA 3.7. *For $s > 0$ and $t > 0$ we have*

$$1 < \frac{A(s+t+1) + A(s)A(t+1)}{A(s+t) + A(s)A(t)} \le 2 + \sqrt{3}. \tag{3.18}$$

The elementary proofs of Lemmas 3.5, 3.6 and 3.7 will be omitted.

To simplify notation we introduce

$$\gamma_{i,k} = \frac{\varepsilon_{i,k}}{\varepsilon_{i-1,k}} \quad \text{for } i = 1, \ldots, N \text{ and } k = 0, \ldots, N, \tag{3.19}$$

where the $\varepsilon_{i,k}$ are as in (3.10), and

$$D_{k,N} = \sum_{i=1}^{N} p_{i,k}\, \phi(\gamma_{i,k}) \quad \text{for } k = 0, \ldots, N. \tag{3.20}$$

Now, using (2.11), (2.12) and (2.15) we find that

$$\|P_N\|_1 = \|P_N\|_\infty = \sup_{0 \le k \le N} D_{k,N}. \tag{3.21}$$

In the proof of Theorem 3.3 the following Lemmas 3.8 and 3.9 are crucial.

LEMMA 3.8. *Let $N = n + \nu$ with $1 \le \nu < n$. Then*

$$D_{2\nu-1,N} < D. \tag{3.22}$$

Proof. We split the sum (3.20) with $k = 2\nu - 1$ into three pieces:

$$D_{2\nu-1,N} = \Big(\sum_{i=1}^{2\nu-1} + \sum_{i=2\nu} + \sum_{i=2\nu+1}^{N} \Big) p_{i,2\nu-1} \phi(\gamma_{i,2\nu-1}). \tag{3.23}$$

The terms of these sums can be handled with the help of formulae (2.13), (3.9), (3.10) and (3.19). In particular for $1 \le i \le 2\nu - 1$ we get

$$\begin{aligned} C(N) p_{i,2\nu-1}\, \phi(\gamma_{i,2\nu-1}) &= \frac{1}{6} (\varepsilon_{i-1,2\nu-1} + \varepsilon_{i,2\nu-1})\, \phi(\gamma_{i,2\nu-1}) \\ &= (A(N-2\nu+1) + 3B(N-2\nu)) \phi\Big(\frac{A(i)}{A(i-1)} \Big) \frac{A(i-1) + A(i)}{3} \\ &< (A(N-2\nu+1) + 3B(N-2\nu)) \phi\Big(\frac{A(2\nu-1)}{A(2\nu-2)} \Big) (B(i) - B(i-1)). \end{aligned} \tag{3.24}$$

The last inequality follows from the monotonicity of $A(m+1)/A(m)$ (cf. (3.5)) and the identity $A(i-1) + A(i) = 3(B(i) - B(i-1))$, which in turn can be obtained from (3.1). Now, Lemma 2.1 gives in particular

$$\phi\Big(\frac{A(m)}{A(m-1)} \Big) < 2 \frac{\sqrt{3}\, B(m)}{A(m)} \quad \text{for } m \ge 1. \tag{3.25}$$

The combination of (3.25) and (3.24) gives the upper bound for the first sum of (3.23):

$$\Sigma_1 = \sum_{i=1}^{2\nu-1} p_{i,2\nu-1}\phi(\gamma_{i,2\nu-1}) < 4\sqrt{3}\cdot\frac{A(N-2\nu)+3\,B(N-2\nu)}{C(N)}\cdot\frac{B(2\nu-1)^2}{A(2\nu-1)}. \tag{3.26}$$

For the second sum in (3.23) we have, by Proposition 3.4,

$$\Sigma_2 = p_{2\nu,2\nu-1}\phi(\gamma_{2\nu,2\nu-1}) = \frac{1}{2\,C(N)}\cdot\frac{\varepsilon^2_{2\nu-1,2\nu}+\varepsilon^2_{2\nu-1,2\nu-1}}{\varepsilon_{2\nu-1,2\nu}+\varepsilon_{2\nu-1,2\nu-1}} = \frac{A(2\nu-1)}{C(N)}\cdot\frac{4(A(N-2\nu)+3B(N-2\nu))^2+A(N-2\nu)^2}{3(A(N-2\nu)+2B(N-2\nu))}. \tag{3.27}$$

To obtain the estimate for the third sum of (3.23) we proceed very much like in the first case:

$$\Sigma_3 = \sum_{i=2\nu+1}^{N} p_{i,2\nu-1}\phi(\gamma_{i,2\nu-1}) = \frac{1}{3}\sum_{i=2\nu+1}^{N}(\varepsilon_{i-1,2\nu-1}+\varepsilon_{i,2\nu-1})\phi(\gamma_{i,2\nu-1}) < 4\sqrt{3}\cdot\frac{A(2\nu-1)B(N-2\nu)^2}{C(N)A(N-2\nu)}. \tag{3.28}$$

