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The classical subspaces of the projective tensor products of ℓ_p and $C(\alpha)$ spaces, $\alpha < \omega_1$

by

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Dedicated to the memory of Professor Aleksander Pełczyński

Abstract. We completely determine the ℓ_q and $C(K)$ spaces which are isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\alpha)$, the projective tensor product of the classical ℓ_p space, $1 \leq p < \infty$, and the space $C(\alpha)$ of all scalar valued continuous functions defined on the interval of ordinal numbers $[1, \alpha]$, $\alpha < \omega_1$. In order to do this, we extend a result of A. Tong concerning diagonal block matrices representing operators from ℓ_p to ℓ_1, $1 \leq p < \infty$.

The first main theorem is an extension of a result of E. Oja and states that the only ℓ_q space which is isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\alpha)$ with $1 \leq p \leq q < \infty$ and $\omega \leq \alpha < \omega_1$ is ℓ_p. The second main theorem concerning $C(K)$ spaces improves a result of Bessaga and Pełczyński which allows us to classify, up to isomorphism, the separable spaces $\mathcal{N}(X, Y)$ of nuclear operators, where X and Y are direct sums of ℓ_p and $C(K)$ spaces. More precisely, we prove the following cancellation law for separable Banach spaces. Suppose that K_1 and K_3 are finite or countable compact metric spaces of the same cardinality and $1 < p, q < \infty$. Then, for any infinite compact metric spaces K_2 and K_4, the following statements are equivalent:

(a) $\mathcal{N}(\ell_p \oplus C(K_1), \ell_q \oplus C(K_2))$ and $\mathcal{N}(\ell_p \oplus C(K_3), \ell_q \oplus C(K_4))$ are isomorphic.
(b) $C(K_2)$ is isomorphic to $C(K_4)$.

1. Introduction. We shall use the standard notations and terminology of Banach space theory (see e.g. [7]). For K a compact Hausdorff space, we denote by $C(K)$ the Banach space of all continuous scalar valued functions defined on K and endowed with the supremum norm. If $\alpha \leq \beta$ are ordinal numbers, then $[\alpha, \beta]$ denotes the interval $\{\gamma;\, \alpha \leq \gamma \leq \beta\}$ endowed with the order topology. ω denotes the first infinite ordinal and ω_1 the first uncountable ordinal. The space $C([1, \alpha])$ will be denoted by $C(\alpha)$. Given Banach spaces X and Y, we write $X \sim Y$ whenever X and Y are isomorphic, and $Y \hookrightarrow X$ when X contains a subspace isomorphic to Y.

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It is not an easy matter to study the projective tensor products $X\widehat{\otimes}_\pi Y$ of two Banach spaces X and Y introduced by Grothendieck in [6]. The starting point of this paper is the widely known fact that these Banach spaces may have some *unexpected subspaces* (see for instance [4]). We only recall that this happens even when $X = \ell_p$ for some $1 \le p < \infty$. Indeed, by [9, Theorem 4], for every $1 \le r, s < \infty$ satisfying $p(r-1) \le r$ and $p(s-1) > s$ we have

$$\ell_1 \hookrightarrow \ell_p \widehat{\otimes}_\pi \ell_r \quad \text{and} \quad \ell_{ps/(p+s)} \hookrightarrow \ell_p \widehat{\otimes}_\pi \ell_s. \tag{1.1}$$

In other words, the projective tensor product of ℓ_p and $Y = \ell_q$ contains *unwanted subspaces* for every $1 \le p, q < \infty$. This is also the case when Y is not necessarily isomorphic to any ℓ_q space, $1 \le q < \infty$. Indeed, it is well known that every finite sum of Y, Y^n, $1 \le n < \omega$, is isomorphic to $\mathbb{K}^n \widehat{\otimes}_\pi Y$, where $\mathbb{K}$ is the field of scalars of Y. Therefore

$$Y^n \hookrightarrow \ell_p \widehat{\otimes}_\pi Y.$$

In particular, when $Y = C(\omega_1)$ we have, by [14, Theorem 2.1],

$$C(\omega_1 \cdot n) \hookrightarrow \ell_p \widehat{\otimes}_\pi C(\omega_1) \quad \text{but} \quad C(\omega_1 \cdot n) \not\hookrightarrow C(\omega_1),\ \forall 1 \le n < \omega. \tag{1.2}$$

The observations (1.1) and (1.2) lead naturally to the following question.

Problem 1.1. *Do the separable $\ell_p \widehat{\otimes}_\pi C(\alpha)$ spaces have any unexpected ℓ_q or $C(\beta)$ subspaces?*

In contrast to the $\ell_p \widehat{\otimes}_\pi \ell_q$ and $\ell_p \widehat{\otimes}_\pi C(\omega_1)$ results mentioned above, our main theorems are as follows.

Theorem 1.2. *Let $1 \le p, q < \infty$ and $\omega \le \alpha < \omega_1$. Then*

$$\ell_q \hookrightarrow \ell_p \widehat{\otimes}_\pi C(\alpha) \;\Leftrightarrow\; p = q.$$

It follows from [9, Theorem 3] that, for $p < q < \infty$, ℓ_q is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\alpha)$. So, Theorem 1.2 completes this result by stating that, for $1 \le q < p$, ℓ_q is also not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\alpha)$.

Theorem 1.3. *Let $1 \le p < \infty$ and $\omega \le \alpha \le \beta < \omega_1$. Then*

$$C(\beta) \hookrightarrow \ell_p \widehat{\otimes}_\pi C(\alpha) \;\Leftrightarrow\; \beta < \alpha^\omega.$$

Bessaga and Pełczyński [1, Theorem 1] completely determined all $C(\beta)$ subspaces of a fixed separable $C(\alpha)$ space. More exactly, they stated that, for $\omega \le \alpha \le \beta < \omega_1$, $C(\beta)$ is isomorphic to a subspace of $C(\alpha)$ if, and only if, $\beta < \alpha^\omega$. Theorem 1.3 shows that there is no change if we consider the question of characterization of the $C(\beta)$ subspaces of $\ell_p \widehat{\otimes}_\pi C(\alpha)$.

