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Fourier transform of Schwartz functions on the Heisenberg group

by

FRANCESCA ASTENGO (Genova), BIANCA DI BLASIO (Milano), and FULVIO RICCI (Pisa)

Abstract. Let H_1 be the 3-dimensional Heisenberg group. We prove that a modified version of the spherical transform is an isomorphism between the space $\mathcal{S}_m(H_1)$ of Schwartz functions of type m and the space $\mathcal{S}(\Sigma_m)$ consisting of restrictions of Schwartz functions on $\mathbb{R}^2$ to a subset Σ_m of the Heisenberg fan with $|m|$ of the half-lines removed. This result is then applied to study the case of general Schwartz functions on H_1.

1. Introduction. One of the most important properties of the Fourier transform $\mathcal{F}$ in $\mathbb{R}^n$ is that $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$, and $\mathcal{F}$ is an isomorphism. The relative closeness between the Heisenberg group H_n and $\mathbb{R}^n$ in many aspects of harmonic analysis raises the question whether a similar property holds in the Heisenberg group setting. A characterization of the image of the Schwartz space $\mathcal{S}(H_n)$ under the group Fourier transform $\mathcal{F}_{H_n}$ was given by D. Geller [6] in terms of "asymptotic series".

Taking $n = 1$ for simplicity, the Fourier transform $\mathcal{F}_{H_1} f$ of an integrable function f can be viewed as a scalar-valued function of several variables. The main variable, denoted by $\lambda \in \mathbb{R}$, defines a character on the center. Two further variables then come out, varying in $\mathbb{R}$ if $\lambda = 0$ and in $\mathbb{N}$ if $\lambda \neq 0$; the latter will be denoted as (j, k). Most of the work concerns the description of $\mathcal{F}_{H_1} f$ on the set where $\lambda \neq 0$, since the case $\lambda = 0$ follows by combining density with our previous result in [1] (see Remark 4.6).

The deep study developed by Geller [6] showed that the "Schwartzness" of the image $\mathcal{F}_{H_1}(\mathcal{S}(H_1))$ relies on a set of rapid decay estimates holding when appropriate differential-difference operators are applied to $\mathcal{F}_{H_1} f$.

This type of analysis emphasizes a preliminary decomposition of the function f into m-types, i.e. $f = \sum_{m\in\mathbb{Z}} f_m$, where $f_m(e^{i\theta} z, t) = e^{im\theta} f(z, t)$ for every $e^{i\theta} \in \mathbb{T}$, $z \in \mathbb{C}$, $t \in \mathbb{R}$.

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Functions which are radial in the variable z, i.e., with $m = 0$, play a special rôle, and their Fourier transform is supported (for $\lambda \neq 0$) on the set of triples (λ, j, k) with $j = k$. Functions of type 0 form a commutative algebra, and their Fourier transforms coincide with their spherical transforms, according to the general theory of Gelfand pairs.

The work of C. Benson, J. Jenkins and G. Ratcliff [3] on the characterization of spherical transforms of K-invariant Schwartz functions on H_n for general Gelfand pairs $(K \ltimes H_n, K)$ is a considerable refinement of Geller's results in the presence of different kinds of invariance.

More recently [2], we have obtained a description of spherical transforms of K-invariant Schwartz functions, of a completely different nature than that of Benson, Jenkins and Ratcliff, and more reminiscent of the original result on $\mathbb{R}^n$. Restricting again ourselves to type-0 functions on H_1, the variables (λ, j) are parameters describing an intrinsic object,

$$\Sigma^* = \{(\xi, \lambda) \in \mathbb{R}^2 : \lambda \neq 0,\, \xi = |\lambda|(2j+1),\, j \in \mathbb{N}\},$$

whose closure Σ is called the *Heisenberg fan*. The set Σ is, at the same time, the Gelfand spectrum of the algebra of type-0 L^1-functions, and the joint L^2-spectrum of the sublaplacian $\mathcal{L}$ and the symmetrized central derivative $i^{-1}T$.

The main theorem of [1] says that, regarding spherical transforms as functions defined on the Heisenberg fan Σ, the image under the spherical transform of type-0 Schwartz functions is the space of Schwartz functions on Σ (meant as restrictions of Schwartz functions on $\mathbb{R}^2$).

In this paper we give an extension of this result to general Schwartz functions on H_1.

We first consider Schwartz functions of type m (Section 3) and show that a modified version $\tilde{\mathcal{G}}_m$ of the spherical transform is an isomorphism between $\mathcal{S}_m(H_1)$ and $\mathcal{S}(\Sigma_m)$, where Σ_m is obtained from Σ by removing $|m|$ of the half-lines in Σ^*.

In Section 4 we associate to a general function $f \in \mathcal{S}(H_1)$ the sequence $\{\tilde{\mathcal{G}}_m f_m\}_{m\in\mathbb{Z}}$, where f_m is the m-type component of f.

This leads to introducing the space $\mathfrak{S}$ of sequences $\mathbf{G} = \{G_m\}_{m\in\mathbb{Z}}$ with $G_m \in \mathcal{S}(\Sigma_m)$. We introduce a Fréchet space structure on $\mathfrak{S}$ which makes it isomorphic to $\mathcal{S}(H_1)$. The family of norms on $\mathfrak{S}$ that gives this isomorphism does not look as a natural combination of quotient norms of the various components, but it brings together features that are already present in [3] and [1].

It would be natural to ask if the various entries G_m of an element $\mathbf{G}$ of $\mathfrak{S}$ admit Schwartz extensions $G_m^\#$ to $\mathbb{R}^2$ such that $\psi_m(re^{i\theta}, t) = e^{im\theta} G_m^\#(r^2, t)$ are the m-types of a single Schwartz function ψ on $\mathbb{C} \times \mathbb{R}$. In this case, a single Schwartz function would subsume all information about the Fourier transform of a given Schwartz function on H_1.

The result in Theorem 4.3 below goes in this direction.

Even though we have restricted ourselves to H_1, we do not expect major difficulties in extending these results to H_n, with the m-types ($m \in \mathbb{Z}^n$) defined in terms of the action of the torus $\mathbb{T}^n$ and the Heisenberg fan replaced by the Heisenberg brush in $\mathbb{R}^{n+1}$.

2. Preliminaries

2.1. Notation and basic facts. We regard the Heisenberg group H_1 as $\mathbb{C} \times \mathbb{R}$ with the product

$$(z,t)(z',t') = \big(z+z', t+t' - \tfrac{1}{2}\operatorname{Im}(z\overline{z'})\big).$$

The left-invariant vector fields

$$X = \partial_x - \frac{y}{2}\partial_t, \qquad Y = \partial_y + \frac{x}{2}\partial_t,$$
$$Z = \tfrac{1}{2}(X - iY), \qquad \bar{Z} = \tfrac{1}{2}(X + iY)$$

satisfy the commutation rules $[X,Y] = \partial_t = T$ and $[Z,\bar{Z}] = iT/2$. The vector field T is central.

The sublaplacian $\mathcal{L}$, defined as $\mathcal{L} = -(X^2+Y^2) = -2(Z\bar{Z}+\bar{Z}Z)$, satisfies the commutation rules

$$[\mathcal{L}, Z] = 2iTZ, \qquad [\mathcal{L}, \bar{Z}] = -2iT\bar{Z}.$$

The basics of Fourier analysis on H_1 are developed, e.g., in [9]. The relevant aspects needed below can be condensed in the inversion formula and in the Plancherel formula,

$$\begin{aligned} f(z,t) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\sum_{j,k\in\mathbb{N}} \hat{f}(\lambda,j,k)\overline{\Phi^\lambda_{j,k}(z,t)}|\lambda|\,d\lambda,\\ \|f\|_2^2 &= \frac{1}{4\pi^2}\int_{\mathbb{R}}\sum_{j,k\in\mathbb{N}} |\hat{f}(\lambda,j,k)|^2|\lambda|\,d\lambda, \end{aligned} \tag{2.1}$$

where

$$\hat{f}(\lambda,j,k) = \int f(z,t)\Phi^\lambda_{j,k}(z,t)\,dz\,dt$$

and the matrix-valued functions $\Phi^\lambda(z,t) = (\Phi^\lambda_{j,k}(z,t))_{j,k}$ are defined for $\lambda \neq 0$ and represent the infinite-dimensional irreducible representations of H_1 in a convenient orthonormal frame in the representation space (the Hermite functions in the Schrödinger model, the monomials in the Bargmann–Fock model).