Moreover,

$$C(N) = A(2\nu-1)A(N-2\nu)+2B(2\nu-1)A(N-2\nu) + 4A(2\nu-1)B(N-2\nu)+6B(2\nu-1)B(N-2\nu). \tag{3.29}$$

To continue the proof we introduce new variables $s = 2\nu-1$, $t = N-2\nu$, $x = A(t)$, $y = B(t)$, $u = A(s)$ and $z = B(s)$. Now,

$$C(N)\Sigma_1 < \frac{f_1(x,y,u,z)}{f_2(x,y,u,z)},$$

where

$$f_1(x,y,u,z) = 4\sqrt{3}\,(x+3y)z^2, \qquad f_2(x,y,u,z) = u;$$

moreover,

$$C(N)\Sigma_2 < \frac{g_1(x,y,u,z)}{g_2(x,y,u,z)},$$

where

$$g_1(x,y,u,z) = u(x^2+4(x+3y)^2), \qquad g_2(x,y,u,z) = 3(x+2y);$$

and finally,

$$C(N)\Sigma_3 < \frac{h_1(x,y,u,z)}{h_2(x,y,u,z)},$$

where

$$h_1(x,y,u,z) = 4\sqrt{3}\,uy^2, \qquad h_2(x,y,u,z) = x.$$

In addition we have

$$C(N) = C(x, y, u, z) = xu + 2xz + 4uy + 6zy.$$

Moreover, let us introduce the function $m = m(x, y, u, z)$ defined as

$$m = (9 - 4\sqrt{3}) \cdot C \cdot f_2 \cdot g_2 \cdot h_2 - f_1 \cdot g_2 \cdot h_2 - f_2 \cdot g_1 \cdot h_2 - f_2 \cdot g_2 \cdot h_1.$$

Now, to prove (3.22) it is enough to show the positivity of the rational function

$$r(a, b) = m\left(\frac{a + 1/a}{2}, \frac{a - 1/a}{2\sqrt{3}}, \frac{b + 1/b}{2}, \frac{b - 1/b}{2\sqrt{3}}\right)$$

for $a = (2 + \sqrt{3})^t \geq 2 + \sqrt{3}$ and $b = (2 + \sqrt{3})^s \geq 2 + \sqrt{3}$, or of the function $w(a, b) = r(a + 2 + \sqrt{3}, b + 2 + \sqrt{3})$ for $a \geq 0$ and $b \geq 0$. However, with the help of MATHEMATICA we find that the numerator of the rational function $w(a, b)$ is equal to

$$\begin{aligned}
&555776 + 320768\sqrt{3} + 784320a + 452800\sqrt{3}a + 463872a^2 + 267776\sqrt{3}a^2\\
&\quad + 148640a^3 + 85856\sqrt{3}\,a^3 + 27528a^4 + 15928\sqrt{3}\,a^4 + 2832a^5 + 1632\sqrt{3}\,a^5\\
&\quad + 128a^6 + 72\sqrt{3}\,a^6 + 406016b + 234368\sqrt{3}\,b + 547008ab + 315808\sqrt{3}\,ab\\
&\quad + 306768a^2b + 177104\sqrt{3}\,a^2b + 93056a^3b + 53744\sqrt{3}\,a^3b + 16440a^4b\\
&\quad + 9508\sqrt{3}\,a^4b + 1644a^5b + 948\sqrt{3}\,a^5b + 74a^6b + 42\sqrt{3}\,a^6b + 104128b^2\\
&\quad + 60032\sqrt{3}\,b^2 + 127888ab^2 + 73824\sqrt{3}\,ab^2 + 63416a^2b^2 + 36600\sqrt{3}\,a^2b^2\\
&\quad + 16456a^3b^2 + 9536\sqrt{3}\,a^3b^2 + 2446a^4b^2 + 1444\sqrt{3}\,a^4b^2 + 222a^5b^2\\
&\quad + 126\sqrt{3}\,a^5b^2 + 11a^6b^2 + 5\sqrt{3}\,a^6b^2 + 13312b^3 + 7680\sqrt{3}\,b^3\\
&\quad + 14624ab^3 + 8448\sqrt{3}\,ab^3 + 5968a^2b^3 + 3456\sqrt{3}\,a^2b^3 + 1072a^3b^3\\
&\quad + 624\sqrt{3}\,a^3b^3 + 68a^4b^3 + 44\sqrt{3}\,a^4b^3 + 896b^4 + 512\sqrt{3}\,b^4\\
&\quad + 976ab^4 + 568\sqrt{3}\,ab^4 + 392a^2b^4 + 236\sqrt{3}\,a^2b^4 + 68a^3b^4\\
&\quad + 44\sqrt{3}\,a^3b^4 + a^4b^4 + 5\sqrt{3}\,a^4b^4,
\end{aligned}$$

and its denominator is equal to

$$8(2 + \sqrt{3} + a)^3(2 + \sqrt{3} + b)^2.$$

Consequently, $w(a, b)$ is positive for $a, b \geq 0$, which completes the proof of (3.22).