From Theorems 1.2 and 1.3 we infer easily when $\ell_q \widehat{\otimes}_\pi C(\beta)$ and $\ell_p \widehat{\otimes}_\pi C(\alpha)$ have the same linear dimension [2, Chapitre XII]. Indeed, we can deduce something more:

THEOREM 1.4. *Let $1 \le p, q < \infty$ and $\omega \le \alpha \le \beta < \omega_1$. Then*

$$\ell_q \mathbin{\widehat{\otimes}_\pi} C(\beta) \hookrightarrow \ell_p \mathbin{\widehat{\otimes}_\pi} C(\alpha) \;\Leftrightarrow\; p = q \text{ and } \beta < \alpha^\omega.$$

The present paper is organized as follows. In Section 2, we present some preliminary results and notation. In Section 3, we extend a result of Tong [15, Theorem 4.6] concerning "diagonal block matrices" representing operators from ℓ_p to ℓ_1, $1 \le p < \infty$ and "normalized diagonal block sequences" in $\ell_p \mathbin{\widehat{\otimes}_\pi} c_0$ (Theorem 3.7). In Section 4, we use this result to show that "normalized diagonal block sequences" in $\ell_p \mathbin{\widehat{\otimes}_\pi} C(\alpha)$ spaces are equivalent to the unit basis of ℓ_p (Theorem 4.2). In Section 5, we prove Theorem 1.2.

In Section 6, we establish Theorem 1.3. In Section 7, we turn our attention to the Banach spaces $\mathcal{N}(X, Y)$ of nuclear operators containing subspaces isomorphic to $C(\alpha)$, $\omega \le \alpha < \omega_1$. As an application of Theorem 1.3 we obtain Theorem 7.1 which is a generalization of the classical isomorphic classification of $C(\alpha)$ spaces, $\omega \le \alpha < \omega_1$, established by Bessaga and Pełczyński [1, Theorem 1].

In Section 8, as another consequence of Theorem 1.3, we accomplish the isomorphic classification of separable spaces of nuclear operators on $\ell_p(\Gamma) \oplus C(K)$ spaces. See the cancellation law stated in Theorem 8.1.

Finally, notice that Theorem 1.3 leaves several questions open on the geometry of projective tensor products of Banach spaces. We only stress the following one:

PROBLEM 1.5. *Let $1 \le p < \infty$, $\omega \le \alpha < \omega_1$ and Y a Banach space. Is it true that*

$$C(\alpha) \hookrightarrow \ell_p \mathbin{\widehat{\otimes}_\pi} Y \;\Rightarrow\; C(\alpha) \hookrightarrow Y?$$

Observe that [10, Corollary 1] provides a positive answer to the above question in the simplest case $\alpha = \omega$.

2. Preliminary results and notation. We start by recalling some basic facts on projective tensor products of Banach spaces [6]. Let E, F be two Banach spaces. We denote by $\mathcal{B}(E, F)$ the space of bounded bilinear functionals on $E \times F$. The projective tensor norm of $u = \sum_{i=1}^n a_i \otimes b_i \in E \otimes F$ is defined by

$$\|u\| = \sup\Big\{\Big|\sum_{i=1}^n \varphi(a_i, b_i)\Big| : \varphi \in \mathcal{B}(E, F),\ \|\varphi\| \le 1\Big\}.$$

As usual, $E \mathbin{\widehat{\otimes}_\pi} F$ denotes the completion of $E \otimes F$ with respect to the projective norm. If necessary we denote by $\| \ \|_{\pi(E\otimes F)}$ the projective norm on $E \mathbin{\widehat{\otimes}_\pi} F$.

Suppose that M is a closed subspace of F. The two norms $\| \ \|_{\pi(E\otimes F)}$ and $\| \ \|_{\pi(E\otimes M)}$ are not necessarily equivalent on the subspace $E \otimes M$ of $E \otimes F$.

Nevertheless we have the following result of Grothendieck ([6, Corollary 1, p. 40]):

THEOREM 2.1. *Let E and F be Banach spaces. Suppose that M is a complemented subspace of F and P is a bounded linear projection from F onto M. Then for every $u \in E\widehat{\otimes}_\pi M$, we have*

$$\|u\|_{\pi(E\otimes F)} \le \|u\|_{\pi(E\otimes M)} \le \|P\|\,\|u\|_{\pi(E\otimes F)}.$$

The following theorem will be useful later. Its proof is straightforward.

THEOREM 2.2. *Let $(p_\alpha)_{\alpha\in\mathcal{A}}$ (resp. $(q_\beta)_{\beta\in\mathcal{B}}$) be a uniformly bounded net of operators on a Banach space E (resp. F) which converge simply to the identity. Then $(p_\alpha\otimes q_\beta)_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$ converges simply to the identity on $E\widehat{\otimes}_\pi F$.*

We also recall that the space $\mathcal{B}(E,F)$ of bounded bilinear functionals on $E\times F$ is canonically isometrically isomorphic to the space $\mathcal{L}(E,F^*)$ of bounded linear operators from E to F^*. Moreover, $(E\widehat{\otimes}_\pi F)^*$ is isometrically isomorphic to $\mathcal{L}(E,F^*)$ in the following manner: to every $v\in\mathcal{L}(E,F^*)$ is associated the continuous linear functional also denoted by v on $E\widehat{\otimes}_\pi F$ such that, for each $a\in E$ and $b\in F$, $v(a\otimes b)=v(a)(b)$.

Finally, throughout this paper we denote by

- $(P_m)_m$ the sequence of natural projections associated to the unit basis of ℓ_p,
- $(Q_m)_m$ the sequence of natural projections associated to the unit basis of ℓ_1,
- $(R_m)_{m\ge1}$ the sequence of natural projections associated to the unit vector basis of c_0.

Observe that $R_m^*=Q_m$ for every $m\ge1$.

3. An extension of a result of Tong on diagonal block sequences in $\ell_p\widehat{\otimes}_\pi c_0$ spaces. We need to establish, in a different setting, results similar to those of Tong on diagonal block matrices [15, Proposition (2.5), Theorem (3.7), Theorem (4.6)]. In the following theorem, we summarize Tong's results that we use in our proof.

Let $(m_k)_{k\ge0}$ be a strictly increasing sequence of integers. Given $\varphi\in\ell_p\widehat{\otimes}_\pi c_0$ and $\psi\in\mathcal{L}(\ell_p,\ell_1)$, we denote, for every integer $k\ge1$,

$$\varphi_k=(P_{m_k}-P_{m_{k-1}})\otimes(R_{m_k}-R_{m_{k-1}})(\varphi),$$
$$\psi_k=(Q_{m_k}-Q_{m_{k-1}})\psi(P_{m_k}-P_{m_{k-1}}).$$

For every $1\le p<\infty$, we denote by p' the conjugate exponent of p, that is, $1/p+1/p'=1$.

