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A_1-regularity and boundedness of Calderón–Zygmund operators

by

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Abstract. The Coifman–Fefferman inequality implies quite easily that a Calderón–Zygmund operator T acts boundedly in a Banach lattice X on $\mathbb{R}^n$ if the Hardy–Littlewood maximal operator M is bounded in both X and X'. We establish a converse result under the assumption that X has the Fatou property and X is p-convex and q-concave with some $1 < p, q < \infty$: if a linear operator T is bounded in X and T is nondegenerate in a certain sense (for example, if T is a Riesz transform) then M is bounded in both X and X'.

The purpose of the present work is to establish the following theorem showing that the boundedness of Calderón–Zygmund singular integral operators T and the boundedness of the Hardy–Littlewood maximal operator M in both the lattice and its dual is actually the same property for a fairly general class of Banach lattices. This constitutes a substantial improvement over the respective results of [24].

The (standard) definitions and basic facts concerning Banach lattices and Calderón–Zygmund operators can be found in Section 1. The notion of an A_2-nondegenerate operator is introduced in Definition 6 below; for now we say that R can be any of the Riesz transforms $\{R_j\}_{j=1}^n$ (or the Hilbert transform H if $n = 1$). We fix a σ-finite measurable space (Ω, μ) which we understand as a space for the second variable ω in $(x, \omega) \in \mathbb{R}^n \times \Omega$ (unless indicated otherwise, all operators are assumed to act in the first variable x only); this allows us to naturally include lattices with mixed norm such as $X(l^r)$ in this setting.

Theorem 1. *Suppose that X is a Banach lattice of measurable functions on $\mathbb{R}^n \times \Omega$ that has the Fatou property and X is p-convex and q-concave with some $1 < p, q < \infty$. Let R be a Calderón–Zygmund operator in $L_2(\mathbb{R}^n)$*

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such that both R and R^* are A_2-nondegenerate. The following conditions are equivalent:

(1) *The Hardy–Littlewood maximal operator M acts boundedly in X and in the order dual X' of X.*
(2) *All Calderón–Zygmund operators act boundedly in X.*
(3) *R acts boundedly in X.*

The implication (1)⇒(2) is Proposition 5 in Section 1. Although it is hard to come by this sufficient condition for boundedness of Calderón–Zygmund operators in the literature, it is certainly not new; see [14, Remark 4.3]. The implication (2)⇒(3) is trivial. The implication (3)⇒(1) is Theorem 16 in Section 3. The argument itself is technically rather simple; however, it relies heavily on the theory of A_p-regular Banach lattices, a part of which we develop further in Section 2, and the proof taken as a whole involves two distinct applications of the Fan–Kakutani fixed point theorem and a variant of the Maurey–Krivine factorization theorem, which is based on the Grothendieck theorem.

We now briefly outline some examples. The classical case of weighted Lebesgue spaces $X = L_p(w)$ is perhaps the best illustration for Theorem 1: the theory of Muckenhoupt weights (see, e.g., [27, Chapter 5]) individually links the conditions of Theorem 1 to the Muckenhoupt condition A_p on the weight w. Another classical type of lattice are rearrangement invariant spaces (also called symmetric spaces) such as Orlicz and Lorentz spaces (and, as a particular case, the Lebesgue spaces L_p, which are a part of the previous example). In this case the conclusion of Theorem 1, at least for $n = 1$ and the Hilbert transform $R = H$, follows from the equivalence of its conditions to the conditions $p_X > 1$ and $p_{X'} > 1$ on the upper Boyd indices p_X and $p_{X'}$; see, e.g., [21], [26], [3], [17, Chapter 2, §6] (by duality $p_{X'} < 1$ if and only if $q_X > 1$ for the lower Boyd index q_X; see, e.g., [17, Chapter 2, Theorem 4.11]).

However, the p-convexity and q-concavity assumptions on X imposed in Theorem 1 imply that if X is symmetric then $p_X \geq p > 1$ and $q_X \leq q < \infty$ (see, e.g., [20, Vol. 2, §2.b]), so the conditions in its conclusion are always satisfied in this case (since X is then an interpolation space between L_r and L_s with some $1 < r < s < \infty$; see, e.g., [4]). Very recently in [1] these classical results were extended to a general case of weighted Lorentz spaces $\Lambda_u^p(w)$ (which also covers the case of weighted Lebesgue spaces; however, the conditions $n = 1$ and $R = H$ are still assumed). A different type of nonhomogeneity arises in the case of variable exponent Lebesgue spaces $X = L_{p(\cdot)}$ (and, more generally, in Musielak–Orlicz spaces), and it was only recently that effective characterizations of boundedness of the Hardy–Littlewood maximal operator M in $L_{p(\cdot)}$ in terms of the exponent $p(\cdot)$ and other interesting properties have been developed; it seems, nonetheless, that a complete characterization of boundedness even for some Calderón–Zygmund operators in these spaces has not been achieved yet, so Theorem 1 seems to provide new information in this extensively researched setting. For more remarks on the case of variable exponent Lebesgue spaces see Section 4 below.

The paper is organized as follows. In Section 1 we provide the definitions and basic properties that will be used in the text and prove the implication (1)⇒(2) of Theorem 1. Section 2 contains a new sufficient condition for A_1-regularity. In Section 3 we prove the converse implication (3)⇒(1) of Theorem 1. Further remarks are given in Section 4.