LEMMA 3.9. *Let* $1 \leq \nu < n$ *and* $k < 2\nu - 1$. *Then*

$$p_{2\nu-1,k}\phi(\beta_1) + p_{2\nu,k}\,\phi(\beta_0) \leq D(p_{2\nu-1,k} + p_{2\nu,k}), \tag{3.30}$$

where β_0 *and* β_1 *are as in Lemma* 3.6.

Proof. It follows by (2.13), Proposition 3.4, (3.6) and (3.1) that for $k < 2\nu - 1$,

$$p_{2\nu-1,k} = \frac{A(k)}{3C(N)}\,(2A(N-2\nu) + 6B(N-2\nu)), \tag{3.31}$$

$$p_{2\nu,k} = \frac{A(k)}{3C(N)}\,A(N-2\nu) \tag{3.32}$$

$$p_{2\nu-1,k} + p_{2\nu,k} = \frac{A(k)}{3C(N)}\,(3A(N-2\nu) + 6B(N-2\nu)). \tag{3.33}$$

Thus (3.30) is equivalent to

$$\begin{aligned}(2A(N-2\nu) + 6B(N-2\nu))\phi(\beta_1) + A(N-2\nu)\phi(\beta_0)&\\ < D(3A(N-2\nu) + 6B(N-2\nu)).&\end{aligned} \tag{3.34}$$

Since $t = N - 2\nu \geq 1$ and $\tanh(\alpha t) = \sqrt{3}\,B(t)/A(t)$, we get in particular $\tanh(\alpha) = \sqrt{3}\,B(1)/A(1) = \sqrt{3}/2$. Thus, inequality (3.34) is equivalent to

$$(2 + 2\sqrt{3}\tanh(\alpha t))\phi(\beta_1) + \phi(\beta_0) < D(3 + 2\sqrt{3}\tanh(\alpha t)). \tag{3.35}$$

Introducing the new variable $y = 2 + 2\sqrt{3}\tanh(\alpha t)$ we get an equivalent inequality

$$y(D - \phi(\beta_1)) > \phi(\beta_0) - D \quad \text{for } y \geq 5, \tag{3.36}$$

which, by (3.17), is equivalent to

$$\frac{5}{6}\,\phi(\beta_1) + \frac{1}{6}\,\phi(\beta_0) < D. \tag{3.37}$$

Now, by Lemma 3.6, it follows that

$$\frac{5}{6}\,\phi(\beta_1) + \frac{1}{6}\,\phi(\beta_0) = 3\,\frac{341 + 76\sqrt{3}}{(9 + 10\sqrt{3})^2} < D = 9 - 4\sqrt{3}\,,$$

and the proof is complete.

Proof of Theorem 3.3. Since $P_{2n} = P_{n,n}$, we may assume by Corollary 3.2 that $1 \leq \nu < n$ or equivalently that $1 < 2\nu < N$. Moreover, according to (2.11), (2.12), (3.20) and by Lemma 3.8 we need to check for fixed N and $0 \leq k \leq N$ that

$$D_{k,N} < D \quad \text{for } k \neq 2\nu - 1. \tag{3.38}$$

Consider the following cases: (A) $k \leq 2\nu$ and (B) $k > 2\nu$. In case (A) the set of indices $E = \{0, \ldots, N\}$ in the sum (3.20) can be split as $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{i \in E : i \leq k\}, \quad E_2 = \{i \in E : k < i \leq 2\nu\}, \quad E_3 = \{i \in E : i > 2\nu\}.$$

Since we are in case (A), it follows from (3.10) that

$$\gamma_{i,k} = \begin{cases} A(i)/A(i-1) & \text{for } i \in E_1,\\ A(N-i)/A(N-i+1) & \text{for } i \in E_3;\end{cases}$$

this implies $2-\sqrt{3} < \gamma_{i,k} < 2+\sqrt{3}$, whence

$$\phi(\gamma_{i,k}) < 2 \quad \text{for } i \in E_1 \cup E_3. \tag{3.39}$$

In particular, if in case (A) the set E_2 is empty, i.e. $k = 2\nu$, then the proof is complete. Now, since $2\nu < N$ we get $s = N - 2\nu \geq 1$ and for $i \in E_2$ we have $t = 2\nu - i \geq 0$. Thus, for $i \in E_2$ we deduce from (3.10) that

$$\phi(\gamma_{i,k}) = \phi(\beta(s,t)), \tag{3.40}$$

where $\beta(s,t)$ is as in Lemma 3.5. Applying Lemma 3.6 we find that for $k < i \leq 2\nu - 1$,

$$\phi(\gamma_{i,k}) \leq \phi(\beta(s,1)) \nearrow \phi(\beta_1) < D \quad \text{as } s \nearrow \infty, \tag{3.41}$$

$$\phi(\gamma_{2\nu,k}) = \phi(\beta(s,0)) \nearrow \phi(\beta_0) = D + 2(2-\sqrt{3})^2 \quad \text{as } s \nearrow \infty. \tag{3.42}$$