THEOREM 3.1. *Let $\varphi\in\ell_p\widehat{\otimes}_\pi c_0$ and $\psi\in\mathcal{L}(\ell_p,\ell_1)$. Then, for every integer N:*

(a) $\Big\|\sum_{k=1}^{N}\varphi_k\Big\| = \Big[\sum_{k=1}^{N}\|\varphi_k\|^p\Big]^{1/p} \le \|\varphi\|$ *if* $1 \le p < \infty$.

(b) $\Big\|\sum_{k=1}^{N}\psi_k\Big\| = \Big[\sum_{k=1}^{N}\|\psi_k\|^{p'}\Big]^{1/p'} \le \|\psi\|$ *if* $1 < p < \infty$.

(c) $\Big\|\sum_{k=1}^{N}\psi_k\Big\| = \max_{1\le k\le N}\|\psi_k\|$ *if* $p = 1$.

The purpose of this section is to prove results analogous to Theorem 3.1 where, instead of a sequence $(m_k)_{k\ge 0}$, we have two strictly increasing sequences of integers.

We introduce a new notation: Let $(m_k)_{k\ge 0}$ and $(n_k)_{k\ge 0}$ be two strictly increasing sequences of integers. For every integer $k \ge 1$ we denote by U_k the operator on $\ell_p \widehat{\otimes}_\pi c_0$ given by

$$(P_{n_k} - P_{n_{k-1}}) \otimes (R_{m_k} - R_{m_{k-1}}).$$

Remark 3.2. Let $V_k = U_k^*$. It is easy to verify that, for every $v \in \mathcal{L}(\ell_p, \ell_1)$,

$$V_k(v) = (Q_{m_k} - Q_{m_{k-1}})v(P_{n_k} - P_{n_{k-1}}).$$

Definition 3.3. We say that a sequence $(u_k)_{k\ge 1}$ (resp. $(v_k)_{k\ge 1}$) in $\ell_p \widehat{\otimes}_\pi c_0$ (resp. $\mathcal{L}(\ell_p, \ell_1)$) is a *diagonal block sequence* if there exist two strictly increasing sequences $(m_k)_{k\ge 0}$ and $(n_k)_{k\ge 0}$ of integers such that $u_k = U_k(u_k)$ (resp. $v_k = V_k(v_k)$) for every integer $k \ge 1$.

Before stating and proving the main result of this section (Theorem 3.7), we state some auxiliary results. By elementary computations we have the following results.

Lemma 3.4. *Let* $v \in \mathcal{L}(\ell_p, \ell_1)$. *For every integer* N,

$$\Big\|\sum_{k=1}^{N} V_k(v)\Big\| = \begin{cases} \Big[\sum_{k=1}^{N}\|V_k(v)\|^{p'}\Big]^{1/p'} & \text{if } 1 < p < \infty, \\ \max_{1\le k\le N}\|V_k(v)\| & \text{if } p = 1. \end{cases}$$

Lemma 3.5. *For every* $v \in \mathcal{L}(\ell_p, \ell_1)$,

$$\|(V_1 + V_2)(v)\| \le \|v\|.$$

Proof. Let $v_1 = V_1(v)$ and $v_2 = V_2(v)$. If $m_1 = n_1$ the result follows directly from Theorem 3.1(b).

Assume first that $m_1 < n_1$. Denote by $(\varepsilon_m)_{m\ge 1}$ the unit basis of ℓ_1. Set $r = n_1 - m_1$. Denote by τ the operator on ℓ_1 defined by

$$\tau(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } 1 \le k \le m_1, \\ \varepsilon_{k+r} & \text{if } m_1 + 1 \le k. \end{cases}$$

Denote $w_1 = Q_{n_1}\tau v P_{n_1}$ and $w_2 = (Q_{m_{2+r}} - Q_{m_{1+r}})\tau v(P_{n_2} - P_{n_1})$. Again by Theorem 3.1(b) we see that $\|w_1 + w_2\| \le \|\tau v\| = \|v\|$. We have $Q_{n_1}\tau = Q_{m_1}$ and $(Q_{m_{2+r}} - Q_{m_{1+r}})\tau = \tau(Q_{m_2} - Q_{m_1})$, so it follows that $\|v_1\| = \|w_1\|$ and $w_2 = \tau v_2$. Moreover, for every $y \in \ell_1$, we have $\|\tau y\| = \|y\|$, so $\|w_2\| = \|v_2\|$. The result follows from Lemma 3.4.

Assume now that $m_1 > n_1$. Denote by $(e_m)_{m\ge 1}$ the unit basis of ℓ_p. Set $\rho = m_1 - n_1$. Let σ be the operator on ℓ_p defined by

$$\sigma(e_k) = \begin{cases} e_k & \text{if } 1 \le k \le n_1, \\ 0 & \text{if } n_1 + 1 \le k \le m_1, \\ e_{k-\rho} & \text{if } m_1 + 1 \le k. \end{cases}$$

It is easy to check that $\sigma P_{m_1} = P_{n_1}$ and $\sigma(P_{n_2+\rho} - P_{n_1+\rho}) = (P_{n_2} - P_{n_1})\sigma$. Denote $w_1' = Q_{m_1} v\sigma P_{m_1}$ and $w_2' = (Q_{m_2} - Q_{m_1})v\sigma(P_{n_2+\rho} - P_{n_1+\rho})$. Once again from Theorem 3.1(b) we infer

$$\|w_1' + w_2'\| \le \|Q_{m_2} v\sigma P_{n_2+\rho}\| \le \|v\|.$$

It is clear that $w_1' = v_1$ and $w_2' = v_2\sigma$.

Now we will prove that $\|w_2'\| = \|v_2\|$. It is obvious that $\|w_2'\| \le \|v_2\|$, so it suffices to show that $\|v_2\| \le \|w_2'\|$. There exists $x = \sum_{k=n_1+1}^{n_2} a_k e_k$ such that $\|x\| = 1$ and $\|v_2\| = \|v_2(x)\|$. We have $x = \sigma(x')$ with $x' = \sum_{k=n_1+1}^{n_2} a_k e_{k+\rho}$. Consequently, $\|v_2\| = \|v_2\sigma(x')\| \le \|w_2'\|$.