1. Preliminaries. In this section we briefly go over the basic definitions and facts used throughout this work. For the generalities on real harmonic analysis see, e.g., [8], [27]; for Banach lattices and their properties see, e.g., [13, Chapter 10], [20]. A *space of homogeneous type* (S, ν) is a quasimetric space equipped with a Borel measure ν that has the *doubling property*, i.e. $\nu(B(x, 2r)) \le c\nu(B(x, r))$ for all $x \in S$ and $0 < r < \infty$ with some constant c, where $B(x, r)$ is the ball of radius r centered at x. The main example here is the Euclidean space $S = \mathbb{R}^n$ equipped with the Lebesgue measure.

A *quasi-normed lattice of measurable functions* is a quasi-normed space X of measurable functions in which the norm is compatible with the natural order: if $|f| \le g$ a.e. for some function $g \in X$ then $f \in X$ and $\|f\|_X \le \|g\|_X$. For simplicity we only work with lattices X such that $\operatorname{supp} X = S \times \Omega$.

For a Banach lattice X of measurable functions, any *order continuous* functional f on X (in the sense that for any sequence $x_n \in X$ such that $\sup_n |x_n| \in X$ and $x_n \to 0$ a.e. one also has $f(x_n) \to 0$) has an integral representation $f(x) = \int x y_f$ for some measurable function y_f which can be identified with f. The set X' of all such functionals is a Banach lattice with the norm defined by $\|f\|_{X'} = \sup_{g \in X,\, \|g\|_X = 1} \int |fg|$. The lattice X' is called the *order dual* of the lattice X.

The norm of a lattice X is said to be *order continuous* if for any non-increasing sequence $x_n \in X$ converging to 0 a.e. one also has $\|x_n\|_X \to 0$. Order continuity of the norm of a Banach lattice X is equivalent to $X^* = X'$, and it is also equivalent to density of the simple functions (i.e. of the linear span of the set $\{\chi_E\}$ where E ranges over the measurable sets) in X.

A lattice X has the *Fatou property* if for any $f_n, f \in X$ such that $\|f_n\|_X \le 1$ and f_n converges to f a.e. we have $f \in X$ and $\|f\|_X \le 1$. The Fatou property of a lattice X is equivalent to $(\nu \times \mu)$-closedness of the unit ball B_X of X (here and elsewhere, $(\nu \times \mu)$-*convergence* means convergence in measure in any measurable set E such that $(\nu \times \mu)(E) < \infty$). If X is a Banach lattice then the Fatou property is equivalent to *order reflexivity* of X, i.e. to $X'' = X$. For a lattice X either the Fatou property or the order continuity of norm is sufficient to guarantee that the lattice X' is *norming* for X, i.e. $\|f\|_X = \sup_{g \in X', \|g\|_{X'}=1} \int fg$ for all $f \in X$.

To illustrate the above properties we briefly consider a couple of examples. Lattices satisfying the Fatou property are ubiquitous. All modular spaces satisfy it (see, e.g., [9, Theorem 2.3.17]). Sometimes the Fatou property is even assumed implicitly or by default in the literature; however, it seems natural not to do so here, since the technique used in some of the core arguments of this work heavily relies on in. The space c_0 of sequences converging to 0 with the uniform norm is an example of a Banach lattice that does not have the Fatou property but has order continuous norm. The Lebesgue space L_p has order continuous norm if and only if $p < \infty$.

For any quasi-normed lattices X and Y on the same measurable space the set of pointwise products

$$XY = \{fg \mid f \in X,\, g \in Y\}$$

is a quasi-normed lattice with the norm defined by

$$\|h\|_{XY} = \inf_{h=fg} \|f\|_X \|g\|_Y.$$

If both X and Y have the Fatou property then so does XY. If either X or Y has order continuous quasi-norm then the quasi-norm of XY is also order continuous.

For any $\delta > 0$ and a quasi-normed lattice X, the lattice X^δ consists of all measurable functions f such that $|f|^{1/\delta} \in X$, with quasi-norm $\|f\|_{X^\delta} = \||f|^{1/\delta}\|_X^\delta$. For example, $\mathrm{L}_p^\delta = \mathrm{L}_{p/\delta}$. It is easy to see that $(XY)^\delta = X^\delta Y^\delta$ for any X, Y and δ, and X^δ naturally inherits many properties from X. For any $0 < \delta \leq 1$, if X is a Banach lattice then so is X^δ. If X and Y are Banach lattices then for any $0 < \delta < 1$ the lattice $X^{1-\delta} Y^\delta$, sometimes called the *Calderón–Lozanovsky product* of X and Y, is also Banach; moreover, one has a useful relation $(X^{1-\delta} Y^\delta)' = (X')^{1-\delta} (Y')^\delta$ (see [5], [22]). If $Z = X^{1-\delta} Y^\delta$ has either the Fatou property or order continuous norm then Z is an exact interpolation space of exponent δ between X and Y; see, e.g., [23], [5], [17].

Let $1 \leq p, q < \infty$. A Banach lattice X is said to be *p-convex* with constant C if

$$\Big\| \Big(\sum_{j=1}^N |f_j|^p \Big)^{1/p} \Big\|_X \leq C \Big(\sum_{j=1}^n \|f_j\|_X^p \Big)^{1/p}$$

for any $\{f_j\}_{j=1}^N \subset X$; and X is said to be *q-concave* with constant c if

$$\Big(\sum_{j=1}^n \|f_j\|_X^q \Big)^{1/q} \leq c \Big\| \Big(\sum_{j=1}^N |f_j|^q \Big)^{1/q} \Big\|_X$$

for any $\{f_j\}_{j=1}^N \subset X$. If X is p-convex then X' is p'-concave, and if X

is q-concave then X' is q'-convex. It is well known (see, e.g., [20, Vol. 2, Proposition 1.d.8]) that a Banach lattice that is p-convex and q-concave can be renormed to make its p-convexity and q-concavity constants equal to 1. It is easy to see that a lattice with the Fatou property which is p-convex and q-concave with some $1 < p, q < \infty$ is reflexive and has order continuous norm.

