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Nontrivial solutions for a class of superquadratic elliptic equations

by

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Abstract. Using a version of the Local Linking Theorem and the Fountain Theorem, we obtain some existence and multiplicity results for a class of superquadratic elliptic equations.

1. Introduction and main results. Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u + a(x)u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain, $a \in L^p(\Omega)$, $p > N/2$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$.

In [9], Li and Willem established the existence of a nontrivial solution for problem (1.1) under the following superquadratic condition: there exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(x,u) \leq u f(x,u) \tag{1.2}$$

for all $|u| \geq L$ and $x \in \Omega$, where

$$F(x,u) = \int_0^u f(x,s)\,ds.$$

Condition (1.2), originally due to Ambrosetti and Rabinowitz [1], has been used extensively in the literature (see [8, 9, 11, 12] and the references therein).

In [7], Jiang and Tang obtained the existence of nontrivial solutions for problem (1.1) under a new superquadratic condition by minimax methods in critical point theory. They established the following with the aid of the Local Linking Theorem (see [9]).

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THEOREM 1.1 ([7, Theorem 1]). *Suppose that $F(x,u)$ satisfies the following conditions:*

(F_1) $F(x,u)/u^2 \to \infty$ *as* $|u| \to \infty$ *uniformly in* $x \in \Omega$.
(F_2) $F(x,u)/u^2 \to 0$ *as* $|u| \to 0$ *uniformly in* $x \in \Omega$.
(F_3) *There are constants* $2 < \lambda < 2N/(N-2) = 2^*$ *and* $a_1 > 0$ *such that*

$$|f(x,u)| \le a_1(1 + |u|^{\lambda-1}) \quad \textit{for all } (x,u) \in \Omega \times \mathbb{R}.$$

(F_4) *There exist constants* $\beta > 2N(\lambda-1)/(N+2), a_2 > 0$ *and* $L > 0$ *such that*

$$f(x,u)u - 2F(x,u) \ge a_2|u|^\beta \quad \textit{for all } x \in \Omega \textit{ and } |u| \ge L.$$

If 0 *is an eigenvalue of* $-\Delta + a$ (*with Dirichlet boundary condition*), *assume also that:*

(F_5) *There exists* $\delta > 0$ *such that either*
  (i) $F(x,u) \ge 0$ *for all* $|u| \le \delta$, $x \in \Omega$, *or*
  (ii) $F(x,u) \le 0$ *for all* $|u| \le \delta$, $x \in \Omega$.

Then problem (1.1) *has a nontrivial solution.*

In this paper, by applying a version of the Local Linking Theorem (see [10]), we can prove the same result under a more general superquadratic condition. Moreover, by using the Fountain Theorem, we get the existence of infinitely many nontrivial solutions of problem (1.1). Our main results are the following theorems.

THEOREM 1.2. *Assume that the nonlinearity* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (F_1)–(F_3) *and*

(F_4') *There exists a constant* $\beta > N(\lambda-2)/2$ *such that*

$$\liminf_{|u|\to\infty} \frac{f(x,u)u - 2F(x,u)}{|u|^\beta} > 0 \quad \textit{uniformly in } x \in \Omega.$$

Assume also that (F_5) *holds if* 0 *is an eigenvalue of* $-\Delta + a$ (*with Dirichlet boundary condition*). *Then problem* (1.1) *has a nontrivial solution.*

REMARK 1.3. Theorem 1.2 extends Theorem 1.1. Obviously, the range of β is extended. There are functions satisfying the assumptions of Theorem 1.2 and not satisfying the assumptions in [7]. For example, fix $x_0 \in \Omega$ and let

$$F(x,u) = \sin^2(|x - x_0|\pi)|u|^\lambda + u^2 \ln(1 + u^2).$$

Let $\lambda = 3$, $N = 3$. Then F satisfies the assumptions of our Theorem 1.2 and does not satisfy (F_4), so it does not satisfy the assumptions of the corresponding results in [7, 9].

Moreover, Theorem 1 in [4] is a special case of our Theorem 1.2 corresponding to $a(x) = 0$.

THEOREM 1.4. *Suppose that the nonlinearity* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (F_1), (F_2) *and the following condition:*

(F_6) *There exist positive constants* L *and* m_1, m_2 *such that*

- (j_1) $f(x,u)u - 2F(x,u) \geq m_1 u^2$ *if* $|u| \geq L$.
- (j_2) $|f(x,u)|^\sigma/|u|^\sigma \leq m_2(f(x,u)u - 2F(x,u))$ *if* $|u| \geq L$, *where* $\sigma > N/2$.