The functions $\Phi^\lambda_{j,k}$ have the following properties:

(i) $\Phi^{-\lambda}_{j,k}(z,t) = \Phi^\lambda_{j,k}(\bar{z},-t)$;

(ii) for $\lambda > 0$, $\Phi^\lambda_{j,k}(z,t) = \Phi^1_{j,k}(\sqrt{\lambda}\,z, \lambda t)$;

(iii) with $L_k^{(m)}$ denoting the Laguerre polynomial of order m and degree k (cf. [9]),

$$\Phi^1_{j,k}(z,t)=\begin{cases} e^{it}e^{-|z|^2/4}\bar z^{j-k}L_k^{(j-k)}(|z|^2/2), & j\ge k,\\ e^{it}e^{-|z|^2/4}(-z)^{k-j}L_j^{(k-j)}(|z|^2/2), & j<k;\end{cases}$$

(iv) $\mathcal{L}\Phi^\lambda_{j,k}=|\lambda|(2k+1)\Phi^\lambda_{j,k}$ and $T\Phi^\lambda_{j,k}=i\lambda\Phi^\lambda_{j,k}$;

(v)

$$Z\Phi^\lambda_{j,k}=\begin{cases} -\sqrt{k\lambda/2}\,\Phi^\lambda_{j,k-1}, & \lambda>0,\\ \sqrt{(k+1)|\lambda|/2}\,\Phi^\lambda_{j,k+1}, & \lambda<0,\end{cases}$$

$$\bar Z\Phi^\lambda_{j,k}=\begin{cases} \sqrt{(k+1)\lambda/2}\,\Phi^\lambda_{j,k+1}, & \lambda>0,\\ -\sqrt{k|\lambda|/2}\,\Phi^\lambda_{j,k-1}, & \lambda<0.\end{cases}$$

For $f\in\mathcal{S}(H_1)$, the following identities follow from (iv) and (v):

$$\widehat{\mathcal{L}f}(\lambda,j,k)=|\lambda|(2k+1)\hat f(\lambda,j,k),\qquad \widehat{Tf}(\lambda,j,k)=-i\lambda\hat f(\lambda,j,k), \tag{2.2}$$

and, for every positive integer r,

$$\widehat{Z^r f}(\lambda,j,k)=\begin{cases} 0, & \lambda>0,\ k\le r-1,\\ \sqrt{\prod_{\ell=0}^{r-1}\frac12\lambda(k-\ell)}\,\hat f(\lambda,j,k-r), & \lambda>0,\ k\ge r,\\ (-1)^r\sqrt{\prod_{\ell=1}^{r}\frac12|\lambda|(k+\ell)}\,\hat f(\lambda,j,k+r), & \lambda<0,\ k\ge 0,\end{cases} \tag{2.3}$$

$$\widehat{\bar Z^r f}(\lambda,j,k)=\begin{cases} (-1)^r\sqrt{\prod_{\ell=1}^{r}\frac12\lambda(k+\ell)}\,\hat f(\lambda,j,k+r), & \lambda>0,\ k\ge 0,\\ 0, & \lambda<0,\ k\le r-1,\\ \sqrt{\prod_{\ell=0}^{r-1}\frac12|\lambda|(k-\ell)}\,\hat f(\lambda,j,k-r), & \lambda<0,\ k\ge r.\end{cases} \tag{2.4}$$

2.2. Schwartz spaces. On the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ (for us $\mathbb{R}^n$ will be either $\mathbb{R}^2$ or $\mathbb{C}\times\mathbb{R}$, the latter meant also as the underlying space of H_1) we consider the following family of norms, parametrized by a nonnegative integer p:

$$\|f\|_{(p,\mathbb{R}^n)}=\max_{N+\alpha\le p}\sup_{x\in\mathbb{R}^n}(1+|x|)^N|\partial^\alpha f(x)|. \tag{2.5}$$

LEMMA 2.1. *Let $\mathcal{A}(z,t)=|z|^2/4+it$. Then the family of norms on $\mathcal{S}(H_1)$,*

$$\|f\|_{(p,H_1)} = \max_{2a+2b\le p} \|\mathcal{L}^b \mathcal{A}^a f\|_2, \qquad p \in \mathbb{N}, \tag{2.6}$$

is equivalent to the family $\{\|f\|_{(p,\mathbb{C}\times\mathbb{R})}\}_{p\in\mathbb{N}}$.

Proof. It is well-known that on $\mathbb{R}^n$ the family (2.5) can be replaced by the equivalent family

$$\|f\|_{[p]} = \max_{|\alpha|+|\beta|\le p} \|x^\alpha \partial^\beta f\|_2, \qquad p \in \mathbb{N}. \tag{2.7}$$

It is known as well that, on a nilpotent group, the partial derivatives in (2.7) can be replaced by products of left-invariant vector fields in some basis of the Lie algebra [5]. This reduces matters to showing the equivalence of the family (2.6) with

$$\|f\|_{[p],H_1} = \max_{2k+2\ell+m+n+2q\le p} \big\| |z|^{2k} |t|^\ell Z^m \bar{Z}^n T^q f \big\|_2, \qquad p \in \mathbb{N}. \tag{2.8}$$

On the other hand, by the L^2-boundedness of the Riesz transforms associated with $\mathcal{L}$, the family (2.6) is equivalent to

$$\|f\|^*_{(p,H_1)} = \max_{2a+m+n+2q\le p} \|Z^m \bar{Z}^n T^q \mathcal{A}^a f\|_2, \qquad p \in \mathbb{N}. \tag{2.9}$$

We show that, for each $p \in \mathbb{N}$, the pth norm in (2.8) is equivalent to the pth norm in (2.9).

Using the identities

$$\begin{gathered} [Z, \mathcal{A}] = \bar{z}/2, \qquad [\bar{Z}, \mathcal{A}] = 0, \qquad [T, \mathcal{A}] = i, \\ [Z, \bar{z}] = [T, \bar{z}] = 0, \qquad [\bar{Z}, \bar{z}] = 1, \end{gathered} \tag{2.10}$$

it is easy to verify that the pth norm (2.9) is controlled by the pth norm (2.8).

To show the converse, we proceed by induction. The cases $p = 0, 1, 2$ are obvious. Assume that, for $p \ge 2$ even, the pth norm (2.8) is controlled by the pth norm (2.9). Consider one of the quantities $\big\| |z|^{2k} |t|^\ell Z^m \bar{Z}^n T^q f \big\|_2$ on the right-hand side of (2.8) with $2k + 2\ell + m + n + 2q = p + 1$.

If $m + n + 2q = 0$, i.e., there are no derivatives, it is sufficient to observe that $|z|^{2k}|t|^\ell \le C_p \mathcal{A}^{k+\ell}$. Suppose therefore that $m + n + 2q > 0$.

Assume first that $q > 0$. Applying the inductive hypothesis to Tf, we obtain ([1])

$$\big\| |z|^{2k} |t|^\ell Z^m \bar{Z}^n T^q f \big\|_2 \le C \max_{2a'+m'+n'+2q'\le p} \|Z^{m'} \bar{Z}^{n'} T^{q'} \mathcal{A}^{a'} T f\|_2.$$

It is then sufficient to apply the identity $[\mathcal{A}^{a'}, T] = -ia'\mathcal{A}^{a'-1}$, which follows from (2.10).

([1]) We shall use C to denote a positive constant which may vary from line to line. When it is relevant, dependence of such constants upon parameters of interest will be indicated through the use of subscripts.

If $q=0$ and $n>0$, we apply the inductive hypothesis to $\bar{Z}f$ and (2.10) to obtain

$$\big\||z|^{2k}|t|^{\ell}Z^m\bar{Z}^n f\big\|_2 \le C \max_{2a'+m'+n'+2q'\le p} \|Z^{m'}\bar{Z}^{n'+1}T^{q'}\mathcal{A}^{a'}f\|_2 \le \|f\|^*_{(p+1,H_1)}.$$

In the last case, $q=n=0$ and $m>0$, we apply the inductive hypothesis to Zf. By (2.10), we have

$$\begin{aligned}\big\||z|^{2k}|t|^{\ell}Z^m f\big\|_2 &\le C \max_{2a'+m'+n'+2q'\le p} \|Z^{m'}\bar{Z}^{n'}T^{q'}\mathcal{A}^{a'}Zf\|_2\\ &\le C \max_{2a'+m'+n'+2q'\le p} \big(\|Z^{m'}\bar{Z}^{n'}T^{q'}Z\mathcal{A}^{a'}f\|_2 + a'\|Z^{m'}\bar{Z}^{n'}T^{q'}\bar{z}\mathcal{A}^{a'-1}f\|_2\big)\\ &\le C\|f\|^*_{(p+1,H_1)} + C \max_{2a'+m'+n'+2q'\le p} \|Z^{m'}\bar{Z}^{n'}T^{q'}\bar{z}\mathcal{A}^{a'-1}f\|_2.\end{aligned}$$

By (2.10), for $g\in\mathcal{S}(H_1)$,

$$Z^{m'}\bar{Z}^{n'}T^{q'}\bar{z}g = \bar{z}Z^{m'}\bar{Z}^{n'}T^{q'}g + n'Z^{m'}\bar{Z}^{n'-1}T^{q'}g.$$

Therefore, if $a'\ge 1$, we can again use the inductive hypothesis with $g=\mathcal{A}^{a'-1}f$. ∎

Remark 2.2. It is easy to verify that, when p is even, the norms (2.6) and (2.9) are equivalent. Moreover using the commutation rules of the vector fields Z and $\bar{Z}$, it is easy to show that for every nonnegative integer p,

$$C_p\|f\|_{[p],H_1} \le \|\bar{f}\|_{[p],H_1} \le C'_p\|f\|_{[p],H_1} \quad \forall f\in\mathcal{S}(H_1).$$

Therefore, arguing as in the proof of Lemma 2.1 we deduce that for every nonnegative integer p there exist positive constants C_p and C'_p such that

$$C_p \max_{2a+2b\le p} \|\mathcal{L}^b\mathcal{A}^a f\|_2 \le \max_{2a+2b\le p} \|\mathcal{L}^b\bar{\mathcal{A}}^a f\|_2 \le C'_p \max_{2a+2b\le p} \|\mathcal{L}^b\mathcal{A}^a f\|_2. \tag{2.11}$$

2.3. Functions of type m. We say that a function f of $z\in\mathbb{C}$ (or of $(z,t)\in\mathbb{C}\times\mathbb{R}$) is *of type* $m\in\mathbb{Z}$ if $f(e^{i\theta}z)=e^{im\theta}f(z)$.