For a quasi-normed lattice X and weight w such that $0 \le w \le \infty$ almost everywhere, the weighted lattice $X(w)$ is defined by

$$X(w) = \{g \mid g/w \in X\}$$

with the quasi-seminorm defined by $\|f\|_{X(w)} = \|fw^{-1}\|_X$. This somewhat cumbersome definition is needed because the more natural definition $X(w) = \{wh \mid h \in X\}$ is meaningless if the weight w takes value $+\infty$ on a set of positive measure, and it seems easier to allow this in the definition and work with weighted lattices that may be quasi-normed rather than negotiate finiteness of w every time. Thus in this setting one has $g = 0$ on the set where $w = 0$, g restricted to the set $\{w = +\infty\}$ is an arbitrary measurable function, and $\|\cdot\|_{X(w)}$ is a quasi-norm for weights w such that $(\nu \times \mu)(\{w = +\infty\}) = 0$. If $w = 0$ on a set of positive measure, we regard $X(w)$ as merely a set of functions with a quasi-seminorm under our conventions, since then $\operatorname{supp} X(w) \ne \operatorname{supp} X$. In majorization arguments it is usually possible to avoid dealing with "bad" weights with the help of the following simple proposition.

PROPOSITION 2 ([24, Proposition 3.2]). *Suppose that X is a Banach lattice on (Σ, μ). Then for every $f \in X$ such that $f \ne 0$ identically and $\varepsilon > 0$, there exists $g \in X$ such that $g > |f|$ a.e. and $\|g\|_X \le (1+\varepsilon)\|f\|_X$.*

The construction of the weighted lattice yields

$$L_\infty(w) = \{f \mid |f| \le Cw \text{ a.e.}\}.$$

It is easy to see that $[X(w)]' = X'(w^{-1})$. Some caution is required since the definition of the weighted Lebesgue space $L_p(w)$ arising from the definition of the weighted lattice above differs from the "classical" one with the norm defined by $\|f\|_{p,w}^p = \int |f|^p w$, which is often used in the literature; the latter norm corresponds to the norm of $L_p(w^{-1/p})$ in our notation. Thus all weighted lattices are defined in the same way everywhere in this paper; however, one has to pay attention to this difference. We adopt the natural conventions $0^{-1} = \infty$ and $\infty^{-1} = 0$ in all expressions involving weights.

The (centered) *Hardy–Littlewood maximal operator*

$$Mf(x,t) = \sup_{r>0} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} |f(z,t)|\, d\nu(z), \qquad x \in S,\ t \in \Omega,$$

is well-defined for a.e. $x \in S$, $t \in \Omega$, and all measurable functions f on $(S \times \Omega, \nu \times \mu)$ that are locally summable in the first variable. We say that a nonnegative measurable function w on $(S \times \Omega, \nu \times \mu)$ belongs to the *Muckenhoupt class* A_p for some $1 \le p < \infty$ with a constant C if

$$\operatorname*{ess\,sup}_{t \in \Omega} \|M\|_{\mathrm{L}_p(w^{-1/p}(\cdot,t)) \to \mathrm{L}_{p,\infty}(w^{-1/p}(\cdot,t))} \le C.$$

In the case $p > 1$ this condition is equivalent to

$$\operatorname*{ess\,sup}_{t \in \Omega} \|M\|_{\mathrm{L}_p(w^{-1/p}(\cdot,t))} \le C'$$

with a constant C' estimated in terms of C and p, and vice versa. The class A_1 is characterized by the estimate $Mw \le C'w$ almost everywhere, while the classes A_p for $p > 1$ are characterized by the well-known Muckenhoupt condition

$$(1) \qquad \operatorname*{ess\,sup}_{x \in S,\, t \in \Omega} \sup_{r > 0} \left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(u,t)\, d\nu(u)\right] \times \left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(u,t)^{-1/(p-1)}\, d\nu(u)\right]^{p-1} < \infty.$$

The class A_∞ may be defined as the class of weights w satisfying the reverse Hölder inequality

$$(2) \qquad \operatorname*{ess\,sup}_{x \in S,\, t \in \Omega} \sup_{r > 0} \left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} [w(u,t)]^q\, d\nu(u)\right]^{1/q} \times \left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(u,t)\, d\nu(u)\right]^{-1} < \infty$$

with some $q > 1$, and we can define the A_∞ constant of the weight w to be the supremum in (2). It is well known that $w \in \mathrm{A}_\infty$ if and only if $w \in \mathrm{A}_p$ with some $1 < p < \infty$, and the value of p and the A_p constant of w can be estimated in terms of the A_∞ constant of w and vice versa.

The following notion was introduced in [24] as an important particular case of so-called BMO-regularity introduced in [12].

Definition 3. A quasi-normed lattice X on $(S \times \Omega, \nu \times \mu)$ is A_p-*regular* with constants (C, m) if for any $f \in X$ there exists a majorant $g \in X$, that is $g \ge |f|$, such that $\|g\|_X \le m\|f\|_X$ and $g \in \mathrm{A}_p$ with constant C.