Now, by using Lemma 3.4 we conclude

$$\|v_1 + v_2\| = [\|v_1\|^{p'} + \|v_2\|^{p'}]^{1/p'} = [\|w_1'\|^{p'} + \|w_2'\|^{p'}]^{1/p'} = \|w_1' + w_2'\| \le \|v\|. \ \blacksquare$$

Proposition 3.6. *Let $v \in \mathcal{L}(\ell_p, \ell_1)$, let $(m_k)_{k\ge 0}$ and $(n_k)_{k\ge 0}$ be strictly increasing sequences of integers and $(V_k)_{k\ge 1}$ the associated sequence of operators on $\mathcal{L}(\ell_p, \ell_1)$. Then, for every $v \in \mathcal{L}(\ell_p, \ell_1)$ and $N \ge 1$,*

$$\Big\| \sum_{k=1}^N V_k(v) \Big\| = \Big[\sum_{k=1}^N \|V_k(v)\|^{p'} \Big]^{1/p'} \le \|v\|.$$

Proof. Use induction on N and Lemmas 3.4 and 3.5. $\blacksquare$

We are now ready to prove the main theorem of this section. Notice that this result was proved in [15, Theorem 4.6] in the case where $m_1 \le n_1 < m_2 \le n_2 < \cdots$.

Theorem 3.7. *Let $(u_k)_{k\ge 1}$ be a normalized diagonal block sequence in $\ell_p \widehat{\otimes}_\pi c_0$. Then, for every integer $N \ge 1$ and for every sequence $(\lambda_k)_k$ of scalars, we have*

$$\Big\| \sum_{k=1}^N \lambda_k u_k \Big\| = \Big[\sum_{k=1}^N |\lambda_k|^p \Big]^{1/p}.$$

Proof. Fix an integer N and scalars $\lambda_1, \ldots, \lambda_N$. There exists $v \in \mathcal{L}(\ell_p, \ell_1)$ satisfying $\|v\| = 1$ and

$$\Big\| \sum_{k=1}^{N} \lambda_k u_k \Big\| = \Big| \sum_{k=1}^{N} \lambda_k v(u_k) \Big|.$$

We have $U_k(u_k) = u_k$ so $v(u_k) = v_k(u_k)$ with $v_k = V_k(v)$.

According to Proposition 3.6 we deduce

$$\Big[\sum_{k=1}^{N} \|v_k\|^{p'} \Big]^{1/p'} \leq \|v\| = 1.$$

Hence, by the Hölder inequality,

$$\Big\| \sum_{k=1}^{N} \lambda_k u_k \Big\| \leq \Big[\sum_{k=1}^{N} |\lambda_k|^p \Big]^{1/p}.$$

In order to prove the reverse inequality fix, for each integer $1 \leq i \leq N$, $w_i \in \mathcal{L}(\ell_p, \ell_1)$ such that $\|w_i\| = w_i(u_i) = 1$. We have $u_i = U_i(u_i)$ so we can suppose that $w_i = U_i^*(w_i)$, that is, $w_i = V_i(w_i)$. Notice that for $1 \leq k \neq i \leq N$, $w_i(u_k) = 0$. Let $\alpha_1, \ldots, \alpha_N$ be scalars such that $\sum_{i=1}^{N} |\alpha_i|^{p'} \leq 1$. Thus, by Proposition 3.6, $\| \sum_{i=1}^{N} \alpha_i w_i \| \leq 1$ and therefore

$$\begin{aligned} \Big\| \sum_{k=1}^{N} \lambda_k u_k \Big\| &\geq \sup \Big\{ \Big| \sum_{k=1}^{N} \sum_{i=1}^{N} \alpha_i \lambda_k w_i(u_k) \Big| ;\ \sum_{i=1}^{N} |\alpha_i|^{p'} \leq 1 \Big\} \\ &\geq \sup \Big\{ \Big| \sum_{k=1}^{N} \alpha_k \lambda_k \Big| ;\ \sum_{k=1}^{N} |\alpha_k|^{p'} \leq 1 \Big\} \geq \Big[\sum_{k=1}^{N} |\lambda_k|^p \Big]^{1/p}. \end{aligned}$$

So the proof is complete. ■

4. On normalized diagonal block sequences in $\ell_p \widehat{\otimes}_\pi C(\alpha)$ spaces. In this section we use Theorem 3.7 to show a similar result on normalized diagonal block sequences in $\ell_p \widehat{\otimes}_\pi C_0(\alpha)$ spaces (Theorem 4.2). This will be the main tool in the proof of Theorem 1.3.

We need to introduce some more notations. We denote by $C_0(\alpha)$ the subspace of $C(\alpha)$ given by $\{f \in C(\alpha);\ f(\alpha) = 0\}$. According to [1, Lemma 1], $C_0(\alpha)$ is isomorphic to $C(\alpha)$ for $\alpha \geq \omega$. Let $0 \leq \beta < \gamma < \alpha$. We denote

$$C([\beta + 1, \gamma]) = \{f \in C_0(\alpha);\ f = f\, 1_{[\beta+1,\gamma]}\},$$

where $1_{[\beta+1,\gamma]}$ is the characteristic function of $[\beta + 1, \gamma]$. Let $1 \leq \gamma \leq \alpha$ be an ordinal. We denote by S_γ the operator from $C_0(\alpha)$ to $C_0(\alpha)$ defined, for every $f \in C_0(\alpha)$, by $S_\gamma(f) = f\, 1_{[1,\gamma]}$. It is obvious that, for $\alpha = \omega$ and $1 \leq m < \omega$, we have $S_m = R_m$.

Let $1 \le p < \infty$, and let $(\gamma_i)_{i\ge 0}$ be a strictly increasing sequence in $[1,\alpha]$ and $(n_i)_{i\ge 0}$ a strictly increasing sequence of integers. For every $1 \le k < \omega$ we denote by Π_k the operator on $\ell_p \mathbin{\widehat{\otimes}_\pi} C_0(\alpha)$ given by

$$P_{n_k} \otimes S_{\gamma_k} - P_{n_{k-1}} \otimes S_{\gamma_k} - P_{n_k} \otimes S_{\gamma_{k-1}} + P_{n_{k-1}} \otimes S_{\gamma_{k-1}}.$$

Definition 4.1. We say that a sequence $(u_k)_{k\ge 1}$ in $\ell_p \mathbin{\widehat{\otimes}_\pi} C_0(\alpha)$ is a *diagonal block sequence* if there exist strictly increasing sequences $(\gamma_k)_{k\ge 0}$ in $[1,\alpha]$ and $(n_k)_{k\ge 0}$ of integers such that $u_k = \Pi_k(u_k)$ for every integer k.