Assume also that (F_5) *holds if* 0 *is an eigenvalue of* $-\Delta + a$ (*with Dirichlet boundary condition*). *Then problem* (1.1) *has a solution.*

REMARK 1.5. For Schrödinger equations, the corresponding condition (F_6) is due to Ding and Luan [5]. Condition (F_6) is weaker than the usual Ambrosetti–Rabinowitz-type condition (1.2) (see [5, 10]).

THEOREM 1.6. *Assume that the nonlinearity* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (F_1), (F_3), (F_4'), *and that* F *is even in* u. *Then problem* (1.1) *has infinitely many nontrivial solutions.*

REMARK 1.7. Theorem 1.6 extends Theorem 1.1 of [6]. Obviously, the range of β is extended. Condition (F_1) is weaker than condition (A_1) of Theorem 1.1 of [6]. Moreover, Theorem 1.6 is a complement of Theorem 3.7 in [12]. Conditions (F_1), (F_4') are more general than condition (1.2) and there are functions F (see Remark 1.3) satisfying the assumptions of Theorem 1.6 and not satisfying the assumptions of the corresponding results in [6, 12].

THEOREM 1.8. *Assume that the nonlinearity* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfies* (F_1), (F_6), *and that* F *is even in* u. *Then problem* (1.1) *has infinitely many nontrivial solutions.*

REMARK 1.9. Theorem 1.8 extends Theorem 3.7 in [12], since (F_6) is weaker than (1.2) (see [5, 10]).

Let $\varphi : H_0^1(\Omega) = E \to \mathbb{R}$ be the functional defined by

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)\,dx - \int_\Omega F(x,u)dx \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_\Omega F(x,u)\,dx \end{aligned} \tag{1.3}$$

where $u^- \in E^-$ and $u^+ \in E^+$; here E^- (resp. E^+) is the space spanned by the eigenvectors corresponding to negative (resp. positive) eigenvalues of $-\Delta + a$. It is easy to see that $\varphi \in C^1(E, \mathbb{R})$ under the conditions of our theorems. It is well known that a critical point of the functional φ in E

corresponds to a weak solution of problem (1.1) and

$$\langle \varphi'(u), v\rangle = \int_{\Omega} (\nabla u \nabla v + a(x)uv)\,dx - \int_{\Omega} f(x,u)v\,dx$$

for any $u, v \in E$.

It is well known that E is continuously embedded in $L^{\theta}(\Omega)$ for every $\theta \in [1, 2N/(N-2)]$. If $1 \le \theta < 2N/(N-2)$, the embedding is compact. It follows from (F_3), (F_4') and (F_6) that

$$\lambda < \frac{2N}{N-2}, \qquad \frac{\lambda N - 2\beta}{N-2} < \frac{2N}{N-2}, \qquad \frac{2\sigma}{\sigma-1} < \frac{2N}{N-2}.$$

Hence, there is a positive constant K such that

$$\|u\|_{L^{\theta}} \le K\|u\|, \quad \forall u \in E, \tag{1.4}$$

for $\theta = 1, 2, \lambda, (\beta+2)/\beta, 2\sigma/(\sigma-1), 2N/(N-2)$, where $\|\cdot\|_{L^{\theta}}$ denotes the norm of $L^{\theta}(\Omega)$.

2. Proof of main results. To prove Theorems 1.2 and 1.4, we shall use the Local Linking Theorem (Theorem 2.2 of [10]). Let X be a real Banach space with $X = X^1 \oplus X^2$ and $X_0^j \subset X_1^j \subset X_2^j \subset \cdots \subset X^j$ such that $X^j = \overline{\bigcup_{n\in\mathbb{N}} X_n^j}$, $j = 1, 2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, let $X_\alpha = X_{\alpha_1}^1 \oplus X_{\alpha_2}^2$. We define $\alpha \le \beta \Leftrightarrow \alpha_1 \le \beta_1$, $\alpha_2 \le \beta_2$. A sequence $\{\alpha_n\} \subset \mathbb{N}^2$ is *admissible* if for every $\alpha \in \mathbb{N}^2$ there is $m \in \mathbb{N}$ such that $n \ge m \Rightarrow \alpha_n \ge \alpha$. We say $\varphi \in C^1(X, \mathbb{R})$ satisfies the (C^*) *condition* if every sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and satisfies

$$u_{\alpha_n} \in X_{\alpha_n}, \qquad \sup_n \varphi(u_{\alpha_n}) < \infty, \qquad (1 + \|u_{\alpha_n}\|)\varphi'_{\alpha_n}(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of φ, where $\varphi_\alpha = \varphi|_{X_\alpha}$.