We need the following version of Hadamard's division lemma. For its proof we refer to [4, Lemma 5.3].

Lemma 2.3. *Let s be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^2)$ such that $\partial_\xi^\alpha u(0,\lambda)=0$ for every $\alpha=0,\dots,s-1$ and for every real λ. Then there exists a function v in $\mathcal{S}(\mathbb{R}^2)$ such that*

$$u(\xi,\lambda)=\xi^s v(\xi,\lambda) \quad \forall(\xi,\lambda)\in\mathbb{R}^2.$$

Proposition 2.4. *For F in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$ and m in $\mathbb{Z}$, denote by*

$$\varTheta_m F(\zeta,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta}\zeta,\lambda)e^{-im\theta}\,d\theta \tag{2.12}$$

the m-type component of F. Then $\Theta_m F$ is in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$ and the following properties hold:

(1) $\|\Theta_m F\|_{(p,\mathbb{C}\times\mathbb{R})} \le C_\ell (1+|m|)^{-\ell}\, \|F\|_{(p+2\ell,\mathbb{C}\times\mathbb{R})}$ *for any nonnegative integers p and ℓ, so that the series $\sum_m \Theta_m F$ converges to F in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$;*

(2) *for every integer m there exists a function F_m in $\mathcal{S}(\mathbb{R}^2)$ such that*

$$\Theta_m F(\zeta,\lambda) = \begin{cases} \zeta^m\, F_m(|\zeta|^2,\lambda), & m\ge 0,\\ \bar\zeta^{|m|}\, F_m(|\zeta|^2,\lambda), & m<0.\end{cases}$$

Proof. It is easy to check that when $m\ne 0$,

$$\Theta_m F(\zeta,\lambda) = \frac{(-im)^{-\ell}}{2\pi}\int_0^{2\pi} \frac{d^\ell}{d\theta^\ell} F(e^{i\theta}\zeta,\lambda)e^{-im\theta}\,d\theta \quad \forall(\zeta,\lambda)\in\mathbb{C}\times\mathbb{R},$$

from which the estimate in (1) follows easily.

As for (2), suppose that $m>0$ and denote by u_m the function $\Theta_m F$ restricted to $\mathbb{R}^2$, i.e.

$$u_m(\xi,\lambda) = \Theta_m F(\xi,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta}\xi,\lambda)e^{-im\theta}\,d\theta \quad \forall(\xi,\lambda)\in\mathbb{R}^2.$$

It is easy to verify that $\partial_\xi^\alpha u_m(0,\lambda)=0$ for every $\alpha=0,\dots,m-1$ and every real λ. Thus by Lemma 2.3 there exists v_m in $\mathcal{S}(\mathbb{R}^2)$ such that

$$\Theta_m F(\xi,\lambda) = u_m(\xi,\lambda) = \xi^m v_m(\xi,\lambda) \quad \forall(\xi,\lambda)\in\mathbb{R}^2.$$

On the other hand, for real ξ,

$$\Theta_m F(e^{i\theta}\xi,\lambda) = e^{im\theta}u_m(\xi,\lambda) = e^{im\theta}\xi^m v_m(\xi,\lambda).$$

In particular if $\theta=\pi$ we obtain

$$v_m(-\xi,\lambda) = v_m(\xi,\lambda) \quad \forall(\xi,\lambda)\in\mathbb{R}^2.$$

By the Whitney–Schwarz Theorem (see [2, Theorem 6.1] for the case of Schwartz functions) there exists a function F_m in $\mathcal{S}(\mathbb{R}^2)$ such that

$$v_m(\xi,\lambda) = F_m(\xi^2,\lambda).$$

Therefore if $\zeta=\xi e^{i\theta}$, then $|\zeta|^2=\xi^2$ and

$$\Theta_m F(\zeta,\lambda) = \Theta_m F(e^{i\theta}\xi,\lambda) = e^{im\theta}\xi^m v_m(\xi,\lambda) = \zeta^m F_m(|\zeta|^2,\lambda)$$

as required. ∎

Lemma 2.5. *Let F be in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$ and let $\{F_m\}_{m\in\mathbb{Z}}$ be the sequence of functions in $\mathcal{S}(\mathbb{R}^+\times\mathbb{R})$ such that*

$$F(\zeta,\lambda) = \sum_m \Theta_m F(\zeta,\lambda) = \sum_{m\ge 0}\zeta^m F_m(|\zeta|^2,\lambda) + \sum_{m<0}\bar\zeta^{-m}F_m(|\zeta|^2,\lambda).$$

Then for all nonnegative integers α, β, N,

$$\sup_{(\xi,\lambda)\in\mathbb{R}_+\times\mathbb{R}} \xi^{|m|/2}(1+|\lambda|+\xi)^N |\partial_\lambda^\beta \partial_\xi^\alpha F_m(\xi,\lambda)| \le C_p \|\Theta_m F\|_{(p,\mathbb{C}\times\mathbb{R})} \qquad \forall m \in \mathbb{Z}$$

with $p = 2\alpha + \beta + 2N$.

Proof. Obviously

$$\begin{aligned}\xi^{|m|/2}\,(1+|\lambda|+\xi)^N |F_m(\xi,\lambda)| &= (1+|\lambda|+|\zeta|^2)^N |\Theta_m F(\zeta,\lambda)| \\ &\le C_N \|\Theta_m F\|_{(2N,\mathbb{C}\times\mathbb{R})}.\end{aligned}$$

Note that

$$\begin{cases} \partial_{\bar\zeta}^\alpha \Theta_m F(\zeta,\lambda) = \zeta^{m+\alpha}\,\partial_\xi^\alpha F_m(|\zeta|^2,\lambda), & m \ge 0, \\ \partial_\zeta^\alpha \Theta_m F(\zeta,\lambda) = \bar\zeta^{|m|+\alpha}\,\partial_\xi^\alpha F_m(|\zeta|^2,\lambda), & m < 0, \end{cases}$$

and denote

$$\partial^{\alpha'} = \begin{cases} \partial_{\bar\zeta}^\alpha, & m \ge 0, \\ \partial_\zeta^\alpha, & m < 0. \end{cases}$$

Thus

$$|\zeta|^{|m|} |\partial_\xi^\alpha F_m(|\zeta|^2,\lambda)| = |\zeta|^{-\alpha} |\partial^{\alpha'} \Theta_m F(\zeta,\lambda)|.$$

When $|\zeta| > 1$ there is a trivial estimate

$$|\zeta|^m |\partial_\xi^\alpha F_m(|\zeta|^2,\lambda)| = |\zeta^{-\alpha} \partial^{\alpha'} \Theta_m F(\zeta,\lambda)| \le |\partial^{\alpha'} \Theta_m F(\zeta,\lambda)|,$$

while when $|\zeta| \le 1$ we can use Taylor's expansion to conclude that

$$\begin{aligned}|\zeta|^{|m|}\,|\partial_\xi^\alpha F_m(|\zeta|^2,\lambda)| &= |\zeta|^{-\alpha} |\partial^{\alpha'} \Theta_m F(\zeta,\lambda)| \\ &\le C_\alpha \sup_{\substack{|\zeta|\le 1 \\ \gamma+\gamma'\le 2\alpha}} |\partial_\zeta^\gamma \partial_{\bar\zeta}^{\gamma'} \Theta_m F(\zeta,\lambda)|.\end{aligned}$$

Putting together these two estimates we obtain

$$\begin{aligned}&\xi^{|m|/2}(1+|\lambda|+\xi)^N |\partial_\lambda^\beta \partial_\xi^\alpha F_m(\xi,\lambda)| \\ &\quad \le C_\alpha \sup_{\substack{|\zeta|\le 1,\,\lambda\in\mathbb{R} \\ \gamma+\gamma'\le 2\alpha}} (2+|\lambda|)^N |\partial_\lambda^\beta \partial_\zeta^\gamma \partial_{\bar\zeta}^{\gamma'} \Theta_m F(\zeta,\lambda)| \\ &\qquad + \sup_{\substack{|\zeta|\ge 1 \\ \lambda\in\mathbb{R}}} (1+|\lambda|+|\zeta|^2)^N |\partial_\lambda^\beta \partial^{\alpha'} \Theta_m F(\zeta,\lambda)| \\ &\quad \le C_p \|\Theta_m F\|_{(p,\mathbb{C}\times\mathbb{R})}. \ \blacksquare\end{aligned}$$

In the remaining part of this section we describe some properties of m-type functions on the Heisenberg group. Note that the function $\Phi_{j,k}^\lambda$ is of type $k-j$ if $\lambda > 0$, and of type $j-k$ if $\lambda < 0$. Therefore a function f in $\mathcal{S}(H_1)$ is of type m if and only if $\hat f(\lambda,j,k) = 0$ for $j-k \ne m \operatorname{sgn}\lambda$.