Theorem 4.2. *Suppose that $(u_k)_{k\ge 1}$ is a normalized diagonal block sequence in $\ell_p \mathbin{\widehat{\otimes}_\pi} C_0(\alpha)$. Then $(u_k)_{k\ge 1}$ is equivalent to the unit basis of ℓ_p.*

Proof. There exists a sequence $(f_i)_i$ in $C_0(\alpha)$ such that, for every integer k, we have $u_k = \sum_{i=n_{k-1}+1}^{n_k} e_i \otimes f_i$ and $f_{n_{k-1}+1},\dots,f_{n_k} \in C([\gamma_{k-1}+1,\gamma_k])$. The space $C([\gamma_{k-1}+1,\gamma_k])$ is an $\mathcal{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$ [7, p. 57], so there exists a finite-dimensional subspace E_k of $C([\gamma_{k-1}+1,\gamma_k])$ such that $f_{n_{k-1}+1},\dots,f_{n_k} \in E_k$ and the Banach–Mazur distance $d(E_k,\ell_\infty^{d_k})$ is less than 2, where d_k is the dimension of E_k. Therefore there exists a linear projection π_k of $C([\gamma_{k-1}+1,\gamma_k])$ onto E_k of norm less than or equal to 2.

Fix $1 \le N < \omega$. The subspace $E = E_1 \oplus \dots \oplus E_N$ of $C_0(\alpha)$ is complemented by a projection of norm less than or equal to 2. Let $a_1,\dots,a_N$ be scalars. We have $\sum_{n=1}^N a_n u_n \in \ell_p \mathbin{\widehat{\otimes}_\pi} E$. So, according to Theorem 2.1,

(4.1)
$$\Big\|\sum_{n=1}^N a_n u_n\Big\|_{\pi(\ell_p\otimes C_0(\alpha))} \le \Big\|\sum_{n=1}^N a_n u_n\Big\|_{\pi(\ell_p\otimes E)} \le 2\Big\|\sum_{n=1}^N a_n u_n\Big\|_{\pi(\ell_p\otimes C_0(\alpha))}.$$

Let $d = d_1 + \dots + d_N$. It is obvious that E is 2-isomorphic to a subspace of c_0. Then, by Theorems 3.7 and 2.1, we obtain

$$\Big[\sum_{n=1}^N |a_n|^p\Big]^{1/p} \le \Big\|\sum_{n=1}^N a_n u_n\Big\|_{\pi(\ell_p\otimes E)} \le 2\Big[\sum_{n=1}^N |a_n|^p\Big]^{1/p}. \tag{4.2}$$

By combining (4.1) with (4.2) we conclude

$$\frac{1}{2}\Big[\sum_{n=1}^N |a_n|^p\Big]^{1/p} \le \Big\|\sum_{n=1}^N a_n u_n\Big\|_{\pi(\ell_p\otimes C_0(\alpha))} \le 2\Big[\sum_{n=1}^N |a_n|^p\Big]^{1/p}. \ \blacksquare$$

5. ℓ_q subspaces of separable $\ell_p \mathbin{\widehat{\otimes}_\pi} C(\alpha)$ spaces. The object of this section is to prove Theorem 1.2. It is convenient to introduce the following notation:

Let $1 \le p < \infty$ and $\omega \le \alpha < \omega_1$. For every $(m,\gamma) \in [1,\omega) \times [1,\alpha)$, we denote by $T_{m,\gamma}$ the operator on $\ell_p \mathbin{\widehat{\otimes}_\pi} C_0(\alpha)$ given by

$$P_m \otimes I_{C_0(\alpha)} + I_{\ell_p} \otimes S_\gamma - P_m \otimes S_\gamma.$$

Lemma 5.1. *Let $1 \le p < \infty$ and $\omega \le \alpha < \omega_1$. Then, for every $(m,\gamma) \in [1,\omega) \times [1,\alpha)$, $\operatorname{Im} T_{m,\gamma}$ is isomorphic to $C_0(\alpha) \oplus \ell_p \widehat{\otimes}_\pi C(\gamma)$.*

Proof. Denote by Ψ_m and $\Phi_{m,\gamma}$ the projections of $\ell_p \widehat{\otimes}_\pi C_0(\alpha)$ given by $P_m \otimes I_{C_0(\alpha)}$ and $(I_{\ell_p} - P_m) \otimes S_\gamma$ respectively. We notice that

(a) $\Psi_m(\ell_p \widehat{\otimes}_\pi C_0(\alpha)) \sim C_0(\alpha)$,
(b) $\Phi_{m,\gamma}(\ell_p \widehat{\otimes}_\pi C_0(\alpha)) \sim \ell_p \widehat{\otimes}_\pi C(\gamma)$,
(c) $T_{m,\gamma} = \Psi_m + \Phi_{m,\gamma}$,
(d) $\Psi_m \Phi_{m,\gamma} = \Phi_{m,\gamma}\Psi_m = 0$

It follows that $\operatorname{Im} T_{m,\gamma}$ is isomorphic to $C_0(\alpha) \oplus \ell_p \widehat{\otimes}_\pi C(\gamma)$. ∎

Proof of Theorem 1.2. We need only show that if ℓ_q is isomorphic to a subspace of $\ell_p \hat{\otimes}_\pi C(\alpha)$ then $p = q$. The converse is obvious.

We suppose $p \ne q$ and we prove by transfinite induction that for every $\alpha < \omega_1$, ℓ_q is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\alpha)$. This is so if $\alpha < \omega$. Now let $\omega \le \alpha < \omega_1$ and suppose that, for every $\gamma < \alpha$, ℓ_q is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C(\gamma)$. It is still the case for $\gamma = \alpha$ if α is a successor. Consider the case where α is a limit ordinal. We shall show that the existence of a linear operator $T : \ell_q \to \ell_p \widehat{\otimes}_\pi C_0(\alpha)$ which is an isomorphism onto its image leads to a contradiction.

