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New limit theorems related to free multiplicative convolution

by

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Abstract. Let $\boxplus$, $\boxtimes$, and $\uplus$ be the free additive, free multiplicative, and boolean additive convolutions, respectively. For a probability measure μ on $[0,\infty)$ with finite second moment, we find a scaling limit of $(\mu^{\boxtimes N})^{\boxplus N}$ as N goes to infinity. The $\mathcal{R}$-transform of its limit distribution can be represented by Lambert's W-function. From this, we deduce that the limiting distribution is freely infinitely divisible, like the lognormal distribution in the classical case. We also show a similar limit theorem by replacing free additive convolution with boolean convolution.

1. Introduction. In probability theory, limit theorems and infinite divisibility are considered in various situations. The classical references are the books by Gnedenko and Kolmogorov [11] and Petrov [17]. One of the most famous limit theorems is the Central Limit Theorem (for short CLT) that gives the scaling limit of the sum of independent, identically distributed (i.i.d.) random variables. Suppose that a random variable Z has the standard normal distribution. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with finite second moment. Then the scaling

$$\frac{X_1+\cdots+X_N-N\mathbb{E}[X_1]}{\sqrt{N\mathbb{V}[X_1]}} \tag{1.1}$$

converges to Z in distribution as N goes to infinity.

When we consider the product of i.i.d. random variables, we also have a CLT type limit theorem. The simplest case is as follows: For a sequence $\{X_k\}_{k=1}^{\infty}$ of i.i.d. random variables with finite second moment, we consider the scaling

$$\prod_{k=1}^{N}\exp\biggl(\frac{X_k-\mathbb{E}[X_k]}{\sqrt{N\mathbb{V}[X_k]}}\biggr). \tag{1.2}$$

By the CLT, this scaling converges to e^Z in distribution as N goes to infinity.

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The distribution of e^Z is called the lognormal distribution. It was proved by Thorin [20] that the lognormal distribution is infinitely divisible. The product limit theorems are also interesting in view of their applications in statistics. For details, see [18] and the book by Galambos and Simonelli [10].

In free probability theory, some limit theorems are also known. The most famous limit theorem is the free CLT, which was found by Voiculescu. If $\{X_k\}_{k\in\mathbb{N}}$ is a sequence of freely independent identically distributed (for short freely i.i.d.) random variables with finite second moment, then the normalized sum (1.1) converges in distribution to the standard Wigner's semi-circle law as N goes to infinity. In addition, we know the Poisson limit theorem, the stable limit theorem and so on; for details, see [12], [6], [4]. Recently new limit theorems with respect to free convolutions [7], [23], [21] have been studied.

In this paper, we shall prove a limit theorem involving not only free additive but also free multiplicative convolutions. We introduce a new normalized sum of products of freely independent random variables. For a double sequence $\{\{X_i^{(j)}\}_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of freely i.i.d. random variables having a distribution μ on $[0,\infty)$ with finite second moment, we consider a new normalization Y_N,

$$Y_N = \sum_{j=1}^{N} \frac{\sqrt{X_N^{(j)}}\cdots\sqrt{X_2^{(j)}}X_1^{(j)}\sqrt{X_2^{(j)}}\cdots\sqrt{X_N^{(j)}}}{m_1^N N}, \tag{1.3}$$

where m_1 is the mean of the distribution μ. We shall see that its limit distribution depends only on the first and second moments. In its proof, we shall investigate a Taylor type expansion of the S-transform. In addition, a formula by Belinschi and Nica [2] suggests that the distribution of (1.3) is equal to that of

$$\widetilde{Y_N} = \frac{\sqrt{\sum_{i=1}^N X_i^{(N)}}\cdots\sqrt{\sum_{i=1}^N X_i^{(1)}}\sqrt{\sum_{i=1}^N X_i^{(1)}}\cdots\sqrt{\sum_{i=1}^N X_i^{(N)}}}{m_1^N N^N},$$

which corresponds to the scaling (1.2). In this sense, we may call it the free lognormal distribution. Compared to the free additive CLT case, it is not exactly the same scaling. The difference may occur because of noncommutativity of random variables. Furthermore a similar limit theorem can be found under boolean independence.

In order to investigate properties of this limit distribution, we show that it is freely infinitely divisible, just as in the classical case the lognormal distribution is infinitely divisible. In the proof, the properties of Lambert's W-function play an important role and we obtain the corresponding Lévy measure.