Proposition 4 ([24, Proposition 1.2]). *A quasi-normed lattice X on $(S \times \Omega, \nu \times \mu)$ is A_1-regular if and only if the maximal operator M is bounded in X.*

We say that T is a *Calderón–Zygmund operator* if T is a singular integral operator that is bounded in $\mathrm{L}_2(\mathbb{R}^n)$, its kernel $K(x,y)$ satisfies

$$(3)\qquad |K(x,s)-K(x,t)| \le C_K \frac{|s-t|^\gamma}{|x-s|^{n+\gamma}}, \qquad x,s,t\in\mathbb{R}^n,\ |x-s|>2|s-t|,$$

with some $\gamma>0$, and the kernel $K^*(y,x)=K(x,y)$ of the adjoint operator T^* satisfies the same estimates. It is well known that Calderón–Zygmund operators T are bounded in L_p for all $1<p<\infty$.

The Coifman–Fefferman inequality [6]

$$(4)\qquad \int |Tf|^p\omega \le C\int (Mf)^p\omega, \qquad 0<p<\infty,$$

holds true for Calderón–Zygmund operators T, any weights $\omega\in\mathrm{A}_\infty$ and all locally summable functions f such that the right-hand side of (4) is finite; the constant C does not depend on f and is estimated in terms of the A_∞ constant of the weight ω.

We are now ready to prove the implication (1)⇒(2) of Theorem 1, which can be stated as follows.

PROPOSITION 5. *Suppose that X is a Banach lattice on $\mathbb{R}^n\times\Omega$ having either the Fatou property or order continuous norm and both X and X' are A_1-regular. Then any Calderón–Zygmund operator T is bounded in X.*

Indeed, let $f\in X$ and $g\in X'$, and let h be an A_1-majorant of g in X'. Then

$$\int (Mf)h \le \|Mf\|_X\|h\|_{X'} \le c_1\|f\|_X\|g\|_{X'}<\infty,$$

and the Coifman–Fefferman inequality (4) with $p=1$ implies that

$$\int (Tf)g \le \int |Tf|h \le c\int (Mf)h \le cc_1\|f\|_X\|g\|_{X'}$$

with certain constants c and c_1 independent of f and g, which shows that T acts boundedly in X.

DEFINITION 6. A mapping $T:\mathrm{L}_2\to\mathrm{L}_2$ is called A_2-*nondegenerate* with constants (C,m) if boundedness of T in a lattice $\mathrm{L}_2(w^{-1/2})$ with norm at most m implies that $w\in\mathrm{A}_2$ with constant C.

We remark that by [24, Proposition 3.7] an A_2-nondegenerate linear operator T in $\mathrm{L}_2(\mathbb{R}^n)$ is also A_2-nondegenerate as an operator in $\mathrm{L}_2(\mathbb{R}^n\times\Omega)$ acting in the first variable. The nature of A_2-nondegeneracy is illustrated by the following well-known result.

PROPOSITION 7 ([27, Chapter 5, §4.6]). *Suppose that T is a Calderón–Zygmund operator with kernel K and there exist some $u\in\mathbb{R}^n$ and a constant c such that for any $x\in\mathbb{R}^n$ and $t\ne 0$ we have*

$$(5)\qquad |K(x,x+tu)| \ge ct^{-n}.$$

Then T is A_2-nondegenerate.

It is easy to see that the Hilbert transform H on $\mathbb{R}$ with kernel $K(x, y) = c_1/(x - y)$ and the Riesz transforms R_j, $1 \le j \le n$, on $\mathbb{R}^n$ with kernels $K_j(x, y) = c_n(y_j - x_j)/|y - x|^{n+1}$, where $c_n \ne 0$ are some constants, satisfy condition (5) for $u = e_j$, e_j being the jth coordinate basis vector of $\mathbb{R}^n$, and thus all these operators are A_2-nondegenerate.

2. A lemma about A_p-regularity

THEOREM 8. *Suppose that X is a Banach lattice of measurable functions on $(S \times \Omega, \mu \times \nu)$ such that X has the Fatou property, and*

(1) *X is A_p-regular with constants (c_1, m_1) for some $1 < p < \infty$,*
(2) *X^δ is A_1-regular with constants (c_2, m_2) for some $\delta > 0$.*

Then X is A_1-regular with an estimate for the constants depending only on the corresponding A_p-regularity constants of X, the A_1-regularity constants of X^δ and the value of δ.

This theorem is easily derived from the corresponding result for A_p weights with the help of a fixed point argument.

LEMMA 9. *Suppose that a weight w on $(S \times \Omega, \mu \times \nu)$ satisfies $w \in \mathrm{A}_p$ and $w^\delta \in \mathrm{A}_1$ with some $1 < p < \infty$ and $\delta > 0$. Then $w \in \mathrm{A}_1$ with an estimate for the constants depending only on δ and on the constants of the A_p condition for w and the A_1 condition for w^δ.*

Lemma 9 is essentially a particular case $X = \mathrm{L}_\infty(w)$ of Theorem 8. To prove Lemma 9, fix some $\omega \in \Omega$ such that $w(\cdot, \omega) \in \mathrm{A}_p$ and $w^\delta(\cdot, \omega) \in \mathrm{A}_1$, and let $B(x, r) \subset S$, $x \in S$, $r > 0$, be an arbitrary ball in S. Then consecutive application of the A_p condition satisfied by w, the Jensen inequality with the convex function $t \mapsto t^{-\delta(p-1)}$, $t > 0$, and the A_1 condition satisfied by w^δ yields

$$
\begin{aligned}
(6)\qquad & \frac{1}{\nu(B(x,r))} \int_{B(x,r)} w(u,\omega)\, d\nu(u) \\
&\le c\left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} [w(u,\omega)]^{-\frac{1}{p-1}}\, d\nu(u)\right]^{-(p-1)} \\
&= c\left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} [w(u,\omega)]^{-\frac{1}{p-1}}\, d\nu(u)\right]^{-\delta(p-1)\cdot\frac{1}{\delta}} \\
&\le c\left[\frac{1}{\nu(B(x,r))} \int_{B(x,r)} [w(u,\omega)]^{\delta}\, d\nu(u)\right]^{1/\delta} \le c' w(x,\omega)
\end{aligned}
$$

for almost all $x \in S$ with some constants c and c' depending only on the constants of the A_p condition for w and the A_1 condition for w^δ, and on the value of δ. Since ω, x and B are arbitrary, (6) implies that $w \in \mathrm{A}_1$ with the necessary estimates of the constants, which concludes the proof of Lemma 9.