For $f \in \mathcal{S}(H_1)$ and $m \in \mathbb{Z}$, let $\Theta_m f$ be the m-type component of f defined as in (2.12). Then $\Theta_m f$ belongs to $\mathcal{S}(H_1)$ and

$$\Theta_m f(z,t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sum_{j-k=m\,\mathrm{sgn}\,\lambda} \hat{f}(\lambda,j,k)\overline{\Phi^\lambda_{j,k}(z,t)}|\lambda|\,d\lambda.$$

LEMMA 2.6. *Let f be in $\mathcal{S}(H_1)$. Then for all nonnegative integers p and ℓ, $\|\Theta_m f\|_{(p,H_1)} \le C_{p,\ell}(1+|m|)^{-\ell}\|f\|_{(p+4\ell,H_1)}$, so that the series $\sum_m \Theta_m f$ converges to f in $\mathcal{S}(H_1)$.*

Proof. Since $\frac{d}{d\theta} = izZ - i\bar{z}\bar{Z} - \frac{|z|^2}{2}T$, we have

$$\begin{aligned}\|\Theta_m f\|_{(p,H_1)} &\le C_\ell(1+|m|)^{-\ell} \sup_{2a+2b\le p} \int_0^{2\pi} \Big\|\mathcal{L}^b\mathcal{A}^a \frac{d^\ell}{d\theta^\ell} f(e^{i\theta}\cdot,\cdot)\Big\|_2\,d\theta \\ &\le C_{p,\ell}(1+|m|)^{-\ell}\|f\|_{(p+4\ell,H_1)}. \ \blacksquare\end{aligned}$$

The Gelfand spectrum of the algebra of type-0 integrable functions may be identified with the *Heisenberg fan* $\Sigma = \overline{\Sigma^*} = \Sigma^* \cup (\mathbb{R}_+ \times \{0\})$, where $\mathbb{R}_+ = [0,\infty)$ and

$$\Sigma^* = \{(\xi,\lambda) \in \mathbb{R}^2 : \lambda \ne 0,\ \xi = |\lambda|(2j+1),\ j \in \mathbb{N}\}.$$

Let f be an integrable function on H_1. For every integer m we define the following functions on Σ^*:

$$\mathcal{G}_m f(|\lambda|(2j+1),\lambda) = \begin{cases} \frac{(-i)^{|m|}}{\prod_{k=1}^{|m|}\sqrt{2|\lambda|(j+k)}}\,\hat{f}(\lambda,j,j+|m|), & m\lambda \le 0,\ j\in\mathbb{N}, \\ \frac{i^{|m|}}{\prod_{k=1}^{|m|}\sqrt{2|\lambda|(j+k)}}\,\hat{f}(\lambda,j+|m|,j), & m\lambda > 0,\ j\in\mathbb{N}. \end{cases} \tag{2.13}$$

Note that $\mathcal{G}_0 f$ is the Gelfand transform of $\Theta_0 f$ relative to the Gelfand pair $(H_1, U(1))$. Moreover $\mathcal{G}_m\Theta_m f = \mathcal{G}_m f$ and $\mathcal{G}_m$ is injective on the space of m-type Schwartz functions on H_1. Indeed,

$$\|\Theta_m f\|_2^2 = \frac{1}{4\pi^2}\sum_{j\in\mathbb{N}}\Big(\prod_{k=1}^{|m|}(j+k)\Big)\int_{\mathbb{R}} |\mathcal{G}_m\Theta_m f(|\lambda|(2j+1),\lambda)|^2(2|\lambda|)^{|m|}|\lambda|\,d\lambda.$$

If g is a type-0 function in $\mathcal{S}(H_1)$, then for every (ξ,λ) in Σ^*,

$$\mathcal{G}_0 g(\xi + 2(\lambda m)_+,\lambda) = \begin{cases} \mathcal{G}_m[(2i\bar{Z})^m g](\xi,\lambda), & m \ge 0, \\ \mathcal{G}_m[(2iZ)^{|m|} g](\xi,\lambda), & m < 0, \end{cases} \tag{2.14}$$

where x_+ denotes the positive part of the real number x.

The purpose of the next proposition is to give an analogue of Proposition 2.4(2) in the case of the Heisenberg group.

Proposition 2.7. *Let f be a Schwartz function on H_1. For every integer m, there exists a type-0 function g_m in $\mathcal{S}(H_1)$ such that*

$$\Theta_m f = \begin{cases} (2i\bar{Z})^m g_m, & m \geq 0, \\ (2iZ)^{|m|} g_m, & m < 0. \end{cases}$$

We will prove this proposition working on the Fourier transform side and we shall use the following result.

Lemma 2.8. *Let m be a positive integer and suppose that H in $\mathcal{S}(\mathbb{R}^2)$ vanishes on the half-lines $\lambda > 0 \mapsto (\lambda(2j+1), \lambda)$ for all $j = 0, \dots, m-1$. Then there exists $\tilde{H}$ in $\mathcal{S}(\mathbb{R}^2)$ such that $\tilde{H}_{|\Sigma^*} = H_{|\Sigma^*}$ and $\tilde{H}$ vanishes on the full lines $\lambda \in \mathbb{R} \mapsto (\lambda(2j+1), \lambda)$ for all $j = 0, \dots, m-1$.*

Proof. Let ψ be a nonnegative smooth function on the real line such that $\psi(0) = 1$ and whose support is contained in $(-1/2, 1/2)$. Define

$$\tilde{H}(\xi, \lambda) = \begin{cases} H(\xi, \lambda) - \displaystyle\sum_{k=0}^{m-1} \psi(\xi/\lambda - (2k+1)) H(\lambda(2k+1), \lambda), & \lambda \neq 0, \\ H(\xi, 0), & \lambda = 0. \end{cases}$$

It is easy to show that $\tilde{H}$ satisfies the required conditions. ∎

Proof of Proposition 2.7. We will focus on the case where $m \geq 0$. The case of $m < 0$ follows easily from the previous one, since $\Theta_m f = \overline{\Theta_{-m} \bar{f}}$ and $\overline{Zf} = \bar{Z}\bar{f}$.

So suppose that $m \geq 0$ and let $h_m = (2iZ)^m (\Theta_m f)$. Then h_m is a type-0 Schwartz function on H_1 and by [1] its Gelfand transform $\mathcal{G}_0 h_m$ can be extended to a function H_m in $\mathcal{S}(\mathbb{R}^2)$. Note that by (2.3),

$$\begin{aligned} H_m(|\lambda|(2j+1), \lambda) &= \widehat{h_m}(\lambda, j, j) \\ &= \begin{cases} (-i)^m \sqrt{\displaystyle\prod_{\ell=1}^{m} (2|\lambda|(j+\ell))}\, \hat{f}(\lambda, j, j+m), & \lambda < 0,\ j \geq 0, \\ i^m \sqrt{\displaystyle\prod_{\ell=1}^{m} (2\lambda(j-\ell+1))}\, \hat{f}(\lambda, j, j-m), & \lambda > 0,\ j \geq m, \end{cases} \end{aligned}$$

and H_m vanishes on the half-lines $\lambda > 0 \mapsto (|\lambda|(2j+1), \lambda)$ when $j = 0, \dots, m-1$.

By Lemma 2.8 we may suppose that H_m vanishes on the full lines, i.e.,

$$H_m(\xi, \lambda) = 0 \quad \text{whenever} \quad \xi = \lambda(2j+1), \quad \lambda \in \mathbb{R}, \quad j = 0, \dots, m-1.$$

Then we apply Lemma 2.3 m times, once for each line of the form $\xi = \lambda(2k+1)$, $k = 0, \dots, m-1$, with the corresponding change of variables. In this way we obtain a function G_m in $\mathcal{S}(\mathbb{R}^2)$ such that

$$H_m(\xi,\lambda) = \Big(\prod_{k=0}^{m-1}(\xi-(2k+1)\lambda)\Big)G_m(\xi,\lambda).$$

Let g_m be the type-0 function such that $\mathcal{G}_0 g_m = G_{m|\Sigma}$. Then by [7, 10, 1] the function g_m is in $\mathcal{S}(H_1)$. We now check that $(2i\bar{Z})^m g_m = \Theta_m f$. Indeed, they are both functions of type m and by (2.14), when $\lambda > 0$,

$$\begin{aligned}\mathcal{G}_m&[(2i\bar{Z})^m g_m](|\lambda|(2j+1),\lambda)\\ &= \mathcal{G}_0 g_m(|\lambda|(2j+1)+2(m\lambda)_+,\lambda) = G_m(|\lambda|(2j+2m+1),\lambda)\\ &= H_m(\lambda(2(j+m)+1),\lambda)\prod_{k=0}^{m-1}\frac{1}{2\lambda(j+m-k)}\\ &= \frac{i^m}{\prod_{k=1}^m\sqrt{2\lambda(j+k)}}\widehat{f}(\lambda,j+m,j) = \mathcal{G}_m\Theta_m f(|\lambda|(2j+1),\lambda).\end{aligned}$$

A similar computation shows that when $\lambda < 0$,

$$\mathcal{G}_m[(2i\bar{Z})^m g_m](|\lambda|(2j+1),\lambda) = \mathcal{G}_m[\Theta_m f](|\lambda|(2j+1),\lambda). \blacksquare$$

3. The Fourier transform of m-type Schwartz functions. In this section we characterize the Fourier transform of the space $\mathcal{S}_m(H_1)$ of m-type Schwartz functions on the Heisenberg group.