We now recall the Local Linking Theorem which extends theorems given by Li and Szulkin [8] and Li and Willem [9].

THEOREM 2.1 (Local Linking Theorem, see also [10, Theorem 2.2]). *Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the following assumptions:*

- (i_1) *$X \ne \{0\}$ and φ has a local linking at 0, that is, for some $r > 0$,*

$$\varphi(u) \ge 0, \quad \forall u \in X^1 \text{ with } \|u\| \le r,$$
$$\varphi(u) \le 0, \quad \forall u \in X^2 \text{ with } \|u\| \le r.$$

- (i_2) *φ satisfies the (C^*) condition.*
- (i_3) *φ maps bounded sets into bounded sets.*
- (i_4) *For every $m \in \mathbb{N}$, $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in $X_m^1 \oplus X^2$.*

Then φ has a nonzero critical point.

In [2], Bartsch established the Fountain Theorem (Theorem 2.5 in [2], Theorem 3.6 in [12]) under the $(\mathrm{PS})_c$ condition.

We say $\varphi \in C^1(X,\mathbb{R})$ satisfies the *Cerami condition* (C) if whenever $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\|(1+\|u_n\|) \to 0$ as $n \to \infty$, then $\{u_n\}$ has a convergent subsequence in X. This condition is due to Cerami [3]. Since the Deformation Theorem is still valid under the Cerami condition (C), we see that like many critical point theorems, the Fountain Theorem is true under the Cerami condition (C).

To state this theorem, let X be a reflexive and separable Banach space. It is well known that there exist $\{v_n\}_{n\in\mathbb{N}} \subset X$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset X^*$ such that:

(1) $\langle \psi_n, v_m\rangle = \delta_{n,m}$.
(2) $\overline{\mathrm{span}}\{v_n \mid n \in \mathbb{N}\} = X$, $\overline{\mathrm{span}}^{\omega^*}\{\psi_n \mid n \in \mathbb{N}\} = X^*$.

Let $X_j = \mathbb{R}v_j$. Then $X = \overline{\bigoplus_{j\ge1} X_j}$. We define

$$Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

THEOREM 2.2 (Fountain Theorem). *Assume that $\varphi \in C^1(X,\mathbb{R})$ satisfies the Cerami condition* (C) *and $\varphi(-u) = \varphi(u)$. If for almost every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A_1) $a_k := \max_{u\in Y_k,\, \|u\|=\rho_k} \varphi(u) \le 0$,
(A_2) $b_k := \inf_{u\in Z_k,\, \|u\|=r_k} \varphi(u) \to \infty$ *as* $k \to \infty$,

then φ has an unbounded sequence of critical values.

Now, we can give the proofs of our theorems.

Proof of Theorem 1.2. The proof is divided into several steps.

STEP 1: We claim that $\varphi \in C^1(X,\mathbb{R})$ and φ maps bounded sets into bounded sets. Let $X = E$, $X^1 = E^+ \oplus E^0$ and $X^2 = E^-$, where $E^0 = \ker(-\Delta + a)$. By (F_3), there exists a positive constant c_1 such that

$$|F(x,u)| \le c_1(|u| + |u|^\lambda), \qquad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{2.1}$$

So, by (1.3), (1.4) and (2.1), we have

$$\begin{aligned}|\varphi(u)| &= \Big|\tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_\Omega F(x,u)\,dx\Big| \\ &\le \tfrac{1}{2}\|u\|^2 + c_1 \int_\Omega (|u| + |u|^\lambda)\,dx \\ &\le \tfrac{1}{2}\|u\|^2 + c_1K\|u\| + c_1K^\lambda\|u\|^\lambda.\end{aligned}$$

Hence, $\varphi \in C^1(X,\mathbb{R})$ and φ maps bounded sets into bounded sets.

STEP 2: We claim that φ has a local linking at zero with respect to (X^1, X^2).

Here, we consider only the case where 0 is an eigenvalue of $-\Delta + a$ and case (ii) of (F_5) holds. Case (i) is similar.

By (F_2), for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|F(x,u)| \le \varepsilon u^2, \quad \forall |u| \le \delta_1.$$

From (2.1) and the above we obtain

$$|F(x,u)| \le \varepsilon u^2 + M|u|^\lambda, \quad \forall (x,u) \in \Omega \times \mathbb{R}, \tag{2.2}$$

where $M = c_1(1 + \delta_1^{1-\lambda})$. From (1.4) and (2.2) we get

$$\begin{aligned} \Big|\int_\Omega F(x,u)\,dx\Big| &\le \int_\Omega \varepsilon u^2\,dx + M\int_\Omega |u|^\lambda\,dx \\ &\le \varepsilon\|u\|_{L^2}^2 + M\|u\|_{L^\lambda}^\lambda \le K^2\varepsilon\|u\|^2 + K^\lambda M\|u\|^\lambda \end{aligned} \tag{2.3}$$

for all $u \in E$.