Now, (3.39) gives

$$\sum_{i \in E_1 \cup E_3} \phi(\gamma_{i,k}) p_{i,k} \leq 2 \sum_{i \in E_1 \cup E_3} p_{i,k}. \tag{3.43}$$

Moreover, (3.41), (3.42) and Lemma 3.9 imply

$$\begin{aligned} \sum_{i \in E_2} \phi(\gamma_{i,k}) p_{i,k} &= \sum_{i=k+1}^{2\nu-2} \phi(\gamma_{i,k}) p_{i,k} + \phi(\gamma_{2\nu-1,k}) p_{2\nu-1,k} + \phi(\gamma_{2\nu,k}) p_{2\nu,k} \\ &\leq \phi(\beta_1) \sum_{i=k+1}^{2\nu-2} p_{i,k} + \phi(\beta_1) p_{2\nu-1,k} + \phi(\beta_0) p_{2\nu,k} \leq D \sum_{i \in E_2} p_{i,k}. \end{aligned} \tag{3.44}$$

Adding inequalities (3.43) and (3.44) we obtain $D_{k,N} < D$, which completes the proof in case (A). In case (B) we find that

$$\gamma_{i,k} = \begin{cases} \dfrac{A(i)}{A(i-1)} & \text{for } i \leq 2\nu, \\[2ex] \dfrac{A(i) + A(2\nu)A(i-2\nu)}{A(i-1) + A(2\nu)A(i-1-2\nu)} & \text{for } 2\nu < i \leq k, \\[2ex] \dfrac{A(N-i)}{A(N-i+1)} & \text{for } i > k; \end{cases}$$

according to Lemma 3.7 we get $2-\sqrt{3} < \gamma_{i,k} < 2+\sqrt{3}$ and consequently $\phi(\gamma_{i,k}) < 2$ for all $i \in E$. Thus, $D_{k,N} < 2$ for $k > 2\nu$ and the proof of (3.7) is complete.

For the lower estimate we have, for given natural m and for large enough $(n-\nu) \wedge \nu$,

$$D_{2\nu-1,N} \geq \sum_{i=m+1}^{N-m} p_{i,2\nu-1} \phi(\gamma_{i,2\nu-1}), \tag{3.45}$$

where according to (3.10),

$$\gamma_{i,2\nu-1} = \begin{cases} A(i)/A(i-1) & \text{for } i = 1, \dots, 2\nu - 1, \\ A(n+\nu-i)/A(n+\nu+1-i) & \text{for } i = 2\nu+1, \dots, n+\nu. \end{cases}$$

Now,

$$\sum_{i=m+1}^{2\nu-1} p_{i,2\nu-1}\phi(\gamma_{i,2\nu-1}) \geq \phi\left(\frac{A(m+1)}{A(m)}\right) \sum_{i=m+1}^{2\nu-1} p_{i,2\nu-1},$$

$$\sum_{i=2\nu+1}^{N-m} p_{i,2\nu-1}\phi(\gamma_{i,2\nu-1}) \geq \phi\left(\frac{A(n+\nu-m)}{A(n+\nu-m-1)}\right) \sum_{i=2\nu+1}^{n+\nu-m} p_{i,2\nu-1},$$

$$p_{2\nu,2\nu-1} = \frac{A(2\nu-1)B(n-\nu+1)}{B(n+\nu)+B(n-\nu)A(2\nu)} \to \frac{1}{3} \quad \text{as } (n-\nu)\wedge\nu \to \infty,$$

$$\phi(\gamma_{2\nu,2\nu-1}) = \phi(\beta(n-\nu,0)) \nearrow \phi(\beta_0) = D + 2(2-\sqrt{3})^3 \quad \text{as } (n-\nu)\wedge\nu \to \infty.$$

Moreover, we have identity (2.14). In addition for fixed m there is a K_m such that

$$\max_{1\leq i\leq m} p_{i,2\nu-1} \leq K_m(2-\sqrt{3})^{2\nu} \quad \text{for } 2\nu > m.$$

Thus,

$$\sum_{i=1}^{m} p_{i,2\nu-1} \leq mK_m(2-\sqrt{3})^{2\nu} \quad \text{for } 2\nu > m.$$

Similarly,

$$\sum_{i=N-m+1}^{N} p_{i,2\nu-1} \leq mK_m(2-\sqrt{3})^{n-\nu} \quad \text{for } n-\nu > m.$$

It now follows from (3.45) and Lemma 2.1 that with some $\eta_k = o(1)$ for large k we get

$$\begin{aligned} D_{2\nu-1,N} &\geq (\phi(\gamma_{2\nu,2\nu-1}) - (2-\eta_{m+1}\vee\eta_{n+\nu-m}))p_{2\nu,2\nu-1} \\ &\quad + (2-\eta_{m+1}\vee\eta_{n+\nu-m}) \sum_{i=m+1}^{N-m} p_{i,2\nu-1} \\ &\geq (\phi(\gamma_{2\nu,2\nu-1}) - (2-\eta_{m+1}\vee\eta_{n+\nu-m}))p_{2\nu,2\nu-1} \\ &\quad + (2-\eta_{m+1}\vee\eta_{n+\nu-m}) - 4mK_m(2-\sqrt{3})^{(n-\nu)\wedge\nu}. \end{aligned}$$