For m in $\mathbb{Z}$, denote by Σ_m the subset of Σ^* defined by

$$\Sigma_m = \Sigma^* \setminus \{(\xi,\lambda)\in\mathbb{R}^2 : m\lambda>0,\ \xi=|\lambda|(2j+1),\ j=0,1,\ldots,|m|-1\}$$

and note that $\Sigma_0 = \Sigma^*$.

Let $\mathcal{S}(\Sigma_m)$ be the space of restrictions to Σ_m of Schwartz functions on $\mathbb{R}^2$. On $\mathcal{S}(\Sigma_m)$ we consider the quotient topology of $\mathcal{S}(\mathbb{R}^2)/\{f : f_{|\Sigma_m}=0\}$ defined by the family $\{\|\cdot\|_{(p,\Sigma_m)}\}_{p\in\mathbb{N}}$ of norms given by

$$\|G\|_{(p,\Sigma_m)} = \inf\{\|\tilde{G}\|_{(p,\mathbb{R}^2)} : \tilde{G}\in\mathcal{S}(\mathbb{R}^2) \text{ and } \tilde{G}_{|\Sigma_m} = G\}. \tag{3.1}$$

Let $\tilde{\mathcal{G}}_m$ be the map defined on $\mathcal{S}_m(H_1)$ by

$$\tilde{\mathcal{G}}_m f(\xi,\lambda) = \mathcal{G}_m f(\xi-2(\lambda m)_+,\lambda) \quad \forall(\xi,\lambda)\in\Sigma_m.$$

THEOREM 3.1. *The map $\tilde{\mathcal{G}}_m$ is a topological isomorphism between $\mathcal{S}_m(H_1)$ and $\mathcal{S}(\Sigma_m)$.*

Proof. For $m=0$ the result is in [1]. Let $m>0$ and let T_m be the linear operator from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}_m(H_1)$ defined by

$$T_m G = (2i\bar{Z})^m g$$

where g is the function in $\mathcal{S}_0(H_1)$ such that $\mathcal{G}_0 g = G_{|\Sigma^*}$.

We shall verify that T_m is a surjective, continuous linear operator with $\ker T_m = \{G : G_{|\Sigma_m} = 0\}$. Therefore we can apply the open mapping theorem to the operator $\tilde{T}_m : \mathcal{S}(\Sigma_m) \to \mathcal{S}_m(H_1)$ and obtain the conclusion since $\tilde{T}_m^{-1} = \tilde{\mathcal{G}}_m$.

T_m is surjective: indeed, given f in $\mathcal{S}_m(H_1)$, by Proposition 2.7, there exists g in $\mathcal{S}_0(H_1)$ such that $f = (2i\bar{Z})^m g$ and, by [1], there exists G in $\mathcal{S}(\mathbb{R}^2)$ such that $\mathcal{G}_0 g = G_{|\Sigma^*}$.

T_m is continuous: indeed, by [10] for every nonnegative integer p there exists p_m such that

$$\|T_m G\|_{(p,H_1)} = \|(2i\bar{Z})^m g\|_{(p,H_1)} \le C_{m,p}\|g\|_{(p+m,H_1)} \le C'_{m,p}\|G\|_{(p_m,\mathbb{R}^2)}.$$

The fact that $\ker T_m = \{G : G_{|\Sigma_m} = 0\}$ follows easily from the observation that $\bar{Z}^m g = 0$ if and only if $\mathcal{G}_0 g_{|\Sigma_m} = 0$. ■

We now introduce a second family of norms on $\mathcal{S}(\Sigma_m)$ which will eventually turn out to be equivalent to the family of the quotient norms (3.1).

Denote by $M_\pm$ the operators acting on a smooth function Ψ on $\mathbb{R}^2$ by the rule

$$\begin{aligned} M_\pm \Psi(\xi,\lambda) &= \partial_\lambda \Psi(\xi,\lambda) \mp \partial_\xi \Psi(\xi,\lambda) \\ &\qquad - \frac{\lambda \pm \xi}{2\lambda^2}(\Psi(\xi \pm 2\lambda,\lambda) - \Psi(\xi,\lambda) \mp 2\lambda\partial_\xi\Psi(\xi,\lambda)) \\ &= \frac{1}{\lambda}(\lambda\partial_\lambda + \xi\partial_\xi)\Psi(\xi,\lambda) - \frac{\lambda \pm \xi}{2\lambda^2}(\Psi(\xi \pm 2\lambda,\lambda) - \Psi(\xi,\lambda)). \end{aligned}$$

Since $\lambda\partial_\lambda + \xi\partial_\xi$ is the derivative in the radial direction, the operators $M_\pm$ can also be applied to functions which are only defined on the Heisenberg fan. In this case, these operators coincide with the operators $M_\pm$ of [3].

The operators $M_\pm$ have the following relevant property. If f is a type-0 Schwartz function on H_1 then [3, 6, 8]

$$\mathcal{G}_0(\mathcal{A}f) = M_+(\mathcal{G}_0 f) \quad \text{and} \quad \mathcal{G}_0(\bar{\mathcal{A}}f) = -M_-(\mathcal{G}_0 f), \tag{3.2}$$

where $\mathcal{A}(z,t) = |z|^2/4 + it$.

For G in $\mathcal{S}(\Sigma_m)$ define

$$\begin{aligned} &\|G\|_{[p,\Sigma_m]} \\ &\quad = \sup_{\substack{2a+2b\le p \\ (\xi,\lambda)\in\Sigma_m}} \sqrt{\prod_{r=0}^{m-1}(\xi - 2(\lambda m)_+ + |\lambda|(2r+1))}\,(\xi - 2m\lambda)^b |M^a_{\operatorname{sgn} m} G(\xi,\lambda)|. \end{aligned} \tag{3.3}$$

The dependence on $\operatorname{sgn} m$ of these norms is justified by Proposition 2.7, formula (3.2), and the fact that we shall need to use the identities $[Z,\bar{\mathcal{A}}] = 0$ and $[\bar{Z},\mathcal{A}] = 0$.

By (2.2) and (3.2), for every f in $\mathcal{S}_m(H_1)$ we have

$$\|\tilde{\mathcal{G}}_m f\|_{[p,\Sigma_m]} = \begin{cases} \sup\limits_{\substack{2a+2b\le p\\ (\xi,\lambda)\in\Sigma^*}} \sqrt{\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))}\,|\mathcal{G}_m(\mathcal{L}^b\mathcal{A}^a f)(\xi,\lambda)|, & m\ge 0,\\ \sup\limits_{\substack{2a+2b\le p\\ (\xi,\lambda)\in\Sigma^*}} \sqrt{\prod_{r=0}^{|m|-1}(\xi+|\lambda|(2r+1))}\,|\mathcal{G}_m(\mathcal{L}^b\bar{\mathcal{A}}^a f)(\xi,\lambda)|, & m< 0.\end{cases} \tag{3.4}$$

Note that here the supremum is taken over Σ^*, while in (3.3) it is taken over Σ_m, simply because

$$(\xi,\lambda)\in\Sigma^* \;\Leftrightarrow\; (\xi+2(m\lambda)_+,\lambda)\in\Sigma_m.$$

Lemma 3.2. *Let m be in $\mathbb{Z}$. For every nonnegative integer p there exist positive constants C_p and C_p' independent of m such that*

$$C_p\|\tilde{\mathcal{G}}_m f\|_{[p,\Sigma_m]} \le \|f\|_{(p+2,H_1)} \le C_p'\|\tilde{\mathcal{G}}_m f\|_{[p+4,\Sigma_m]} \quad \forall f\in\mathcal{S}_m(H_1).$$

Proof. Let f be in $\mathcal{S}_m(H_1)$, $m>0$ and a,b nonnegative integers. By (2.13) we have

$$\Big|\Big(\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))\Big)^{1/2}\mathcal{G}_m(\mathcal{L}^b\mathcal{A}^a f)(\xi,\lambda)\Big| \le \|\mathcal{L}^b\mathcal{A}^a f\|_1$$
$$\le \|(1+\mathcal{A})^{-1}\|_2\|(1+\mathcal{A})\mathcal{L}^b\mathcal{A}^a f\|_2.$$

Since by (2.10) we have $[\mathcal{A},\mathcal{L}]=1+2\bar{z}\bar{Z}$, the first inequality follows from (3.4) and the equivalence between the families of norms (2.8) and (2.6).