Denote by $(x_n)_{n\ge1}$ the unit basis of ℓ_q. Fix a number $\varepsilon > 0$. We construct by induction a normalized block basic sequence $(y_k)_{k\ge1}$ of $(x_n)_{n\ge1}$, a strictly increasing sequence $(m_k)_{k\ge1}$ of integers and a strictly increasing sequence $(\gamma_k)_{k\ge1}$ in $[1,\alpha[$ such that, for every integer k,

$$\|T(y_k) - (P_{n_k} - P_{n_{k-1}}) \otimes (S_{\gamma_k} - S_{\gamma_{k-1}})T(y_k)\| \le \varepsilon/2^k\cdot \tag{5.1}$$

We take $y_1 = x_1$. By Theorem 2.2 we fix an integer m_1 and $\gamma_1 < \alpha$ such that

$$\|T(y_1) - (P_{n_1} \otimes S_{\gamma_1})T(y_1)\| \le \varepsilon/2\cdot$$

Now let $i \ge 1$ be an integer and suppose that we have a finite normalized block basic sequence $(y_1,\dots,y_i)$ of $(x_n)_{n\ge1}$, $m_1 < \dots < m_i$ and $\gamma_1 < \dots < \gamma_i < \alpha$ such that (5.1) is satisfied for $1 \le k \le i$. There exists an integer k_i such that $y_1,\dots,y_i \in \operatorname{span}\{x_k;\, 1 \le k \le k_i\}$. It follows from Lemma 5.1 that $\operatorname{Im} T_{n_i,\gamma_i}$ is isomorphic to $C_0(\alpha) \oplus \ell_p \widehat{\otimes}_\pi C(\gamma_i)$. Hence, $\operatorname{Im} T_{n_i,\gamma_i}$ does not contain a subspace isomorphic to ℓ_q. So there is $y_{i+1} \in \operatorname{span}\{x_l;\, l \ge k_i + 1\}$ which satisfies $\|y_{i+1}\| = 1$ and

$$\|T_{n_i,\gamma_i} T(y_{i+1})\| \le \varepsilon/2^{i+2}.$$

There exist an integer $n_{i+1} > n_i$ and an ordinal $\gamma_{i+1} \in]\gamma_i,\alpha[$ such that

$$\|T(y_{i+1}) - (P_{n_{i+1}} \otimes S_{\gamma_{i+1}})T(y_{i+1})\| \le \varepsilon/2^{i+2}. \tag{5.2}$$

We have

$$\|(P_{n_{i+1}} \otimes S_{\gamma_{i+1}})T_{n_i,\gamma_i} T(y_{i+1})\| \le \varepsilon/2^{i+2}, \tag{5.3}$$

so, by (5.2) and (5.3), (5.1) holds for $i+1$.

Let $z_1 = (P_{n_1} \otimes S_{\gamma_1})T(y_1)$ and, for $k \geq 2$,

$$z_k = (P_{n_k} - P_{n_{k-1}}) \otimes (S_{\gamma_k} - S_{\gamma_{k-1}})T(y_k).$$

On one hand, for $\varepsilon > 0$ small enough the sequence $(z_k)_{k\geq 1}$ is equivalent to the unit basis of ℓ_q; on the other hand, $(z_k)_{k\geq 1}$ is a seminormalized diagonal block sequence in $\ell_p \widehat{\otimes}_\pi C_0(\alpha)$. Thus, by Theorem 4.2, it is equivalent to the unit basis of ℓ_p, which is a contradiction.

6. $C(\beta)$ subspaces of separable $\ell_p \widehat{\otimes}_\pi C(\alpha)$ spaces. In this section we prove Theorem 1.3. We begin with some auxiliary results.

LEMMA 6.1. *Suppose that $1 \leq p < \infty$ and $\omega \leq \alpha < \omega_1$. If, for every ordinal $\gamma < \alpha$, the space $C_0(\alpha)$ is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C_0(\gamma)$, then for every operator $L : C_0(\alpha^\omega) \to \ell_p \widehat{\otimes}_\pi C_0(\alpha)$ and for every $(n,\gamma) \in [1,\omega) \times [1,\alpha)$, the operator $T_{n,\gamma}L$ is not an isomorphism onto its image.*

Proof. Suppose that there exist $L : C_0(\alpha^\omega) \to \ell_p \widehat{\otimes}_\pi C_0(\alpha)$ and $(m,\gamma) \in [1,\omega) \times [1,\alpha)$ such that $T_{m,\gamma}L$ is an isomorphism onto its image. Lemma 5.1 shows that $C(\alpha^\omega)$ is isomorphic to a subspace of $C_0(\alpha) \oplus \ell_p \widehat{\otimes}_\pi C(\gamma)$. According to [1, Theorem 1], $C_0(\alpha)$ contains no subspace isomorphic to $C(\alpha^\omega)$. Therefore by [11, Theorem 1] and [5, Theorem 2.4] we infer that $\ell_p \widehat{\otimes}_\pi C(\gamma)$ contains a subspace isomorphic to $C(\alpha^\omega)$, a contradiction. ∎

The next lemma is a direct consequence of Theorem 2.2.

LEMMA 6.2. *Let $1 \leq p < \infty$ and $\omega \leq \alpha < \omega_1$. For all $u \in \ell_p \widehat{\otimes}_\pi C_0(\alpha)$ and $\varepsilon > 0$ there exist $1 \leq n_0 < \omega$ and $1 \leq \gamma_0 < \alpha$ such for all $n_0 \leq n < \omega$ and $\gamma_0 \leq \gamma < \alpha$,*

$$\|u - (P_n \otimes S_\gamma)(u)\| \leq \varepsilon.$$

The following proposition is a key result to prove Theorem 1.3.

PROPOSITION 6.3. *Let $1 \leq p < \infty$ and $\omega \leq \alpha < \omega_1$. If, for every ordinal $\gamma < \alpha$, the space $C_0(\alpha)$ is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C_0(\gamma)$, then $C_0(\alpha^\omega)$ is not isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C_0(\alpha)$.*

Proof. Towards a contradiction, suppose that $L : C_0(\alpha^\omega) \to \ell_p \widehat{\otimes}_\pi C_0(\alpha)$ is an isomorphism onto its image. There exist $0 < a \leq b$ such that

$$a\|f\| \leq \|L(f)\| \leq b\|f\|$$

for every $f \in C_0(\alpha^\omega)$. Let $0 < \varepsilon < a$. By using Lemmas 6.1 and 6.2 we will construct by induction a normalized sequence $(f_i)_i$ in $C_0(\alpha^\omega)$, two strictly increasing sequences $(k_i)_i$, $(n_i)_i$ of integers and a strictly increasing sequence $(\gamma_i)_i$ in $[1,\alpha]$ such that

(a) $f_1 = f_1 1_{[1,\alpha^{k_1}]}$ and $f_i = f_i 1_{[\alpha^{k_{i-1}}+1,\alpha^{k_i}]}$ for every integer $i \geq 2$,
(b) $\|T_{n_{i-1},\gamma_{i-1}}L(f_i)\| \leq \varepsilon/2^{i+1}$ for every integer $i \geq 2$,

(c) $\|L(f_1) - (P_{n_1} \otimes S_{\gamma_1})L(f_1)\| \le \varepsilon/2$,
(d) $\|L(f_i) - (P_{n_i} \otimes S_{\gamma_i})L(f_i)\| \le \varepsilon/2^{i+1}$ for every $i \ge 2$.