In order to reduce Theorem 8 to Lemma 9 we need to show that under the conditions of Theorem 8 every function $f \in X$ has a majorant w such that w is an A_p-majorant of f in X and simultaneously w^δ is an A_1-majorant of $|f|^\delta$ in X^δ, with appropriate estimates on the constants. At first glance it may seem that there is little reason to suspect existence of a common majorant in sets that look vastly different (for example, a majorant w such that $w^\delta \in \mathrm{A}_1$ may not even be locally summable in the first variable, while on the other hand a majorant $w \in \mathrm{A}_p$ may vanish near some points); however, careful application of the celebrated Fan–Kakutani fixed point theorem allows us to establish the existence of a common majorant in this setting with relative ease.

Theorem ([10]). *Suppose that K is a compact set in a locally convex linear topological space. Let Φ be a mapping from K to the set of nonempty convex compact subsets of K. If the graph*

$$\Gamma(\Phi) = \{(x, y) \in K \times K \mid y \in \Phi(x)\}$$

of Φ is closed in $K \times K$ then Φ has a fixed point, i.e. $x \in \Phi(x)$ for some $x \in K$.

We will also need the following sets of nonnegative a.e. measurable functions w on $(S \times \Omega, \nu \times \mu)$ (see also [24, Section 3]):

$$B\mathrm{A}_p(C) = \Big\{ w \;\Big|\; \operatorname*{ess\,sup}_{\omega \in \Omega} \|M\|_{\mathrm{L}_p(w^{-1/p}(\cdot,\omega))} \le C \Big\},$$

$$B\mathrm{A}_1(C) = \Big\{ w \;\Big|\; \operatorname{ess\,sup} \frac{Mw}{w} \le C \Big\}.$$

These are the sets of Muckenhoupt weights with fixed bounds on the constants ("the ball of A_p").

Proposition 10 ([24, Proposition 3.4]; see also [12, Lemma 4.2]). *Suppose that $1 \le p < \infty$ a.e. and $C \ge 0$. The set $B\mathrm{A}_p(C)$ is a nonempty convex cone which is also logarithmically convex and closed in measure.*

We are now ready to prove Theorem 8. The technical details of this proof as well as the general pattern are similar to those for the main result of [24]. By using [24, Proposition 3.6] it is sufficient to establish the existence of a suitable majorant for every function $f \in X$ with $\|f\|_X \le 1$ such that $E = \operatorname{supp} f$ has positive finite measure and $f \ge \beta$ on E with some $\beta > 0$, since the set of such functions is dense in measure in the nonnegative part of the closed unit ball B of X. We fix such a function f.

By Proposition 2 there exists some function $a \in X'$ with $\|a\|_{X'} = 1$ such that $a > 0$ almost everywhere. This implies that for any $u \in B$ we have $\int |u|a \leq \|u\|_X \|a\|_{X'} \leq 1$, i.e. $\|u\|_{\mathrm{L}_1(a^{-1})} \leq 1$. Let $0 < \alpha \leq \beta \leq 1$ be a sufficiently small number to be determined later, and let

$$D = \{\chi_E \log g \mid g \in B,\, g \geq \chi_E \alpha\}.$$

It is easy to see that D is a bounded set in $Y = \mathrm{L}_2(a^{-1/2})$ for any given E and α, because

$$\int\limits_{E \cap \{g<1\}} |\log g|^2 a \leq |\log \alpha|^2 \|\chi_E\|_X \|a\|_{X'} \leq \frac{1}{\beta} |\log \alpha|^2$$

and

$$\int\limits_{E \cap \{g \geq 1\}} |\log g|^2 a = \int\limits_{E \cap \{g \geq 1\}} 4 |\log(g^{1/2})|^2 a \leq 4 \int |g| a \leq 4$$

for any $\chi_E \log g \in D$; D is convex because B is logarithmically convex, and D is closed in measure, so D is compact in the weak topology of Y.

Observe that since A_1-regularity of X implies A_1-regularity of X^γ for all $0 < \gamma < 1$, we may assume that $0 < \delta < 1$, otherwise the conclusion of Theorem 8 is immediate. We define a set-valued map Φ in $D \times D$ by

$$\begin{aligned} \Phi((\log u, \log v)) = \big\{ &(\log u_1, \log v_1) \mid u_1, v_1 \in X, \\ &u_1 \in B \cap B\mathrm{A}_p(c_1),\, v_1 \in B,\, v_1^\delta \in B\mathrm{A}_1(c_2), f \vee (u \vee v) \leq A(u_1 \wedge v_1) \big\}. \end{aligned}$$

For any $(\log u, \log v) \in D \times D$ we have $w = f \vee u \vee v \in X$ with $\|w\|_X \leq 3$, and by the assumptions there exist some $a, b \in X$ such that $a \in B\mathrm{A}_p(c_1)$, $b^\delta \in B\mathrm{A}_1(c_2)$, $a \geq w$, $b \geq w$ and $\|a\|_X \leq 3m_1$, $\|b\|_X \leq (3m_2)^{1/\delta}$. Thus choosing $A = (3m_1) \vee (3m_2)^{1/\delta}$ and $\alpha = \beta A^{-1}$ yields $(\log u_1, \log v_1) \in \Phi((\log u, \log v))$ with $u_1 = (1/A)a$ and $v_1 = (1/A)b$, so Φ takes nonempty values. The condition $f \vee (u \vee v) \leq A(u_1 \wedge v_1)$ is of course equivalent to (and a shorthand for) the six inequalities $f \leq Au_1$, $f \leq Av_1$, $u \leq Au_1$, $v \leq Au_1$, $u \leq Av_1$ and $v \leq Av_1$. It is easy to see using Proposition 10 that the graph Γ of Φ is a convex set and Γ is closed with respect to convergence in measure.