Choose a Hilbertian basis $\{e_n\}_{n\ge 0}$ for X^1 and define

$$\begin{aligned} X_n^1 &:= \operatorname{span}\{e_0, \dots, e_n\}, \quad n \in \mathbb{N}, \\ X_n^2 &:= X^2,\ n \in \mathbb{N}, \\ X^1 &= \overline{\bigcup_n X_n^1}. \end{aligned}$$

Now, by (2.3), for each $u \in X^2 = E^-$, one has

$$\varphi(u) = -\tfrac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx \le -\tfrac{1}{2}\|u\|^2 + K^2\varepsilon\|u\|^2 + K^\lambda M\|u\|^\lambda.$$

Letting $\varepsilon = 1/(8K^2)$, since $\lambda > 2$, we have

$$\varphi(u) \le 0, \quad \forall u \in X^2 \text{ with } \|u\| \le \delta_2$$

for $\delta_2 > 0$ small enough.

It follows from the equivalence of norms on the finite-dimensional space E^0 that there exists $K_1 > 0$ such that

$$\|u\|_\infty \le K_1\|u\|, \quad \|u\| \le K_1\|u\|_{L^1}, \quad \|u\| \le K_1\|u\|_{L^2}, \quad \forall u \in E^0. \tag{2.4}$$

Let $u = u^0 + u^+ \in E^0 \oplus E^+ = X^1$ be such that $\|u\| \le \delta_3 \triangleq \delta/(2K_1)$. Put

$$\Omega_1 = \{x \in \Omega \mid |u^+(x)| \le \delta/2\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Then, for all $\|u\| \le \delta_3$ and $x \in \Omega$, by (2.4), one has

$$|u^0(x)| \le \|u^0\|_\infty \le K_1\|u^0\| \le K_1\|u\| \le \delta/2. \tag{2.5}$$

On one hand, from (2.5), for each $x \in \Omega_1$, we have

$$|u(x)| \le |u^0(x)| + |u^+(x)| \le \|u^0\|_\infty + \delta/2 \le \delta.$$

Hence, by condition (ii) of (F_5),

$$\int_{\Omega_1} F(x,u)\,dx \le 0.$$

On the other hand, by (2.5), for every $x \in \Omega_2$,

$$|u(x)| \le |u^0(x)| + |u^+(x)| \le \delta/2 + |u^+(x)| \le 2|u^+(x)|.$$

Hence, for all $x \in \Omega_2$ and $u \in X^1$ with $\|u\| \le \delta_3$, we infer from (2.2) that

$$F(x,u) \le \varepsilon u^2 + M|u|^\lambda \le 4\varepsilon|u^+(x)|^2 + 2^\lambda M|u^+(x)|^\lambda,$$

which implies that

$$\begin{aligned}\int_{\Omega_2} F(x,u)\,dx &\le 4\varepsilon \int_{\Omega_2} |u^+(x)|^2\,dx + 2^\lambda M \int_{\Omega_2} |u^+(x)|^\lambda\,dx \\ &\le 4\varepsilon\|u^+\|^2_{L^2} + 2^\lambda M\|u^+\|^\lambda_{L^\lambda} \le 4K^2\varepsilon\|u^+\|^2 + (2K)^\lambda M\|u^+\|^\lambda.\end{aligned}$$

Letting $\varepsilon = 1/(16K^2)$ in the above expression, for all $x \in \Omega_2$ and $u \in X^1$ with $\|u\| \le \delta_3$ we have

$$\begin{aligned}\varphi(u) &= \tfrac{1}{2}\|u^+\|^2 - \int_{\Omega_2} F(x,u)\,dx - \int_{\Omega_1} F(x,u)\,dx \\ &\ge \tfrac{1}{2}\|u^+\|^2 - 4K^2\varepsilon\|u^+\|^2 - (2K)^\lambda M\|u^+\|^\lambda \ge \tfrac{1}{4}\|u^+\|^2 - (2K)^\lambda M\|u^+\|^\lambda,\end{aligned}$$

and consequently

$$\varphi(u) \ge 0, \qquad \forall u \in X^1 \text{ with } \|u\| \le \delta_4,$$

for $\delta_4 > 0$ small enough. Hence, φ has a local linking at zero with respect to (X^1, X^2) for $\delta_5 = \min\{\delta_2, \delta_4\}$ small enough.