To begin, we fix $f_1 = 1_{[1,\alpha]} \in C_0(\alpha^\omega)$ and $k_1 = 1$. By Lemma 6.2 there exist an integer n_1 and an ordinal $\gamma_1 < \alpha$ such that

$$\|L(f_1) - (P_{n_1} \otimes S_{\gamma_1})L(f_1)\| \le \varepsilon/2.$$

Let $i \ge 1$, and suppose that $f_1, \ldots, f_i$, $n_1 < \cdots < n_i < \omega$, $k_1 < \cdots < k_i < \omega$ and $\gamma_1 < \cdots < \gamma_i < \alpha$ have been chosen satisfying (a)–(d). It is clear that $C_0([\alpha^{k_i} + 1, \alpha^\omega])$ is isomorphic to $C_0(\alpha^\omega)$. So, by Lemma 6.1, there exists $f'_{i+1} \in C_0([\alpha^{k_i} + 1, \alpha^\omega])$ such that

$$\|f'_{i+1}\| = 1 \quad \text{and} \quad \|T_{n_i,\gamma_i} L(f'_{i+1})\| < \varepsilon/2^{i+2}.$$

Now we fix k_{i+1} such that

$$\|f'_{i+1} 1_{[\alpha^{k_i}+1,\alpha^{k_{i+1}}]}\| = 1 \quad \text{and} \quad \|T_{n_i,\gamma_i} L(f'_{i+1} 1_{[\alpha^{k_i}+1,\alpha^{k_{i+1}}]})\| \le \varepsilon/2^{i+2}.$$

We take $f_{i+1} = f'_{i+1} 1_{[\alpha^{k_i}+1,\alpha^{k_{i+1}}]}$. Then, by Lemma 6.2, we choose $n_{i+1} > n_i$ and $\gamma_i < \gamma_{i+1} < \alpha$ satisfying (d).

This sequence $(f_i)_i$ leads to a contradiction. Indeed, let $\Pi_1 = P_{n_1} \otimes S_{\gamma_1}$ and, for $i \ge 2$,

$$\Pi_i = P_{n_i} \otimes S_{\gamma_i} - P_{n_{i-1}} \otimes S_{\gamma_i} - P_{n_i} \otimes S_{\gamma_{i-1}} + P_{n_{i-1}} \otimes S_{\gamma_{i-1}}.$$

For every integer $i \ge 2$ we have

$$(P_{n_i} \otimes S_{\gamma_i}) T_{n_{i-1},\gamma_{i-1}} = P_{n_{i-1}} \otimes S_{\gamma_i} + P_{n_i} \otimes S_{\gamma_{i-i}} - P_{n_{i-1}} \otimes S_{\gamma_{i-1}}.$$

Consequently,

$$\|\Pi_i L(f_i) - L(f_i)\| \le \|(P_{n_i} \otimes S_{\gamma_i})L(f_i) - L(f_i)\| + \|(P_{n_i} \otimes S_{\gamma_i}) T_{n_{i-1},\gamma_{i-1}} L(f_i)\|,$$

and therefore

$$\|\Pi_i L(f_i) - L(f_i)\| \le \varepsilon/2^i.$$

The sequence $(L(f_i))_i$ is equivalent to the unit basis of c_0. On one hand, for $\varepsilon > 0$ enough small, the sequence $(\Pi_i L(f_i))_i$ is equivalent to the unit basis of c_0. On the other hand, $(\Pi_i L(f_i))_i$ is a seminormalized diagonal block sequence in $\ell_p \widehat{\otimes}_\pi C_0(\alpha)$, so by Theorem 4.2, it is equivalent to the unit basis of ℓ_p. We have the required contradiction. ■

Proof of Theorem 1.3. Let

$$I = \{\alpha \in [\omega, \omega_1[;\ C_0(\alpha^\omega) \text{ is not isomorphic to a subspace of } \ell_p \widehat{\otimes}_\pi C_0(\alpha)\}.$$

It is well known that c_0 is not isomorphic to a subspace of ℓ_p. So, by Proposition 6.3, $\omega \in I$. Now we suppose that $I \neq [\omega, \omega_1[$ and show that this leads to a contradiction.

Let $\alpha_0 = \min([\omega, \omega_1[\setminus I)$. This means that $C_0(\alpha_0^\omega)$ is isomorphic to a subspace of $\ell_p \widehat{\otimes}_\pi C_0(\alpha_0)$ and by Proposition 6.3 there exists $\beta_0 < \alpha_0$ such that

(6.1) $C_0(\alpha_0)$ is isomorphic to a subspace of $\ell_p\widehat{\otimes}_\pi C_0(\beta_0)$.

We have $\beta_0 \in I$, thus

(6.2) $C_0(\beta_0^\omega)$ is not isomorphic to a subspace of $\ell_p\widehat{\otimes}_\pi C_0(\beta_0)$.

It follows from (6.1) and (6.2) that $\alpha_0 < \beta_0^\omega$. We have $\beta_0 < \alpha_0 < \beta_0^\omega$ and so $\alpha_0^\omega = \beta_0^\omega$. The spaces $C_0(\alpha_0)$ and $C_0(\beta_0)$ are isomorphic by [1, Theorem 1]; we also have $C_0(\alpha_0^\omega) = C_0(\beta_0^\omega)$; hence a contradiction between $\alpha_0 \notin I$ and $\beta_0 \in I$. ∎

7. An extension of a result of Bessaga and Pełczyński's on $C(\alpha)$ spaces. The main aim of this section is to prove Theorem 7.1. Notice that the case where ξ and Γ are finite and Y is a finite-dimensional space is exactly [1, Theorem 1]. We denote by $\overline{\xi}$ the cardinality of the ordinal ξ.