On the other hand, by (2.1), (2.2) and (3.4), we have

$$\|\mathcal{L}^b\mathcal{A}^a f\|_2 = \frac{1}{2\pi}\bigg\{\int_0^\infty\sum_{j=0}^\infty\bigg|\frac{[(1+\mathcal{L}^2)\mathcal{L}^b\mathcal{A}^a f]\widehat{\ }(\lambda,j+m,j)}{1+(|\lambda|(2j+1))^2}\bigg|^2\lambda\,d\lambda$$
$$+\int_{-\infty}^0\sum_{j=0}^\infty\bigg|\frac{[(1+\mathcal{L}^2)\mathcal{L}^b\mathcal{A}^a f]\widehat{\ }(\lambda,j,j+m)}{1+(|\lambda|(2j+2m+1))^2}\bigg|^2|\lambda|\,d\lambda\bigg\}^{1/2}$$
$$\le C\sup_{(\xi,\lambda)\in\Sigma^*}\Big|\Big(\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))\Big)^{1/2}\mathcal{G}_m((1+\mathcal{L}^2)\mathcal{L}^b\mathcal{A}^a f)(\xi,\lambda)\Big|,$$

where

$$C=\frac{1}{\pi}\bigg\{\int_0^\infty\sum_{j=0}^\infty\frac{1}{(1+\lambda^2(2j+1)^2)^2}\lambda\,d\lambda\bigg\}^{1/2}.$$

For the case $m\le 0$ we can apply the same arguments using inequality (2.11). The conclusion follows. ∎

Corollary 3.3. *The families* $\{\|\cdot\|_{[p,\Sigma_m]}\}_{p\geq 0}$ *and* $\{\|\cdot\|_{(p,\Sigma_m)}\}_{p\geq 0}$ *of norms are equivalent on* $\mathcal{S}(\Sigma_m)$.

4. The Fourier transform of Schwartz functions on H_1

Definition 4.1. We define $\mathfrak{G}$ to be the space of sequences $\mathbf{G}=\{G_m\}_{m\in\mathbb{Z}}$ of functions in $\mathcal{S}(\Sigma_m)$ such that for any nonnegative integers ℓ and p,

$$\|\mathbf{G}\|_{\ell,p,\mathfrak{G}} = \sup_{m\in\mathbb{Z}} (1+|m|)^{\ell}\|G_m\|_{[p,\Sigma_m]} < \infty.$$

Denote by $\mathcal{G}$ the linear operator from $\mathcal{S}(H_1)$ to $\mathfrak{G}$ defined by

$$\mathcal{G} : f = \sum_{m\in Z} \Theta_m f \mapsto \mathcal{G}f = \{\tilde{\mathcal{G}}_m(\Theta_m f)\}_{m\in\mathbb{Z}}.$$

Our characterization of the Fourier transform of Schwartz functions on H_1 is the following.

Theorem 4.2. *The map* $\mathcal{G}$ *is a topological isomorphism between* $\mathcal{S}(H_1)$ *and* $\mathfrak{G}$. *Moreover for every* f *in* $\mathcal{S}_m(H_1)$ *and* $p\geq 0$ *we have*

$$\|\mathcal{G}f\|_{\ell,p,\mathfrak{G}} \leq C_{p,\ell}\|f\|_{(p+4\ell+2,H_1)} \qquad \forall\ell\geq 0,$$
$$\|f\|_{(p,H_1)} \leq C_p\|\mathcal{G}f\|_{\ell,p+2,\mathfrak{G}} \qquad \forall\ell\geq 2.$$

Proof. By Lemmata 3.2 and 2.6,

$$\begin{aligned}\|\mathcal{G}f\|_{\ell,p,\mathfrak{G}} &= \sup_{m\in\mathbb{Z}} (1+|m|)^{\ell}\|\tilde{\mathcal{G}}_m(\Theta_m f)\|_{[p,\Sigma_m]}\\ &\leq C_p \sup_{m\in\mathbb{Z}} (1+|m|)^{\ell}\|\Theta_m f\|_{(p+2,H_1)}\\ &\leq C_{p,\ell}\|f\|_{(p+4\ell+2,H_1)}.\end{aligned}$$

Vice versa, let $\mathbf{G}=\{G_m\}_{m\in\mathbb{Z}}$ be in $\mathfrak{G}$. By Theorem 3.1 and Lemma 3.2, for every integer m there exists f_m in $\mathcal{S}(H_1)$ such that $\tilde{\mathcal{G}}_m f_m = G_m$ and

$$\|f_m\|_{(p,H_1)} \leq C_p\|\tilde{\mathcal{G}}_m f_m\|_{[p+2,\Sigma_m]}.$$

Therefore for every nonnegative integer p,

$$\sum_m \|f_m\|_{(p,H_1)} \leq C_p\sum_m \|G_m\|_{[p+2,\Sigma_m]} \leq C_p\|\mathbf{G}\|_{\ell,p+2,\mathfrak{G}}\sum_m (1+m)^{-\ell},$$

so that $\sum_m f_m$ converges in $\mathcal{S}(H_1)$ to a function f such that $\mathcal{G}f=\mathbf{G}$, and the assertion follows. ∎

As mentioned in the introduction, it would be interesting to relate the m-type components of a single Schwartz function F on $\mathbb{C}\times\mathbb{R}$ to the m-type components of a unique Schwartz function f on H_1. This is the result of the following theorem, where starting from F in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$ we build a corresponding f in $\mathcal{S}(H_1)$. However, the nature of the specific norms given in $\mathfrak{G}$ does not allow the reverse correspondence.

THEOREM 4.3. *Let F be in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$. Then there exists f in $\mathcal{S}(H_1)$ such that for any integer m,*

$$\tilde{\mathcal{G}}_m \Theta_m f(\xi,\lambda) = F_m(\xi - m\lambda, \lambda) \qquad \forall(\xi,\lambda)\in\Sigma_m,$$

where F_m are the functions in $\mathcal{S}(\mathbb{R}_+\times\mathbb{R})$ defined as in Proposition 2.4. *Moreover for every nonnegative integer p there exists p' in $\mathbb{N}$ such that*

$$\|f\|_{(p,H_1)} \le C_p \|F\|_{(p',\mathbb{C}\times\mathbb{R})}.$$

In order to prove Theorem 4.3, we shall use the following lemmata.

LEMMA 4.4. *For all integers $j = 0,1,2,\ldots$ and $m = 1,2,\ldots,$*

$$\frac{(j+1)\cdots(j+m)}{\left(j-k+\frac{m+1}{2}\right)^m} \le (k+1)m^k, \qquad k = 0,1,\ldots,j.$$

Proof. First let $k=0$. Then controlling the geometric mean with the arithmetic mean we obtain

$$\sqrt[m]{(j+1)\cdots(j+m)} \le \frac{1}{m}\sum_{\ell=1}^{m}(j+\ell) = j + \frac{m+1}{2}.$$

When $k = 1,\ldots,j$, we write

$$\frac{(j+1)\cdots(j+m)}{\left(j-k+\frac{m+1}{2}\right)^m} = \frac{(j-k+1)\cdots(j-k+m)}{\left(j-k+\frac{m+1}{2}\right)^m}\prod_{\ell=1}^{m}\frac{j+\ell}{j-k+\ell},$$

and we verify that

$$\prod_{\ell=1}^{m}\frac{j+\ell}{j-k+\ell} \le (k+1)m^k.$$

Indeed, this estimate is trivial when $m=1$, and when $m\ge 2$,

$$\begin{aligned}\prod_{\ell=1}^{m}\frac{j+\ell}{j-k+\ell} &= \prod_{\ell=1}^{m}\left(1+\frac{k}{j-k+\ell}\right)\\ &\le \left(1+\frac{k}{j-k+1}\right)\prod_{\ell=2}^{m} e^{\frac{k}{j-k+\ell}}\\ &\le (1+k)e^{\sum_{\ell=2}^{m}\frac{k}{j-k+\ell}}\\ &\le (1+k)e^{k\ln(j-k+m)-k\ln(j-k+1)}\\ &= (1+k)\left(1+\frac{m-1}{j-k+1}\right)^k \le (1+k)m^k. \ \blacksquare\end{aligned}$$

For the statement of the following lemma we need to introduce some notation. Let W denote the operator acting on a smooth function Ψ on $\mathbb{R}^2$ by

$$W\Psi(\xi,\lambda) = \frac{1}{2\lambda^2}\big(\Psi(\xi+2\lambda,\lambda) - \Psi(\xi,\lambda) - 2\lambda\partial_\xi\Psi(\xi,\lambda)\big)$$
$$= 2\int_0^1 \partial_\xi^2\Psi(\xi+2\lambda t,\lambda)(1-t)\,dt.$$

For every $j \ge 0$ let η_j and V_j be the function and the operator defined by

$$\eta_j(\xi,\lambda) = \xi + (2j+1)\lambda, \qquad V_j = \partial_\lambda - (2j+1)\partial_\xi.$$

With this notation, $M_+ = V_0 - \eta_0 W$. Note that the V_j's commute while, for each j, the operator V_j does not commute with W.

LEMMA 4.5. *For every $a \ge 1$,*

$$M_+^a = V_0^a + \sum_{k=1}^{a} \eta_0\cdots\eta_{k-1}D_{k,a}, \tag{4.1}$$

where $D_{k,a}$ is a polynomial in $V_0,\ldots,V_k,W$ of degree a such that in each monomial the operator W appears k times.