Let us verify that Γ is closed in $Y \times Y$. Indeed, the weak topology of $Y \times Y$ is metrizable on a bounded set $D \times D$. If $x_j \in \Gamma$ and $x_j \to x \in Y \times Y$ then there exists some sequence y_j of convex combinations of x_j such that $y_j \to x$ in the strong topology of $Y \times Y$, and $y_j \in \Gamma$ by the convexity of Γ. Strong convergence in Y implies convergence in measure, so $y_j \to x$ in measure. Since Γ is closed in measure, it follows that $x \in \Gamma$ and thus Γ is indeed closed in $Y \times Y$. From this we also infer that the values of Φ are convex and closed in the compact set $D \times D$, and thus they are compact in $Y \times Y$.

By the Fan–Kakutani fixed point theorem there exists some

$$(\log u, \log v) \in D \times D$$

such that $(\log u, \log v) \in \Phi((\log u, \log v))$. This implies that u and v are pointwise equivalent to one another with a constant of equivalence depending only on A (which, in turn, only depends on the values of m_1, m_2 and δ), and so $w = Au$ is a majorant of f such that $w \in \mathrm{A}_p$ and $w^\delta \in \mathrm{A}_1$ with appropriate estimates on the constants. By Lemma 9 it follows that $w \in \mathrm{A}_1$ with suitable estimates on the constants, which concludes the proof of Theorem 8.

We will need the following proposition, which is a simple consequence of duality and the properties of A_p weights.

Proposition 11 ([24, Proposition 2.3]). *Suppose that X is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ such that X' is a norming space for X. If X' is A_1-regular then $X^{1/q}$ is A_1-regular for all $q > 1$. If X' is A_p-regular with some $p > 1$ then $X^{1/p}$ is A_1-regular.*

Theorem 8 has an interesting immediate application.

Proposition 12. *Let X be a Banach lattice on $(S \times \Omega, \nu \times \mu)$ having the Fatou property. Suppose that both X and X' are A_∞-regular. Then X and X' are also A_1-regular.*

Indeed, since X and X' are A_∞-regular, they are also A_p-regular with some $p < \infty$, which by Proposition 11 means that both $(X')^{1/p}$ and $X^{1/p}$ are A_1-regular, and it remains to apply Theorem 8 to X and X' with $\delta = 1/p$.

Corollary 13. *Suppose that X is a Banach lattice on $\mathbb{R}^n$ having the Fatou property, and both X and X' are A_∞-regular. Then any Calderón–Zygmund operator T is bounded in X.*

This corollary, which strengthens Proposition 5, immediately follows from Propositions 12 and 5.

3. Necessity of A_1-regularity. The proof of the following result can be found in [24, Theorem 2.6].

Theorem 14. *Suppose that Y is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ with an order continuous norm. If a linear operator T is bounded in $Y^{1/2}$ then for every $f \in Y'$, $m > 1$ and $a > K_G \|T\|_{Y^{1/2} \to Y^{1/2}}$, K_G being the Grothendieck constant, there exists a majorant $w \geq |f|$ with $\|w\|_{Y'} \leq \frac{m}{m-1} \|f\|_{Y'}$ such that*

$$\|T\|_{\mathrm{L}_2(w^{-1/2}) \to \mathrm{L}_2(w^{-1/2})} \leq a\sqrt{m}.$$

The following result is a direct precursor to a very deep and nontrivial fact that the so-called BMO-regularity is self-dual at least for Banach lattices having the Fatou property; see [12], [15], [24].

Theorem 15 ([24, Theorem 1.6]). *Suppose that X is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ having the Fatou property. Suppose also that $X\mathrm{L}_q$ for some $1 < q < \infty$ is a Banach lattice and $X\mathrm{L}_q$ is A_p-regular for some $1 \leq p < \infty$. Then X is A_{p+1}-regular.*

We are now ready to prove the implication (3)⇒(1) of Theorem 1, which can be stated as follows.

Theorem 16. *Suppose that X is a Banach lattice of measurable functions on $(S \times \Omega, \nu \times \mu)$ such that X is p-convex and q-concave for some $1 < p, q < \infty$ and X has the Fatou property. Let T be a linear operator on $\mathrm{L}_2(S \times \Omega)$ such that both T and T^* are A_2-nondegenerate and T acts boundedly in X and in all L_s for $1 < s < \infty$. Then the lattices X and X' are A_1-regular.*

By p-convexity X^p is also a Banach lattice with the Fatou property, and so $X^{p(1-\theta)}\mathrm{L}_t^{\theta}$ is also a Banach lattice for all $1 \leq t \leq \infty$ and $0 < \theta < 1$. Choosing $\theta = 1 - 1/p$ shows that $Y_s = X\mathrm{L}_s$ is a Banach lattice for all sufficiently large s. The lattice Y_s has the Fatou property and has order continuous norm (because L_s has order continuous norm for $s < \infty$). Since T is bounded in X and in L_s for all $1 < s < \infty$, by the interpolation property mentioned in Section 1 the operator T is also bounded in $X^{1/2}\mathrm{L}_s^{1/2} = Y_s^{1/2}$ for all $1 < s < \infty$. Theorem 14 and A_2-nondegeneracy of T then imply that the lattice $Y_s' = X'\mathrm{L}_{s'}$ is A_2-regular for all sufficiently large s. By Theorem 15 it follows that the lattice X' is A_3-regular, and furthermore by Proposition 11 the lattice $X^{1/3}$ is A_1-regular.