THEOREM 7.1. *Let $1 < p < \infty$, $1 \le \xi, \eta < \omega_1$ with $\overline{\xi} = \overline{\eta}$, Γ a countable set and Y a Banach space containing no subspace isomorphic to c_0. Then, for any ordinals $\omega \le \alpha \le \beta < \omega_1$,*

$$\mathcal{N}(\ell_p(\Gamma) \oplus C(\xi), Y \oplus C(\alpha)) \sim \mathcal{N}(\ell_p(\Gamma) \oplus C(\eta), Y \oplus C(\beta)) \;\Leftrightarrow\; \beta < \alpha^\omega.$$

Proof. We begin by noticing that if $1 \le p < \infty$, $1 \le \lambda, \mu < \omega_1$, Λ is a set and Y is an arbitrary Banach space, then by [6, Proposition 35, p. 164],

(7.1) $\mathcal{N}(\ell_p(\Lambda) \oplus C(\lambda), Y \oplus C(\mu)) \sim (\ell_p(\Lambda) \oplus C(\lambda))^* \,\widehat{\otimes}_\pi\, (Y \oplus C(\mu))$.

Moreover, by [6, Proposition 6, p.46], this space is isomorphic to

(7.2) $(\ell_{p'}(\Lambda) \,\widehat{\otimes}_\pi\, Y) \oplus (\ell_{p'}(\Lambda) \,\widehat{\otimes}_\pi\, C(\mu)) \oplus (\ell_1(\overline{\lambda}) \,\widehat{\otimes}_\pi\, Y) \oplus (\ell_1(\overline{\lambda}) \,\widehat{\otimes}_\pi\, C(\mu))$,

where $1/p + 1/p' = 1$.

To prove the sufficiency, suppose that $\beta < \alpha^\omega$. Then by [1, Theorem 1], $C(\alpha)$ is isomorphic to $C(\beta)$. Hence by (7.1) and (7.2) we deduce

(7.3) $\mathcal{N}(\ell_p(\Gamma) \oplus C(\xi), Y \oplus C(\alpha)) \sim \mathcal{N}(\ell_p(\Gamma) \oplus C(\eta), Y \oplus C(\beta))$.

Conversely, assume that (7.3) holds. For contradiction suppose $\beta \ge \alpha^\omega$. Since $C(\beta)$ is isomorphic to a subspace of $\mathcal{N}(\ell_p(\Gamma) \oplus C(\eta), X \oplus C(\beta))$, it follows by (7.1), (7.2) and our hypothesis that $C(\alpha^\omega)$ is isomorphic to a subspace of the space in (7.2) with $\lambda = \xi$, $\mu = \alpha$ and $\Lambda = \Gamma$.

Therefore [1, Theorem 1] and [5, Theorem 2.4] imply that $C(\alpha^\omega)$ is isomorphic to a subspace of some of the four summands in (7.2). However, an appeal to [10, Corollary 1] shows that $C(\alpha^\omega)$ is isomorphic to no subspace of the first summand in (7.2). Furthermore, since the third summand is a subspace of $\ell_1(\mathbb{N}, Y)$, a standard gliding hump argument shows that c_0 and therefore $C(\alpha^\omega)$ is not isomorphic to any subspace of this (see for instance [3]). Finally, by Theorem 1.3, $C(\alpha^\omega)$ is isomorphic to no subspace of the second or fourth summands, completing the proof. ∎

8. Separable spaces of nuclear operators on $\ell_p(\Gamma) \oplus C(K)$ spaces. The purpose of this last section is to classify, up to isomorphisms, all separable spaces $\mathcal{N}(X,Y)$ of nuclear operators where X and Y are direct sums of ℓ_p and $C(K)$ spaces. Namely, we have:

THEOREM 8.1. *Let $1 < p, q < \infty$, $1 \leq \xi, \eta < \omega_1$ with $\overline{\xi} = \overline{\eta}$, and let Γ and Λ countable sets. Then, for any infinite compact metric spaces K_1 and K_2, the following statements are equivalent:*

(a) $\mathcal{N}(\ell_p(\Gamma) \oplus C(\xi), \ell_q(\Lambda) \oplus C(K_1)) \sim \mathcal{N}(\ell_p(\Gamma) \oplus C(\eta), \ell_q(\Lambda) \oplus C(K_2))$.
(b) $C(K_1)$ *is isomorphic to* $C(K_2)$.

Proof. It is clear that (b) implies (a). Next, suppose that (a) holds. It is convenient to distinguish two cases.

CASE 1: *K_1 and K_2 are countable.* In this case, by the Mazurkiewicz and Sierpiński theorem [8] there exist ordinals $\omega \leq \alpha, \beta < \omega_1$ such that K_1 is homeomorphic to $[1, \alpha]$ and K_2 is homeomorphic to $[1, \beta]$. Then by Theorem 7.1 and [1, Theorem 1], $C(K_1)$ is isomorphic to $C(K_2)$.

CASE 2: *K_1 or K_2 is uncountable.* Without loss of generality we assume that K_2 is uncountable. To prove that $C(K_1)$ is isomorphic to $C(K_2)$ it suffices by Milyutin's theorem [12, Theorem 21.5.10] to show that K_1 is uncountable. Suppose the contrary. Then, again by the Mazurkiewicz and Sierpiński theorem [8], there exists an ordinal $\omega \leq \alpha < \omega_1$ such that $C(K_1)$ is isomorphic to $C(\alpha)$. Hence the first space of (a) is isomorphic to

$$(\ell_{p'}(\Lambda) \widehat{\otimes}_\pi \ell_q(\Lambda)) \oplus (\ell_{p'}(\Lambda) \widehat{\otimes}_\pi C(\alpha)) \oplus (\ell_1(\overline{\xi}) \widehat{\otimes}_\pi \ell_q(\Lambda)) \oplus (\ell_1(\overline{\xi}) \widehat{\otimes}_\pi C(\alpha)), \tag{8.1}$$

where $1/p + 1/p' = 1$.

Pick $r > \max\{p', q\}$. Since $C(K_2)$ is universal for separable Banach spaces and is isomorphic to a subspace of the second space in (a), it follows that ℓ_r is isomorphic to a subspace of the space in (8.1). Therefore ℓ_r is isomorphic to any of the four summands of (8.1) [13, Theorem 1]. But, by [9, Theorem 3], this is impossible. ∎

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