Letting $(n-\nu)\wedge\nu \to \infty$ while m remains fixed we obtain

$$\liminf_{(n-\nu)\wedge\nu\to\infty} D_{2\nu-1,N} \geq (\phi(\beta_0) - (2-\eta_{m+1}))\frac{1}{3} + (2-\eta_{m+1}) = D - \frac{2}{3}\eta_{m+1},$$

and this completes the proof.

4. The norms of orthogonal projections on $\mathbb{R}$. P. Bechler [1] has proved that the norm of the orthogonal projection onto piecewise linear functions on $\mathbb{R}$ with integer knots is 2, and that the Lebesgue constant for the Franklin–Strömberg wavelet is $2 + (2 - \sqrt{3})^2$. The aim of this section is to show how these results on $\mathbb{R}$ can be obtained from the result on $[0, 1]$. Namely, the entries of the inverse to the corresponding Gram matrix on $\mathbb{R}$ are obtained as limits of the corresponding entries on $[0, 1]$ (cf. Propositions 4.1 and 4.4).

4.1. *Equally spaced knots on* $\mathbb{R}$. In this section we consider the set of integer knots $\{t_i = i : i \in \mathbb{Z}\}$. The corresponding piecewise linear continuous B-splines in this case have a simple formula

$$N_i(t) = N_0(t - i) \quad \text{for } i \in \mathbb{Z},\ t \in \mathbb{R},$$

where

$$N_0(t) = (1 - |t|) \vee 0 \quad \text{for } t \in \mathbb{R}. \tag{4.1}$$

Let $\mathcal{S}^p = \text{span}[N_i : i \in \mathbb{Z}] \cap L^p(\mathbb{R})$, where $1 \le p \le \infty$. The orthogonal projection of $L^1(\mathbb{R})$ onto $\mathcal{S}^1$ will be obtained as a weak limit of properly transformed corresponding orthogonal projections on the interval $I = [0, 1]$. The affine map $s = n \cdot (2t - 1)$ takes the interval $I = [0, 1]$ onto $I_n = [-n, n]$ and the knots $(i/2n : i = 0, \ldots, 2n)$ onto the knots $(k = -n, \ldots, n)$. The relation of the new B-splines, corresponding to I_n, and old ones, corresponding to I, is as follows:

$$N_{k,n}(s) = N_i\left(\frac{s+n}{2n}\right) \quad \text{with } k = i - n,\ s, k \in I_n.$$

For the entries of the Gram matrix of the new B-splines we have

$$\begin{aligned}(N_{k,n}, N_{k',n})_{I_n} &= \int_{I_n} N_i\left(\frac{s+n}{2n}\right) N_{i'}\left(\frac{s+n}{2n}\right) ds \\ &= 2n \int_I N_i(t) N_{i'}(t)\, dt = (N_i, N_{i'})_I \quad \text{with } i = k + n,\ i' = k' + n.\end{aligned}$$

Thus, the Gram matrix $\mathbf{B} = [b_{i,i'} : i, i' = 0, \ldots, 2n]$ for $(N_i : i = 0, \ldots, 2n)$ and the Gram matrix $\mathbf{B}^{(n)} = [b^{(n)}_{k,k'} : k, k' \in I_n]$ for $(N_{k,n} : k \in I_n)$ differ by a factor of $2n$, i.e. $\mathbf{B}^{(n)} = 2n\mathbf{B}$. More explicitly, according to formula (2.2),

$$b^{(n)}_{k,k'} = b^{(\infty)}_{|k-k'|} \quad \text{for } n > |k| \vee |k'|, \tag{4.2}$$

where $b^{(\infty)}_i = b^{(\infty)}_{-i}$ and

$$b^{(\infty)}_i = \begin{cases} 2/3 & \text{for } i = 0, \\ 1/6 & \text{for } i = 1, \\ 0 & \text{for } i > 1. \end{cases} \tag{4.3}$$

However, we are interested in the inverse $\mathbf{A}^{(n)} = (\mathbf{B}^{(n)})^{-1} = \frac{1}{2n}\mathbf{A}$ and after replacing n by $2n$ in (3.2) we find the following formula for the entries of $\mathbf{A}^{(n)}$:

$$a_{k,k'}^{(n)} = 2\cdot(-1)^{k+k'}\,\frac{A(n+k\wedge k')A(n-k\vee k')}{B(2n)}. \tag{4.4}$$