Since the convexity assumptions of Theorem 16 imply that X and X' have order continuous norm, we have $X' = X^*$ and $X = (X')^*$, and moreover $X \cap \mathrm{L}_2$ is dense in X and $X' \cap \mathrm{L}_2$ is dense in X', so the duality relation

$$\int (Tf)g = \int f(T^*g) \quad \text{for } f \in X \cap \mathrm{L}_2 \text{ and } g \in X' \cap \mathrm{L}_2$$

shows that boundedness of T in X implies boundedness of T^* in X' and vice versa.

Repeating the argument above with X' in place of X (X' is q'-convex since X is q-concave) and with T^* in place of T shows that X is A_3-regular and $(X')^{1/3}$ is A_1-regular.

Finally, we apply Theorem 8 to X and to X' with $p = 3$ and $\delta = 1/3$, which establishes that X and X' are both A_1-regular. The proof of Theorem 16 is complete.

4. Concluding remarks. The theory of Calderón–Zygmund operators naturally generalizes to spaces of homogeneous type. However, we cannot just replace $\mathbb{R}^n$ by a space of homogeneous type in the statement of Theorem 1 because it is not clear for which of these spaces there is at least one suitably nondegenerate linear operator R. The individual implications, however, still work in the form of Proposition 5 and Theorem 16.

The p-convexity and q-concavity assumptions of Theorem 1 are probably superfluous; so far we can only say for sure that they are not used in the implication (1)⇒(2). We conjecture that these assumptions are actually a consequence of any of the conditions of Theorem 1; that condition (1) implies p-convexity and q-concavity with some $1 < p, q < \infty$ is known to hold true at least for variable exponent Lebesgue spaces (see, e.g., [9, Theorem 4.7.1]), and it seems that it is possible to adapt the same argument to cover suitable nondegenerate singular integral operators as well. Recently in [7, Theorem 5.42] it was established that if all Riesz transforms R_j are bounded in $L_{p(\cdot)}$ then the exponent $p(\cdot)$ is bounded away from 1 and ∞.

Furthermore, it is easy to see that the assumptions of p-convexity and q-concavity could be eliminated from Theorem 16 if we assume boundedness of T and T^* in $X^{1/2}$ and $(X')^{1/2}$ instead of just X. It is, however, unclear whether boundedness of a Calderón–Zygmund operator T in X implies its boundedness in $X^{1/2}$; this seems plausible because T acts boundedly from L_∞ to BMO, but as far as we know, all available interpolation results that make it possible to replace L_∞ by BMO as an endpoint in the appropriate interpolation scale (see, e.g., [16], [25]) work only under the assumption that both X^α and $(X')^\alpha$ are A_1-regular for some $\alpha > 0$, which is a consequence of what we are trying to establish in this setting.

There is a different approach to the implication (3)⇒(1) of Theorem 1 that works at least in certain cases. Let $\mathcal{S} = \{Q_l\}$ be a collection of cubes or balls. We define operators

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} f_Q \chi_Q(x) \quad \text{and} \quad \mathcal{A}_{\mathcal{S}}^{\square} f(x) = \Big(\sum_{Q\in\mathcal{S}} (f_Q)^2 \chi_Q(x)\Big)^{1/2}$$

for all locally summable functions f; here $f_Q = |Q|^{-1}\int_Q f(z)\,dz$ for $Q \in \mathcal{S}$. It is easy to see that if the cubes or balls from $\mathcal{S}$ are pairwise disjoint then $\mathcal{A}_{\mathcal{S}}^{\square} f = \mathcal{A}_{\mathcal{S}} f$ almost everywhere for nonnegative functions f.

Proposition 17. *Suppose that a singular integral operator T with kernel K satisfying* (5) *is bounded with norm C in a Banach lattice X having the Fatou property. Then for any collection $\mathcal{S} = \{Q_l\}$ of cubes or balls we have*

$$\|\mathcal{A}_{\mathcal{S}}^{\square} f\|_X \le c_a \Big\| f\Big(\sum_l \chi_{Q_l}\Big)^{1/2}\Big\|_X \tag{7}$$

for all f such that the right-hand side of (7) *is well-defined, with a constant c_a independent of f and $\mathcal{S}$.*

Observe that Proposition 17 also implies that if a suitably nondegenerate operator T acts boundedly in X then all operators $\mathcal{A}_{\mathcal{S}}$ with disjoint collections $\mathcal{S}$ of cubes or balls are uniformly bounded in X. In particular, it is well known that taking collections $\mathcal{S}$ consisting of a single cube implies that if (7) holds true for $X = \mathrm{L}_p(w^{-1/p})$, $1 < p < \infty$, then $w \in \mathrm{A}_p$, and thus X is A_1-regular. It is not clear in general whether either (7) or uniform boundedness of $\mathcal{A}_{\mathcal{S}}$ is related to other properties of interest. Of course, one immediately observes that such operators $\mathcal{A}_{\mathcal{S}}$ are bounded in L_p for both $p = 1$ and $p = \infty$, so their uniform boundedness in a lattice X does not necessarily mean that X is A_1-regular. However, and somewhat surprisingly, this implication holds true at least in the case of variable exponent Lebesgue spaces $X = \mathrm{L}_{p(\cdot)}$ if we also assume that X is p-convex and q-concave for some $1 < p, q < \infty$; see, e.g., [9, Theorem 5.7.2]. Thus not only the converse to [7, Theorem 5.39] is true for nondegenerate operators, which answers positively [7, Problem A.17], but there is also no need to involve the complicated machinery of the main results of the present work.