STEP 3: We claim that φ satisfies the (C^*) condition. Consider a sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \qquad \sup_n \varphi(u_{\alpha_n}) < \infty, \qquad (1 + \|u_{\alpha_n}\|)\varphi'_{\alpha_n}(u_{\alpha_n}) \to 0.$$

Then there exists a constant $M_0 > 0$ such that

$$\varphi(u_{\alpha_n}) \le M_0, \qquad (1 + \|u_{\alpha_n}\|)\|\varphi'_{\alpha_n}(u_{\alpha_n})\| \le M_0. \tag{2.6}$$

By a standard argument, we only need to prove that $\{u_{\alpha_n}\}$ is a bounded sequence in X.

Indeed, otherwise we can assume that $\|u_{\alpha_n}\| \to \infty$ as $n \to \infty$. From (F'_4), there exist constants $c_2, c_3 > 0$ such that

$$f(x,u)u - 2F(x,u) \ge c_2|u|^\beta - c_3, \qquad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{2.7}$$

So, we conclude from (2.6) and (2.7) that

$$\begin{aligned}3M_0 &\ge 2\varphi(u_{\alpha_n}) - \langle \varphi'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n}\rangle \\ &= \int_\Omega (f(x,u_{\alpha_n})u_{\alpha_n} - 2F(x,u_{\alpha_n}))\,dx \ge c_2 \int_\Omega |u_{\alpha_n}|^\beta\,dx - c_3|\Omega|,\end{aligned}$$

which implies that

$$\int_\Omega |u_{\alpha_n}|^\beta\,dx < c_4 \tag{2.8}$$

for all α_n and some positive constant c_4.

We have

$$\beta > \frac{N}{2}(\lambda-2) \quad \text{and} \quad \frac{N}{2}(\lambda-2) < \frac{2N}{N+2}(\lambda-1).$$

Here, we consider only the case

$$\frac{N}{2}(\lambda-2) < \beta < \frac{2N}{N+2}(\lambda-1).$$

Put

$$\alpha = \frac{2(\lambda-1)N-(N+2)\beta}{2N-(N-2)\beta}.$$

Then $0<\alpha<1$. Let

$$p = \frac{\beta}{\lambda-1-\alpha} > 1 \quad \text{and} \quad u_{\alpha_n} = u^+_{\alpha_n} + u^-_{\alpha_n} + u^0_{\alpha_n} \in E^+ \oplus E^- \oplus E^0.$$

From Hölder's inequality, (1.4) and (2.8) we obtain

$$\begin{aligned}
\int_\Omega |u_{\alpha_n}|^{\lambda-1}|u^+_{\alpha_n}|\,dx &= \int_\Omega |u_{\alpha_n}|^{\lambda-1-\alpha}|u_{\alpha_n}|^\alpha|u^+_{\alpha_n}|\,dx \\
&= \int_\Omega |u_{\alpha_n}|^{\beta/p}|u_{\alpha_n}|^\alpha|u^+_{\alpha_n}|\,dx \\
&\le \Big(\int_\Omega (|u_{\alpha_n}|^{\beta/p})^p\,dx\Big)^{1/p}\Big(\int_\Omega (|u_{\alpha_n}|^\alpha|u^+_{\alpha_n}|)^q\,dx\Big)^{1/q} \\
&\le \Big(\int_\Omega |u_{\alpha_n}|^\beta\,dx\Big)^{1/p}\Big(\int_\Omega (|u_{\alpha_n}|^{q\alpha})^{2^*/(q\alpha)}\,dx\Big)^{\alpha/2^*}\Big(\int_\Omega (|u^+_{\alpha_n}|^q)^{2^*/q}\,dx\Big)^{1/2^*} \\
&\le c_4^{1/p}\|u_{\alpha_n}\|^\alpha_{L^{2^*}}\|u^+_{\alpha_n}\|_{L^{2^*}} \le c_4^{1/p}K^{\alpha+1}\|u_{\alpha_n}\|^\alpha\|u^+_{\alpha_n}\|
\end{aligned} \tag{2.9}$$

for all n, where $q = p/(p-1) = 2^*/(\alpha+1)$.