Now, (4.3), (4.4) and the asymptotic formulae

$$2A(m)\simeq(2+\sqrt{3})^m,\qquad 2\sqrt{3}\,B(m)\simeq(2+\sqrt{3})^m$$

for large m imply

PROPOSITION 4.1. *For fixed $k,k'\in\mathbb{Z}$ we have*

$$\lim_{n\to\infty} b_{k,k'}^{(n)} = b_{|k-k'|}^{(\infty)}, \tag{4.5}$$

and

$$\lim_{n\to\infty} a_{k,k'}^{(n)} = a_{|k-k'|}^{(\infty)} = \sqrt{3}\cdot(-1)^{k+k'}(2-\sqrt{3})^{|k-k'|} = \sqrt{3}\cdot(\sqrt{3}-2)^{|k-k'|}. \tag{4.6}$$

COROLLARY 4.2. *For $i,k\in\mathbb{Z}$ we have*

$$\sum_{m\in\mathbb{Z}} a_{|i-m|}^{(\infty)} b_{|m-k|}^{(\infty)} = \delta_{i,k}.$$

We may now define the dual basis in $\mathcal{S}^1$ to the B-splines $(N_i(\cdot): i\in\mathbb{Z})$, i.e.

$$N_i^*(s) = \sum_{k\in\mathbb{Z}} a_{|i-k|}^{(\infty)}\, N_i(s)\qquad\text{for } i\in\mathbb{Z},\ s\in\mathbb{R}. \tag{4.7}$$

The duality relation $(N_i^*, N_k) = \delta_{i,k}$ for $i,k\in\mathbb{Z}$, with respect to the scalar product $(f,g)=\int_{\mathbb{R}} f(s)g(s)\,ds$ follows from Corollary 4.2. Let us take a look at the operator $P^{(1)}: L^1(\mathbb{R})\to\mathcal{S}^1$ defined by

$$P^{(1)}(f) = \sum_{i\in\mathbb{Z}} (f, N_i^*)N_i\qquad\text{for } f\in L^1(\mathbb{R}). \tag{4.8}$$

Since $\phi(t)\le 3$, the formula analogous to (2.15) for knots on $\mathbb{R}$ implies that

$$\|P^{(1)}(f)\|_1 \le 3\|f\|_1\qquad\text{for } f\in L^1(\mathbb{R}), \tag{4.9}$$

and that $P^{(1)}$ is the orthogonal projection onto $\mathcal{S}^1$ i.e.

$$(f-P^{(1)}f, g) = 0\qquad\text{for } f\in L^1(\mathbb{R}),\ g\in\mathcal{S}^\infty. \tag{4.10}$$

LEMMA 4.3. *For the $L^1(\mathbb{R})$ norm of $P^{(1)}$ we have $\|P^{(1)}\|_1 = 2$. Consequently*, $\lim_{n\to\infty}\|P_{n,n}\|_1 = \|P^{(1)}\|_1$, *where $P_{n,n}$ is as in Section* 3.2.

Proof. Formulae (2.11) and (2.12) extended to equally spaced knots on $\mathbb{R}$ give

$$\|P^{(1)}\|_1 = \frac{1}{2}\sum_{i\in\mathbb{Z}} \frac{|a_{i-1-k}^{(\infty)}|^2 + |a_{i-k}^{(\infty)}|^2}{|a_{i-1-k}^{(\infty)}| + |a_{i-k}^{(\infty)}|}\qquad\text{for } k\in\mathbb{Z}.$$

Using the notation of Section 2 and formula (4.6) we find that

$$\frac{1}{2}\sum_{i\in\mathbb{Z}}\frac{|a^{(\infty)}_{i-1-k}|^2+|a^{(\infty)}_{i-k}|^2}{|a^{(\infty)}_{i-1-k}|+|a^{(\infty)}_{i-k}|}=\sum_{i\in\mathbb{Z}}p_{i,0}\phi(2-\sqrt{3})=2,$$

since the weights $(p_{i,0} : i \in \mathbb{Z})$ add to 1 and since according to Lemma 2.1 we also have $\phi(2-\sqrt{3}) = 2$. Combining these equalities and Corollary 3.2 completes the proof.

4.2. *Partially equally spaced knots on* $\mathbb{R}$. Let us start with given $N = n+\nu$, $n \geq 1$ and $1 \leq \nu \leq n$. Now the affine map $\psi_N(t) = 2(nt-\nu)$ transfers the knots (3.6) from $I = [0, 1]$ to the knots

$$\psi_N(t_i)=\begin{cases} i-2\nu & \text{for } i=0,\ldots,2\nu,\\ 2(i-2\nu) & \text{for } i=2\nu+1,\ldots,N.\end{cases}\tag{4.11}$$

After reindexing we get the set of knots

$$t_{i,N}=\begin{cases} i & \text{for } i=-2\nu,\ldots,0,\\ 2i & \text{for } i=1,\ldots,N-2\nu.\end{cases}\tag{4.12}$$