Let us prove Proposition 17. First, observe that (5) implies by [27, Chapter 5, §4.6] that there exists a constant $c > 0$ and some $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and any locally summable nonnegative function f supported on B we have

$$|Tf(x)| \geq c f_B \tag{8}$$

for all $x \in B \pm r x_0$. Let $\mathcal{S}' = \{Q'_l\}$ with $Q'_l = Q_l + x_0$ being the cubes or balls Q_l shifted by x_0, and set $f_l = f \chi_{Q_l}$. We may assume that f is nonnegative and the right-hand side of (7) is finite. It follows that the sequence valued function $F = \{f_l\}$ belongs to $X(l^2)$ with $\|F\|_{X(l^2)} = \|f(\sum_l \chi_{Q_l})^{1/2}\|_X$. Using the nondegeneracy assumption (8) and the Grothendieck theorem (which shows that T is bounded in $X(l^2)$; see, e.g., [18]) we can easily obtain an estimate

$$\begin{aligned} c^{-1}\Big\|\Big(\sum_l \chi_{Q'_l}(f_{Q_l})^2\Big)^{1/2}\Big\|_X &\leq \Big\|\Big(\sum_l \chi_{Q'_l}|Tf_l|^2\Big)^{1/2}\Big\|_X \\ &\leq \|TF\|_{X(l^2)} \leq CK_G\|F\|_{X(l^2)} = CK_G\Big\|f\Big(\sum_l \chi_{Q_l}\Big)^{1/2}\Big\|_X, \end{aligned} \tag{9}$$

K_G being the Grothendieck constant. On the other hand, repeating this estimate for $G = \{g_l\}$, $g_l = \chi_{Q'_l} f_{Q_l}$, in place of F and with the order of Q_l and Q'_l reversed shows that

$$\begin{aligned} c^{-1}\|\mathcal{A}^{\square}_{\mathcal{S}} f\|_X = c^{-1}\Big\|\Big(\sum_l \chi_{Q_l}(f_{Q_l})^2\Big)^{1/2}\Big\|_X &\leq \Big\|\Big(\sum_l \chi_{Q_l}|Tg_l|^2\Big)^{1/2}\Big\|_X \\ &\leq \|TG\|_{X(l^2)} \leq CK_G\|G\|_{X(l^2)} = CK_G\Big\|\Big(\sum_l \chi_{Q'_l}(f_{Q_l})^2\Big)^{1/2}\Big\|_X. \end{aligned} \tag{10}$$

Combining (9) and (10) yields (7) with $c_a = (CcK_G)^2$.

There is an interesting generalization of Proposition 5 which is easily obtained from certain less classical results. It is well known (see, e.g., [2], [11]) that

$$M^{\sharp}_{\lambda}(Tf) \le cMf \tag{11}$$

almost everywhere for a wide variety of operators T including Calderón–Zygmund operators and all locally summable functions f with c independent of f, where $M^{\sharp}_{\lambda}$ is the Strömberg local sharp maximal function. S_0 denotes the set of all measurable functions f on $\mathbb{R}^n$ whose nonincreasing rearrangement f^* satisfies $f^*(+\infty) = 0$. The following result is similar to the well-known duality relation between H_1 and BMO.

Theorem 18 ([19, Theorem 1]).

$$\int |fg| \le c\int (M^{\sharp}_{\lambda} f)(Mg)$$

for any $f \in S_0$ *and locally summable function* g*, with some* c *and* λ *independent of* f *and* g.

Now we are ready to obtain the following extension of Proposition 5.

Theorem 19. *Suppose that* X, Y *and* Z *are Banach lattices on* $\mathbb{R}^n$ *having the Fatou property,* S_0 *is dense in* X*, and the Hardy–Littlewood maximal operator* M *acts boundedly from* X *to* Z *and from* Y' *to* Z'*. Then any operator* T *that satisfies estimate* (11) *acts boundedly from* X *to* Y.

Theorem 19 follows at once from Theorem 18, since for any $f \in X \cap S_0$ and $g \in Y'$ we have the estimate

$$\begin{aligned} \int |(Tf)g| &\le c\int [M^{\sharp}_{\lambda}(Tf)][Mg] \le c_1 \int (Mf)\,(Mg) \\ &\le c_1 \|Mf\|_Z \|Mg\|_{Z'} \le c_2 \|f\|_X \|g\|_{Y'} \end{aligned} \tag{12}$$

with some c, c_1 and c_2 independent of f and g.

Since M is a positive operator and $Mg \ge g$ almost everywhere for any locally summable g, the assumptions of Theorem 19 imply that $X \subset Z$ and $Y' \subset Z'$, which in turn implies that $X \subset Z \subset Y$. Unlike the case $X = Y = Z$, it is presently unclear whether Theorem 19 admits a converse similar to Theorem 16 below. In other words, if a suitably nondegenerate Calderón–Zygmund operator T acts boundedly from X to Y, does it follow that $X \subset Y$ and there exists a lattice Z satisfying the conditions of Theorem 19?

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