By (F_3), (1.4), (2.8) and (2.9),

$$\begin{aligned}
\langle \varphi'_{\alpha_n}(u_{\alpha_n}), u^+_{\alpha_n}\rangle &= \|u^+_{\alpha_n}\|^2 - \int_\Omega f(x,u_{\alpha_n})u^+_{\alpha_n}\,dx \\
&\ge \|u^+_{\alpha_n}\|^2 - \int_\Omega |f(x,u_{\alpha_n})|\,|u^+_{\alpha_n}|\,dx \\
&\ge \|u^+_{\alpha_n}\|^2 - a_1\int_\Omega (|u_{\alpha_n}|^{\lambda-1}|u^+_{\alpha_n}| + |u^+_{\alpha_n}|)\,dx
\end{aligned}$$

$$= \|u_{\alpha_n}^+\|^2 - a_1 \int_\Omega |u_{\alpha_n}|^{\lambda-1} |u_{\alpha_n}^+| \, dx - a_1 \int_\Omega |u_{\alpha_n}^+| \, dx$$
$$\geq \|u_{\alpha_n}^+\|^2 - a_1 c_4^{1/p} K^{\alpha+1} \|u_{\alpha_n}\|^\alpha \|u_{\alpha_n}^+\| - a_1 K \|u_{\alpha_n}^+\|$$

for all n.

Since $\alpha < 1$, we have

$$\frac{\|u_{\alpha_n}^+\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

Similarly,

$$\frac{\|u_{\alpha_n}^-\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

It follows from (2.4), (2.8) and Hölder's inequality that

$$\begin{aligned} \frac{1}{K_1^2} \|u_{\alpha_n}^0\|^2 &\leq \int_\Omega |u_{\alpha_n}^0|^2 \, dx \leq \int_\Omega |u_{\alpha_n}|^2 \, dx \\ &= \int_\Omega |u_{\alpha_n}|^{\beta/(\beta+1)} |u_{\alpha_n}|^{(\beta+2)/(\beta+1)} \, dx \\ &\leq \Big(\int_\Omega |u_{\alpha_n}|^\beta \, dx \Big)^{1/(\beta+1)} \Big(\int_\Omega |u_{\alpha_n}|^{(\beta+2)/\beta} \, dx \Big)^{\beta/(\beta+1)} \\ &\leq c_4^{1/(\beta+1)} K^{(\beta+2)/(\beta+1)} \|u_{\alpha_n}\|^{(\beta+2)/(\beta+1)}, \end{aligned}$$

and consequently

$$\frac{\|u_{\alpha_n}^0\|}{\|u_{\alpha_n}\|} \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

Hence, by (2.10)–(2.12),

$$1 = \frac{\|u_{\alpha_n}\|}{\|u_{\alpha_n}\|} \leq \frac{\|u_{\alpha_n}^+\| + \|u_{\alpha_n}^0\| + \|u_{\alpha_n}^-\|}{\|u_{\alpha_n}\|} \to 0$$

as $n \to \infty$, which is a contradiction. So, $\{u_{\alpha_n}\}$ is bounded in X. By a standard argument, we deduce that φ satisfies the (C*) condition.

STEP 4: Now, we claim that for each $m \in \mathbb{N}$,

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ in } X_m^1 \oplus X^2.$$

Since $\dim E^0 < \infty$ and $\dim X_m^1 < \infty$, all norms are equivalent. There exists a constant $c_5 > 0$ such that for all $u \in X_m^1 \oplus X^2$,

$$\|u\| \leq c_5 \|u\|_{L^2}. \tag{2.13}$$

From condition (F$_1$), there exists $c_6 > 0$ such that

$$F(x,u) \geq c_5^2 |u|^2 - c_6, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{2.14}$$

For $u \in X_m^1 \oplus X^2$, it follows from (2.13) and (2.14) that

$$\begin{aligned}\varphi(u) &= \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_\Omega F(x,u)\,dx \\ &\le \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - c_5^2\|u\|_{L^2}^2 + c_6|\Omega| \\ &\le \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - (\|u^+\|^2 + \|u^0\|^2) + c_6|\Omega| \\ &\le -\tfrac{1}{2}\|u^+\|^2 - \tfrac{1}{2}\|u^-\|^2 - \|u^0\|^2 + c_6|\Omega| \le -\tfrac{1}{2}\|u\|^2 + c_6|\Omega|,\end{aligned}$$

which implies that

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ in } X_m^1 \oplus X^2.$$

So, the proof of Theorem 1.2 is complete. ∎

Proof of Theorem 1.4. It is easy to see that φ satisfies (i_1), (i_3) and (i_4) of Theorem 2.1. The proof is similar to that of Theorem 1.2. Here, we only need to prove that φ satisfies (i_2), i.e., the (C^*) condition.

Consider a sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \qquad \sup_n \varphi(u_{\alpha_n}) < \infty, \qquad (1 + \|u_{\alpha_n}\|)\varphi'_{\alpha_n}(u_{\alpha_n}) \to 0.$$

By a standard argument, we only need to prove that $\{u_{\alpha_n}\}$ is a bounded sequence in X.