Let $I_N = [t_{-2\nu,N}, t_{N-2\nu,N}] = [-2\nu, 2(N-2\nu))]$ and $(f, g)_{I_N} = \int_{I_N} fg$. The limiting set of knots (4.12) as $N \to \infty$ with $\nu \wedge (n-\nu) \to \infty$ is

$$t_i=\begin{cases} i & \text{for } i\le 0,\\ 2i & \text{for } i>0.\end{cases}\tag{4.13}$$

For the B-splines $(B_i : i \in \mathbb{Z})$ corresponding to the knots (4.13) we have for $t \in \mathbb{R}$ the formulae

$$B_i(t)=\begin{cases} N_0(t-i) & \text{for } i\le -1,\\ N_0(t)+\frac{1}{2}N_0(t-1) & \text{for } i=0\ ,\\ N_0(t/2-i) & \text{for } i\ge 1;\end{cases}\tag{4.14}$$

where $N_0(t)$ is as in (4.1). Now, for each integer N we have the Gram matrix $\mathbf{B}^{(N)} = [b^{(N)}_{i,k} : i, k \in I_N]$ with $b^{(N)}_{i,k} = (B_{i,N}, B_{k,N})_{I_N}$ for $i, k \in I_N$. It is useful to introduce now the infinite matrix $\mathbf{B}$ with the entries

$$b_{i,k}=\lim_{\nu\wedge(n-\nu)\to\infty} b^{(N)}_{i,k}=(B_i,B_k)_{\mathbb{R}} \quad \text{for } i,k\in\mathbb{Z}.\tag{4.15}$$

Each $\mathbf{B}^{(N)}$ has an inverse $\mathbf{A}^{(N)} = [a^{(N)}_{i,k} : i, k \in I_N]$. For its entries, after transforming the knots from I to I_N and reindexing them, we get, from Proposition 3.4,

$$a^{(N)}_{i,k}=\frac{(-1)^{i+k}}{C(N)}\,\varepsilon_{i+2\nu,k+2\nu} \quad \text{for } i,k=-2\nu,\ldots,N-2\nu,\tag{4.16}$$

with the $\varepsilon_{\cdot,\cdot}$ as in (3.10). Clearly, $a^{(N)}_{i,k}$ depends on $N = n+\nu = 2\nu+(n-\nu)$.

PROPOSITION 4.4. *Let $i, k \in \mathbb{Z}$ be fixed. Then*

$$\lim_{\nu\wedge(n-\nu)\to\infty} a_{i,k}^{(N)} = a_{i,k} = (-1)^{i+k}\lambda_{i,k}\cdot(2-\sqrt{3})^{|i-k|}, \tag{4.17}$$

where $\lambda_{i,k}$ is uniformly bounded and

$$\lambda_{i,k} = \begin{cases} \dfrac{2}{\sqrt{3}} & \text{if } i\vee k > 0 \geq i\wedge k, \\ \dfrac{1}{2\sqrt{3}}\,[3+(2-\sqrt{3})^{2(i\wedge k)}] & \text{if } i\wedge k > 0, \\ \dfrac{1}{\sqrt{3}}\,[3-(2-\sqrt{3})^{-2(i\vee k)}] & \text{if } i\vee k \leq 0. \end{cases} \tag{4.18}$$

Proof. Use (4.16).

COROLLARY 4.5. *For the infinite matrices* **B** *and* **A** *we have*

$$\sum_{j\in\mathbb{Z}} a_{i,j}b_{j,k} = \delta_{i,k} \quad \text{for } i,k\in\mathbb{Z}.$$

As in the equally spaced case on $\mathbb{R}$ we define the biorthogonal system

$$B_i^* = \sum_{k\in\mathbb{Z}} a_{i,k}B_k \tag{4.19}$$

and the orthogonal projection

$$P_0 f = \sum_{i\in\mathbb{Z}} (f, B_i^*)_{\mathbb{R}} B_i \quad \text{for } f\in L^1(\mathbb{R}).$$

Applying the argument standard by now we find that

$$\|P_0 f\|_{L^1(\mathbb{R})} \leq 3\|f\|_{L^1(\mathbb{R})} \quad \text{for } f\in L^1(\mathbb{R}).$$

Moreover,

$$(f-P_0 f, g)_{\mathbb{R}} = 0 \quad \text{for } g\in \operatorname{span}[B_i : i\in\mathbb{Z}]\cap L^\infty(\mathbb{R}).$$

Now, if $P_0^{(N)}$ is the orthogonal projection corresponding to $N = n+\nu$ and to the interval I_N, then

$$\|P_0^{(N)}\|_{L^1(I_N)} = \|P_{\nu,n}\|_{L^1(I)} \quad \text{for } N\geq 2,$$

where $P_{\nu,n}$ is as in Section 3.2. Using (2.12) and Proposition 4.4 we can show that

$$\lim_{\nu\wedge(n-\nu)\to\infty} \|P_0^{(N)}\|_{L^1(I_N)} \geq \|P_0\|_{L^1(\mathbb{R})}.$$