Proof. Let $M_j = V_j - \eta_j W$ and note that $M_0 = M_+$. The proof is based on the identity

$$M_j(\eta_j\Psi) = \eta_j\, M_{j+1}\Psi \quad \forall j\ge 0,\, \Psi\in C^\infty(\mathbb{R}^2), \tag{4.2}$$

which will be proved at the end. Note that by (4.2),

$$M_+(\eta_0\cdots\eta_{k-1}\Psi) = M_0(\eta_0\cdots\eta_{k-1}\Psi) = \eta_0\cdots\eta_{k-1}M_k\Psi, \quad \Psi\in C^\infty(\mathbb{R}^2).$$

We shall prove (4.1) by induction on a. Formula (4.1) holds if $a = 1$, since $M_+ = M_0 = V_0 + \eta_0 D_{1,1}$, with $D_{1,1} = -W$. Suppose that (4.1) holds for $a-1$ and let us verify it for a. We have

$$\begin{aligned}
M_+^a &= M_+\Big(V_0^{a-1} + \sum_{k=1}^{a-1}\eta_0\cdots\eta_{k-1}D_{k,a-1}\Big)\\
&= (V_0 - \eta_0 W)V_0^{a-1} + \sum_{k=1}^{a-1} M_+(\eta_0\cdots\eta_{k-1}D_{k,a-1})\\
&= V_0^a - \eta_0 W V_0^{a-1} + \sum_{k=1}^{a-1}\eta_0\cdots\eta_{k-1}M_k D_{k,a-1}\\
&= V_0^a - \eta_0 W V_0^{a-1} + \sum_{k=1}^{a-1}\eta_0\cdots\eta_{k-1}V_k D_{k,a-1} - \sum_{k=1}^{a-1}\eta_0\cdots\eta_{k-1}\eta_k W D_{k,a-1}
\end{aligned}$$

$$= V_0^a - \eta_0 W V_0^{a-1} + \sum_{k=1}^{a-1} \eta_0 \cdots \eta_{k-1} V_k D_{k,a-1} - \sum_{k=2}^{a} \eta_0 \cdots \eta_{k-2}\eta_{k-1} W D_{k-1,a-1}$$

$$= V_0^a + \sum_{k=1}^{a} \eta_0 \cdots \eta_{k-1} D_{k,a},$$

where $D_{1,a} = -WV_0^{a-1} + V_1 D_{1,a-1}$, $D_{a,a} = -WD_{a-1,a-1}$ and $D_{k,a} = V_k D_{k,a-1} - WD_{k-1,a-1}$, $k = 2, \ldots, a-1$.

We now prove (4.2). We have

$$W(\eta_j \Psi) = \eta_{j+1} W\Psi + 2\partial_\xi \Psi \quad \forall j \geq 0.$$

Indeed,

$$\begin{aligned}
&W(\eta_j\Psi)(\xi,\lambda)\\
&\quad = \frac{1}{2\lambda^2}\big[\eta_j(\xi+2\lambda,\lambda)\Psi(\xi+2\lambda,\lambda) - \eta_j(\xi,\lambda)\Psi(\xi,\lambda) - 2\lambda\partial_\xi(\eta_j\Psi)(\xi,\lambda)\big]\\
&\quad = \frac{1}{2\lambda^2}\big[(\xi+(2j+3)\lambda)\Psi(\xi+2\lambda,\lambda) - (\xi+(2j+1)\lambda)\Psi(\xi,\lambda)\\
&\qquad - 2\lambda(\Psi(\xi,\lambda) + (\xi+(2j+1)\lambda)\partial_\xi\Psi(\xi,\lambda)\big]\\
&\quad = \frac{1}{2\lambda^2}(\xi+(2j+3)\lambda)\big[\Psi(\xi+2\lambda,\lambda) - \Psi(\xi,\lambda) - 2\lambda\partial_\xi\Psi(\xi,\lambda)\big]\\
&\qquad + \frac{2\lambda(\xi+(2j+3)\lambda - \xi - (2j+1)\lambda)}{2\lambda^2}\partial_\xi\Psi(\xi,\lambda)\\
&\quad = \eta_{j+1}(\xi,\lambda)W\Psi(\xi,\lambda) + 2\partial_\xi\Psi(\xi,\lambda).
\end{aligned}$$

Moreover, since $V_j\eta_j = 0$,

$$\begin{aligned}
M_j(\eta_j\Psi) &= V_j(\eta_j\Psi) - \eta_j W(\eta_j\Psi) = \eta_j V_j\Psi - \eta_j\eta_{j+1}W\Psi - \eta_j 2\partial_\xi\Psi\\
&= \eta_j[V_j - 2\partial_\xi]\Psi - \eta_j\eta_{j+1}W\Psi\\
&= \eta_j V_{j+1}\Psi - \eta_j\eta_{j+1}W\Psi = \eta_j\, M_{j+1}\Psi.
\end{aligned}$$

This proves (4.2) and so the lemma is proved. ∎

Proof of Theorem 4.3. Let F be in $\mathcal{S}(\mathbb{C}\times\mathbb{R})$ and, as in Proposition 2.4, define a sequence $\{F_m\}$ by

$$\Theta_m F(\zeta,\lambda) = \begin{cases} \zeta^m F_m(|\zeta|^2,\lambda), & m \geq 0,\\ \bar{\zeta}^{|m|} F_m(|\zeta|^2,\lambda), & m < 0. \end{cases}$$

We now introduce the change of variables

$$\tau_m(\xi,\lambda) = (\xi - m\lambda, \lambda) \quad \forall(\xi,\lambda)\in\mathbb{R}^2,\ m\in\mathbb{Z},$$

so that

$$\tau_m(\xi + 2(m\lambda)_+, \lambda) = (\xi + |m\lambda|, \lambda) \quad \forall(\xi,\lambda)\in\mathbb{R}^2,\ m\in\mathbb{Z}.$$

We want to apply Theorem 4.2 to the sequence $\{G_m = F_m\circ\tau_m\}_{m\in\mathbb{Z}}$, i.e., we show that $\mathbf{G} = \{F_m\circ\tau_m\}$ is in the space $\mathfrak{S}$ of sequences. Since F_m

are Schwartz functions, we have to check the required rapid decay in m of $\|F_m \circ \tau_m\|_{[p,\Sigma_m]}$ for any fixed p. When $m \geq 0$,

$$\begin{aligned}
&\|F_m \circ \tau_m\|_{[p,\Sigma_m]} \\
&= \sup_{\substack{2a+2b\leq p \\ (\xi,\lambda)\in\Sigma_m}} \Big(\prod_{r=0}^{m-1}(\xi - 2m\lambda_+ + |\lambda|(2r+1))\Big)^{1/2}(\xi - 2m\lambda)^b |M_+^a(F_m \circ \tau_m)(\xi,\lambda)| \\
&= \sup_{\substack{2a+2b\leq p \\ (\xi,\lambda)\in\Sigma^*}} \Big(\prod_{r=0}^{m-1}(\xi + |\lambda|(2r+1))\Big)^{1/2}(\xi + 2m\lambda_-)^b |M_+^a(F_m \circ \tau_m)(\xi + 2m\lambda_+,\lambda)|.
\end{aligned}$$

By Lemma 4.5,

$$\begin{aligned}
&M_+^a(F_m \circ \tau_m)(\xi,\lambda) \\
&\qquad = V_0^a(F_m \circ \tau_m)(\xi,\lambda) + \sum_{k=1}^{a} \eta_0(\xi,\lambda)\cdots\eta_{k-1}(\xi,\lambda)[D_{k,a}(F_m \circ \tau_m)](\xi,\lambda),
\end{aligned}$$

where $\eta_j(\xi,\lambda) = \xi + (2j+1)\lambda$ and $D_{k,\ell}$ is a polynomial in $V_0,\dots,V_k,W$ of degree ℓ such that in each monomial the operator W appears k times. We treat the two terms above separately.

Since

$$V_j(\Psi \circ \tau_m) = (\partial_\lambda - (2j+1)\partial_\xi)(\Psi \circ \tau_m) = [\partial_\lambda \Psi - (2j+1+m)\partial_\xi \Psi] \circ \tau_m,$$

it is easy to see that

$$|V_0^a(F_m \circ \tau_m)| \leq C_a(1+m)^a \Big| \sum_{\alpha+\beta=a} (\partial_\lambda^\beta \partial_\xi^\alpha F_m) \circ \tau_m \Big|.$$

Moreover, by Lemma 4.4,

$$\Big(\prod_{r=0}^{m-1}(\xi + |\lambda|(2r+1))\Big)^{1/2} \leq (\xi + m|\lambda|)^{m/2} \qquad \forall (\xi,\lambda) \in \Sigma^*,$$

therefore for every (ξ,λ) in Σ^*,

$$\begin{aligned}
&\Big(\prod_{r=0}^{m-1}(\xi + |\lambda|(2r+1))\Big)^{1/2}(\xi + 2m\lambda_-)^b |V_0^a(F_m \circ \tau_m)(\xi + 2m\lambda_+,\lambda)| \\
&\qquad \leq C_a\,(1+m)^a(\xi + m|\lambda|)^{m/2}(\xi + 2m\lambda_-)^b \sum_{\alpha+\beta=a} |\partial_\lambda^\beta \partial_\xi^\alpha F_m(\xi + m|\lambda|,\lambda)| \\
&\qquad \leq C_{a,b}(1+m)^a(\xi + m|\lambda|)^{m/2+b} \sum_{\alpha+\beta=a} |\partial_\lambda^\beta \partial_\xi^\alpha F_m(\xi + m|\lambda|,\lambda)| \\
&\qquad \leq C_{a,b}(1+m)^a \sum_{\alpha+\beta=a} \sup_{\xi\geq|\lambda|(m+1)} \xi^{m/2+b} |\partial_\lambda^\beta \partial_\xi^\alpha F_m(\xi,\lambda)|.
\end{aligned}$$

This takes care of the first term.