Indeed, otherwise, we can assume that $\|u_{\alpha_n}\| \to \infty$ as $n \to \infty$.

From assumption (F_6), there exist positive constants m_3 and m_4 such that

$$\begin{aligned}m_3 &\ge 2\varphi(u_{\alpha_n}) - \langle \varphi'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n}\rangle \\ &= \int_\Omega (f(x,u_{\alpha_n})u_{\alpha_n} - 2F(x,u_{\alpha_n}))\,dx \ge m_1 \int_\Omega u_{\alpha_n}^2\,dx - m_4|\Omega|.\end{aligned} \tag{2.15}$$

So,

$$\int_\Omega u_{\alpha_n}^2\,dx \le m_5 \tag{2.16}$$

for all n and some positive constant m_5.

Let $v_{\alpha_n} = u_{\alpha_n}/\|u_{\alpha_n}\|$. Then $\|v_{\alpha_n}\| = 1$ and $\|v_{\alpha_n}\|_{L^r} \le C_r$ for all $r \in [1, 2N/(N-2))$. By (2.16), we have

$$\int_\Omega v_{\alpha_n}^2\,dx = \frac{1}{\|u_{\alpha_n}\|^2}\int_\Omega u_{\alpha_n}^2\,dx \le \frac{m_5}{\|u_{\alpha_n}\|^2} \to 0,$$

as $n \to \infty$. So, for $r \in (2, 2N/(N-2))$, Hölder's inequality yields

$$\int_\Omega |v_{\alpha_n}|^r\,dx \le \Big(\int_\Omega |v_{\alpha_n}|^{2(r-1)}\,dx\Big)^{1/2}\Big(\int_\Omega |v_{\alpha_n}|^2\,dx\Big)^{1/2} \to 0 \tag{2.17}$$

as $n \to \infty$. Since $\dim E^0 < \infty$, by (2.4) one has

$$\begin{aligned}\langle \varphi'_{\alpha_n}(u_{\alpha_n}), u^+_{\alpha_n} - u^-_{\alpha_n}\rangle &= \|u^+_{\alpha_n} - u^-_{\alpha_n}\|^2 - \int_\Omega f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})\,dx \\ &= \|u_{\alpha_n}\|^2 - \|u^0_{\alpha_n}\|^2 - \int_\Omega f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})\,dx \\ &= \|u_{\alpha_n}\|^2\biggl(1 - \int_\Omega \frac{f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})}{\|u_{\alpha_n}\|^2}\,dx\biggr) - \|u^0_{\alpha_n}\|^2 \\ &\geq \|u_{\alpha_n}\|^2\biggl(1 - \int_\Omega \frac{f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})}{\|u_{\alpha_n}\|^2}\,dx\biggr) - K_1^2\|u^0_{\alpha_n}\|^2_{L^2}.\end{aligned}$$

Hence,

$$1 - \int_\Omega \frac{f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})}{\|u_{\alpha_n}\|^2}\,dx = o(1). \tag{2.18}$$

From (F_6), (2.15) and (2.17), there exists a positive constant m_6 such that

$$\begin{aligned}\biggl|\int_\Omega \frac{f(x, u_{\alpha_n})(u^+_{\alpha_n} - u^-_{\alpha_n})}{\|u_{\alpha_n}\|^2}\,dx\biggr| &\leq 2\int_\Omega \frac{|f(x, u_{\alpha_n})|}{|u_{\alpha_n}|}|v_{\alpha_n}|^2\,dx \\ &\leq 2\biggl(\int_\Omega \biggl(\frac{|f(x, u_{\alpha_n})|}{|u_{\alpha_n}|}\biggr)^\sigma dx\biggr)^{1/\sigma}\biggl(\int_\Omega |v_{\alpha_n}|^{2\sigma'}\,dx\biggr)^{1/\sigma'} \\ &\leq m_6\biggl(\int_\Omega |v_{\alpha_n}|^{2\sigma'}\,dx\biggr)^{1/\sigma'} \to 0\end{aligned} \tag{2.19}$$

as $n \to \infty$, where $\sigma' = \sigma/(\sigma - 1)$. Therefore, from (2.18) and (2.19), one sees $1 = o(1)$, a contradiction. Hence, $\{u_n\}$ is bounded.

By a standard argument, we deduce that φ satisfies the (C^*) condition. ∎

Proof of Theorem 1.6. First, we claim that φ satisfies the Cerami condition (C).