Let $\gamma_{i,k} = |a_{i,k}|/|a_{i-1,k}|$ and $\delta_i = t_i - t_{i-1}$. Then

$$\|P_0\|_{L^1(\mathbb{R})} \geq \sum_{i\in\mathbb{Z}} p_{i,k}\phi(\gamma_{i,k}) \quad \text{for } k\in\mathbb{Z},$$

where $p_{i,k} = (\delta_i/6)(|a_{i,k}| + |a_{i-1,k}|)$ and $\sum_{i\in\mathbb{Z}} p_{i,k} = 1$ for each $k \in \mathbb{Z}$. In particular, for $k = -1$ we infer from (4.18) that

$$\lambda_{i,-1} = \begin{cases} \dfrac{2}{\sqrt{3}} & \text{for } i \geq 0, \\ 4\Big(1 - \dfrac{1}{\sqrt{3}}\Big) & \text{for } i \leq -1. \end{cases}$$

Thus,

$$\gamma_{i,-1} = \begin{cases} 2-\sqrt{3} & \text{for } i \geq 1, \\ \dfrac{\sqrt{3}-1}{4} & \text{for } i = 0, \\ 2+\sqrt{3} & \text{for } i \leq -1. \end{cases}$$

Consequently, $p_{0,-1} = 1/3$, $\gamma_{0,-1} = (\sqrt{3}-1)/4$ and since $\phi(2\pm\sqrt{3}) = 2$ we obtain

$$\begin{aligned}\sum_{i\in Z} p_{i,-1}\,\phi(\gamma_{i,-1}) &= p_{0,-1}\,\phi(\gamma_{0,-1}) + 2\sum_{i\neq 0} p_{i,-1} \\ &= 2 + p_{0,-1}(\phi(\gamma_{0,-1}) - 2) = 2 + (2-\sqrt{3})^2 = D,\end{aligned}$$

whence we infer

COROLLARY 4.6. $\|P_0\|_{L^1(\mathbb{R})} = D$.

REMARK. Note that if we knew that for each $N = n + \nu$ and $-2\nu \leq i, k \leq N - 2\nu$,

$$\delta_i \frac{|a_{i-1,k}|^2 + |a_{i,k}|^2}{|a_{i-1,k}| + |a_{i,k}|} \geq \delta_i \frac{|a_{i-1,k}^{(N)}|^2 + |a_{i,k}^{(N)}|^2}{|a_{i-1,k}^{(N)}| + |a_{i,k}^{(N)}|}, \tag{4.20}$$

then the estimate on $[0,1]$ would be a simple consequence of the result on $\mathbb{R}$. However, inequality (4.20) is not true in general. For example, for $i = -1$ and $k = 1$, and with $s = 2\nu$, $t = N - 2\nu$, (4.20) takes the form

$$\frac{4(2-\sqrt{3})}{3} \geq \frac{A(t) + 3B(t)}{A(t)B(s) + 2B(t)A(s)} \cdot \frac{A(s-1)^2 + A(s-2)^2}{A(s-1) + A(s-2)}. \tag{4.21}$$

It is easy to see that this inequality fails e.g. for $s = 2$ and $t = 1$. In addition, one can find infinite sequences of t, s (e.g. with $s = t + 2$) for which (4.21) fails to hold.

References

[1] P. Bechler, *Lebesgue constant for the Strömberg wavelet*, J. Approx. Theory 122 (2003), 13–23.

[2] S. V. Bochkarev, *A Fourier series divergent on a set of positive measure for an arbitrary bounded orthonormal system*, Mat. Sb. 98 (140) (1975), 436–449 (in Russian).

[3] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), 141–157.
[4] —, *Properties of the orthonormal Franklin system, II*, ibid. 27 (1966), 289–323.
[5] —, *The $C(I)$ norms of orthogonal projections onto subspaces of polygonals*, Trudy Mat. Inst. Steklova 134 (1975), 366–369.
[6] Z. Ciesielski and A. Kamont, *Survey on the orthogonal Franklin system*, in: Approximation Theory. A volume dedicated to B. Sendov, B. Bojanov (ed.), DARBA, Sofia, 2002, 84–132.
[7] Z. Ciesielski and E. Niedźwiecka, *A conversation with the Odra 1204 computer about approximation by polygonal functions*, Wiadomości Mat. 20 (1977), 29–34 (in Polish).
[8] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (in Russian).
[9] A. M. Olevskiĭ, *Fourier series and Lebesgue functions*, Uspekhi Mat. Nauk 22 (1967), no. 3, 237–239 (in Russian).
[10] K. Oskolkov, *The upper bound of the norms of orthogonal projections onto subspaces of polygonals*, in: Approximation Theory, Banach Center Publ. 4, PWN, Warszawa, 1979, 177–183.
[11] P. Oswald, *The norm in C of orthoprojections onto subspaces of piecewise linear functions*, Mat. Zametki 21 (1977), 495–502 (in Russian).
[12] A. Yu. Shadrin, *The L_∞-norm of the L_2 spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture*, Acta Math. 187 (2001), 59–137.

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