For the second term, note that, since $\partial_\xi(\Psi\circ\tau_m)=(\partial_\xi\Psi)\circ\tau_m$, we have

$$W(\Psi\circ\tau_m)(\xi,\lambda)=2\int_0^1\partial_\xi^2(\Psi\circ\tau_m)(\xi+2\lambda t,\lambda)(1-t)\,dt=(W\Psi)\circ\tau_m(\xi,\lambda),$$

so that

$$\begin{aligned}&|[D_{k,a}(F_m\circ\tau_m)](\xi,\lambda)|\\&\qquad\le C_a\sum_{\alpha+\beta+k=a}(1+m)^\alpha\int_0^k|(\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m)\circ\tau_m(\xi+2\lambda t,\lambda)|\,dt.\end{aligned}$$

We treat the cases where $\lambda>0$ and $\lambda<0$ separately.

First, let $\lambda>0$ and (ξ,λ) in Σ^*. By Lemma 4.4 we obtain

$$\begin{aligned}&\Big(\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))\Big)^{1/2}\xi^b|(\eta_0\cdots\eta_{k-1})\,D_{k,a}(F_m\circ\tau_m)|(\xi+2m\lambda,\lambda)\\&\le C_a(\xi+m\lambda)^{m/2}\xi^b\sum_{\alpha+\beta+k=a}(1+m)^\alpha(\eta_0\cdots\eta_{k-1})(\xi+2m\lambda,\lambda)\\&\qquad\qquad\cdot\int_0^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m\lambda+2\lambda t,\lambda)|\,dt\\&\le C_a\sum_{\alpha+\beta+k=a}(1+m)^\alpha(\xi+m\lambda)^{m/2+b}(\xi+2m\lambda+2\lambda k)^k\\&\qquad\qquad\cdot\int_0^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m\lambda+2\lambda t,\lambda)|\,dt\\&\le C_a\sum_{\alpha+\beta+k=a}(1+m)^\alpha\\&\cdot\int_0^k(\xi+m\lambda+2\lambda t)^{m/2+b}(1+\xi+m\lambda+2\lambda t+\lambda)^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m\lambda+2\lambda t,\lambda)|\,dt\\&\le C_a(1+m)^a\sum_{\alpha+\beta+k=a}\sup_{\substack{\xi\ge\lambda(m+1)\\ \lambda>0}}\xi^{m/2+b}(1+\xi+\lambda)^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi,\lambda)|.\end{aligned}$$

On the other hand, if $\lambda<0$ then $\eta_j(-\lambda(2j+1),\lambda)=0$ and

$$(\eta_0\cdots\eta_{k-1})(-\lambda(2j+1),\lambda)=0\quad\forall j=0,1,\dots,k-1.$$

So when $\lambda<0$, it is enough to consider $\xi=|\lambda|(2j+1)$ with $j\ge k\ge 1$. In this case, by Lemma 4.4 we have

$$\prod_{r=0}^{m-1}(\xi+\lambda(2r+1))=(2|\lambda|)^m\frac{(j+m)!}{j!}\le(k+1)(m+1)^k\,(\xi+m|\lambda|-2k|\lambda|)^m$$

and

$$\Big(\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))\Big)^{1/2}(\xi+2m|\lambda|)^b|(\eta_0\cdots\eta_{k-1})D_{k,a}(F_m\circ\tau_m)|(\xi,\lambda)$$
$$\le C_a\sum_{\alpha+\beta+k=a}(k+1)(1+m)^{\alpha+k/2}(\xi+m|\lambda|-2k|\lambda|)^{m/2}(\xi+2m|\lambda|)^b$$
$$\cdot(\eta_0\cdots\eta_{k-1})(\xi,\lambda)\int_0^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m|\lambda|+2\lambda t,\lambda)|\,dt$$
$$\le C_a\sum_{\alpha+\beta+k=a}(1+m)^{\alpha+k/2}(\xi+m|\lambda|-2k|\lambda|)^{m/2}(\xi+2m|\lambda|)^b$$
$$\cdot(\xi-|\lambda|)^k\int_0^k|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m|\lambda|+2\lambda t,\lambda)|\,dt$$
$$\le C_{a,b}(1+m)^a\sum_{\alpha+\beta+k=a}\int_0^k(\xi+m|\lambda|+2\lambda t)^{m/2}$$
$$\cdot(\xi+m|\lambda|+2\lambda t+|\lambda|)^{b+k}|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi+m|\lambda|+2\lambda t,\lambda)|\,dt$$
$$\le C_{a,b}(1+m)^a\sum_{\alpha+\beta+k=a}\sup_{\substack{\xi\ge|\lambda|(m+1)\\ \lambda<0}}\xi^{m/2}(1+\xi+|\lambda|)^{b+k}|\partial_\lambda^\beta\partial_\xi^{2k+\alpha}F_m(\xi,\lambda)|.$$

Putting together all these estimates, we conclude that when $m\ge 0$,

$$\|F_m\circ\tau_m\|_{[p,\Sigma_m]}$$
$$=\sup_{\substack{a+b\le p\\ (\xi,\lambda)\in\Sigma^*}}\Big(\prod_{r=0}^{m-1}(\xi+|\lambda|(2r+1))\Big)^{1/2}(\xi+2(m\lambda)_-)^b|[M_+^a(F_m\circ\tau_m)](\xi+2(m\lambda)_+,\lambda)|$$
$$\le C_p(1+|m|)^p\sup_{\substack{\alpha+\beta\le 2p\\ \xi\ge|\lambda|(|m|+1)\\ \lambda\ne 0}}\xi^{|m|/2}(1+\xi+|\lambda|)^p|(\partial_\lambda^\beta\partial_\xi^\alpha F_m)(\xi,\lambda)|.$$

The same estimate holds for $m<0$. Indeed, one can check that if $\check\Psi(\xi,\lambda)=\Psi(\xi,-\lambda)$, then $M_+\check\Psi=-[M_-\Psi]\check{}$. From this observation the estimate follows easily.

So for every integer m, by Lemma 2.5 and Proposition 2.4,

$$\begin{aligned}\|F_m\circ\tau_m\|_{[p,\Sigma_m]}&\le C_p(1+|m|)^p\sup_{\substack{\alpha+\beta\le 2p\\ (\xi,\lambda)\in\mathbb{R}_+\times\mathbb{R}}}\xi^{|m|/2}(1+\xi+|\lambda|)^p|\partial_\lambda^\beta\partial_\xi^\alpha F_m(\xi,\lambda)|\\ &\le C_p(1+|m|)^p\|\Theta_m F\|_{(6p,\mathbb{C}\times\mathbb{R})}\\ &\le C_{p,\ell}(1+|m|)^{-\ell}\|F\|_{(8p+2\ell,\mathbb{C}\times\mathbb{R})},\end{aligned}$$

for every nonnegative integer p. Thus, by Theorem 4.2 there exists a function f in $\mathcal{S}(H_1)$ such that

$$\|f\|_{(p,H_1)} \le C_p \sum_{m\in\mathbb{Z}} \|F_m \circ \tau_m\|_{[p+2,\Sigma_m]} \le C_p \|F\|_{(8p+20,\mathbb{C}\times\mathbb{R})}.$$

Finally, f satisfies

$$\tilde{\mathcal{G}}_m \Theta_m f(\xi,\lambda) = F_m \circ \tau_m(\xi,\lambda) = F_m(\xi - m\lambda, \lambda)$$

for every (ξ,λ) in Σ^*, as required. ∎

REMARK 4.6. In this paper we never focus our attention on the representations of H_1 which are trivial on the center, i.e. the characters $\eta_\zeta(z,t) = e^{i\,\mathrm{Re}(z\bar\zeta)}$ which correspond to the horizontal half-line $\{(|\zeta|^2,0)\in\mathbb{R}^2 : \zeta\in\mathbb{C}\}$ of the Heisenberg fan Σ. Indeed, given f in $\mathcal{S}(H_1)$, we define $\mathcal{G}_m f$ only on Σ^*, without discussing its possible extension to the whole Heisenberg fan Σ. However, because of the equality (2.14), the smooth behavior of the extension of $\mathcal{G}_m f$ to all Σ is guaranteed by the result in [1].

In particular, denoting

$$(\eta f)(\zeta) = \int_{H_1} f(z,t) e^{i\,\mathrm{Re}(z\bar\zeta)}\,dz\,dt \qquad \forall f \in \mathcal{S}(H_1),$$

we have

$$(\eta(2i\bar Z)^m g)(\zeta) = \zeta^m (\eta g)(\zeta), \qquad (\eta(2iZ)^{|m|} g)(\zeta) = \bar\zeta^{|m|}(\eta g)(\zeta).$$

Therefore if F is in $\mathcal{S}(\mathbb{R}\times\mathbb{C})$ and $f\in\mathcal{S}(H_1)$ is associated to F as in Theorem 4.3, then

$$(\eta f)(\zeta) = F(\zeta,0) \qquad \forall\zeta\in\mathbb{C}.$$

This equality justifies our normalization by $2i$ of the differential operators $\bar Z$ and Z.

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Francesca Astengo
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146 Genova, Italy
E-mail: astengo@dima.unige.it

Bianca Di Blasio
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano-Bicocca
Via Cozzi 53
20125 Milano, Italy
E-mail: bianca.diblasio@unimib.it

Fulvio Ricci
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa, Italy
E-mail: fricci@sns.it

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