Indeed, let $\{u_n\}$ be a sequence in E such that $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$. Then there exists a constant $M_1 > 0$ such that

$$|\varphi(u_n)| < M_1, \qquad \|\varphi'(u_n)\|(1 + \|u_n\|) \leq M_1$$

for all $n \in \mathbb{N}$.

In a way similar to the proof of Theorem 1.2, we find that $\{u_n\}$ is a bounded sequence in E. By a standard argument, we deduce that φ satisfies the Cerami condition (C).

Since $\dim Y_k < \infty$, all norms are equivalent. For each $u \in Y_k$, there exists a constant $M_2 > 0$ such that

$$\|u\| \leq M_2\|u\|_{L^2}. \tag{2.20}$$

From condition (F_1), there exists $M_3 > 0$ such that

$$F(x,u) \geq M_2^2 u^2 - M_3, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{2.21}$$

For $u \in Y_k$, it follows from (2.20) and (2.21) that

$$\begin{aligned}\varphi(u) &= \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_\Omega F(x,u)\,dx \\ &\leq \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - M_2^2\|u\|_{L^2}^2 + M_3|\Omega| \\ &\leq \tfrac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - (\|u^+\|^2 + \|u^0\|^2) + M_3|\Omega| \\ &\leq -\tfrac{1}{2}\|u^+\|^2 - \tfrac{1}{2}\|u^-\|^2 - \|u^0\|^2 + M_3|\Omega| \leq -\tfrac{1}{2}\|u\|^2 + M_3|\Omega|,\end{aligned}$$

which implies that

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ in } Y_k.$$

So, (A_1) of Theorem 2.2 is satisfied for every $\rho_k > 0$ large enough.

After integrating, we obtain from (F_3) the existence of $M_4 > 0$ such that

$$|F(x,u)| \leq M_4(1 + |u|^\lambda), \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{2.22}$$

Let us define

$$\beta_k = \sup_{u \in Z_k,\, \|u\|=1} \|u\|_{L^\lambda}.$$

Then, for k large enough such that $Z_k \subset E^+$, by (2.22), on Z_k we have

$$\begin{aligned}\varphi(u) &= \frac{\|u\|^2}{2} - \int_\Omega F(x,u)\,dx \geq \frac{\|u\|^2}{2} - M_4\|u\|_{L^\lambda}^\lambda - M_4|\Omega| \\ &\geq \frac{\|u\|^2}{2} - M_4\beta_k^\lambda\|u\|^\lambda - M_4|\Omega|.\end{aligned}$$

Choosing $r_k = (M_4\lambda\beta_k^\lambda)^{1/(2-\lambda)}$, we obtain, for $u \in Z_k$ and $\|u\| = r_k$,

$$\varphi(u) \geq \left(\frac{1}{2} - \frac{1}{\lambda}\right)(M_4\lambda\beta_k^\lambda)^{2/(2-\lambda)} - M_4|\Omega|.$$

Since, by Lemma 3.8 of [12], $\beta_k \to 0$ as $k \to \infty$, relation (A_2) is proved. Hence, the proof is completed by using the Fountain Theorem. ■

Proof of Theorem 1.8. Firstly, we prove that φ satisfies the Cerami condition (C).

Let $\{u_n\}$ be a sequence in E such that $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$. Then there exists a constant $M_5 > 0$ such that

$$|\varphi(u_n)| < M_5, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \leq M_5$$

for all $n \in \mathbb{N}$.

In a way similar to the proof of Theorem 1.4, we find that $\{u_n\}$ is a bounded sequence in E. By a standard argument, φ satisfies the Cerami condition (C).

It follows from (F_1) that there is an $L_1 > 0$ such that

$$F(x,u) \geq |u|^2 \quad \text{if } |u| \geq L_1. \tag{2.23}$$

Then, by (F_6) and (2.23), for $|u| \geq L_2 = \max\{L, L_1\}$, $x \in \Omega$, one has

$$|f(x,u)|^\sigma \leq m_2(f(x,u)u - 2F(x,u))|u|^\sigma \leq m_2|f(x,u)|\,|u|^{\sigma+1}. \tag{2.24}$$

By (2.24), we get

$$|f(x,u)| \leq m_2^{1/(\sigma-1)}|u|^{(\sigma+1)/(\sigma-1)} \quad \text{for } |u| \geq L_2.$$

Therefore, there exists a positive constant m_7 such that

$$|F(x,u)| \leq m_7(1 + |u|^{2\sigma/(\sigma-1)}), \qquad \forall (x,u) \in \Omega \times \mathbb{R},$$

where $2\sigma/(\sigma-1) < 2^* = 2N/(N-2)$.

Now, by a standard argument as in the proof of Theorem 1.6, the conclusion follows immediately. ∎

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