This paper is organized as follows. In Section 2, we shall gather the tools from free and boolean probability. Especially, we recall the $\mathcal{R}$-, S-, and Σ-transforms and infinite divisibility in free probability theory. In Section 3, we shall give Taylor type expansions for S- and Σ-transforms and prove our limit theorems. Finally, in Section 4, we shall discuss the limit distributions, focusing on infinite divisibility and moments.

2. Preliminaries. Let $\mathbb{R}_+$ be the half-line $[0,\infty)$ and $\mathbb{C}^+$ be the upper half-plane $\{z = x+iy \in \mathbb{C};\, y>0\}$. We write $\mathcal{P}$ and $\mathcal{P}_+$ for the sets of all Borel probability measures on $\mathbb{R}$ and on $\mathbb{R}_+$, respectively. We denote by $\boxplus$, $\boxtimes$, and $\uplus$ the free additive, free multiplicative, and boolean additive convolutions, respectively; for details on convolutions, see [19], [22], and [15]. Hereafter, δ_0 stands for the Dirac probability measure concentrated at 0.

2.1. Analytic tools for free and boolean convolutions. Here, we shall gather the analytic tools from free and boolean probability and mention some relevant facts.

We denote the Cauchy transform of a probability measure μ on $\mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\,\mu(dx), \qquad z\in\mathbb{C}^+,$$

and

$$\Psi_\rho(z) = \int_{\mathbb{R}} \frac{xz}{1-xz}\,\rho(dx), \qquad z\in\mathbb{C}\setminus\mathbb{R},$$

denotes the moment generating function of ρ on $\mathbb{R}_+$. Then Speicher's R-transform and Voiculescu's $\mathcal{R}$-transform of μ are defined as follows: for any given $\alpha>0$, one can find $\beta>0$ so that

$$R_\mu(z) = z\mathcal{R}_\mu(z) = zG_\mu^{-1}(z) - 1, \qquad 1/z \in \Gamma_{\alpha,\beta},$$

where $G_\mu^{-1}(z)$ is the right inverse of $G_\mu(z)$ with respect to composition and $\Gamma_{\alpha,\beta} = \{z = x+iy\in\mathbb{C}^+;\, y>\beta,\, |y|>\alpha x\}$. Note that we will use both R- and $\mathcal{R}$-transforms for convenience. The S- and Σ-transforms of ρ are defined by

$$S_\rho(z) = \frac{(z+1)\Psi_\rho^{-1}(z)}{z}, \qquad z\in\Psi_\rho(i\mathbb{C}^+),$$

and

$$\Sigma_\rho(z) = S_\rho\left(\frac{z}{1-z}\right), \qquad \frac{z}{1-z}\in\Psi_\rho(i\mathbb{C}^+),$$

respectively, where $\Psi_\rho^{-1}(z)$ is the right inverse of $\Psi_\rho(z)$ with respect to composition. Now, we summarize the relations between the transforms and convolutions; for proofs see [6] and [2].

Proposition 2.1. *For $\mu_1, \mu_2 \in \mathcal{P}$ and $\rho_1, \rho_2 \in \mathcal{P}_+$ which are not δ_0, there exist $\alpha, \beta > 0$ such that*

$$\begin{aligned}
R_{\mu_1 \boxplus \mu_2}(z) &= R_{\mu_1}(z) + R_{\mu_2}(z), && 1/z \in \Gamma_{\alpha,\beta},\\
S_{\rho_1 \boxtimes \rho_2}(z) &= S_{\rho_1}(z) S_{\rho_2}(z), && z \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+),\\
S_{\rho_1^{\boxplus t}}(z) &= \frac{1}{t} S_{\rho_1}\left(\frac{z}{t}\right),\\
\Sigma_{\rho_1 \boxtimes \rho_2}(z) &= \Sigma_{\rho_1}(z) \Sigma_{\rho_2}(z), && z/(1-z) \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+),\\
\Sigma_{\rho_1^{\uplus t}}(z) &= \frac{1}{t} \Sigma_{\rho_1}\left(\frac{z}{t}\right).
\end{aligned}$$

For $c > 0$, the dilation operator D_c on $\mathcal{P}$ is defined by

$$D_c(\mu)(B) = \mu\left(\frac{1}{c} B\right)$$

for any Borel set B in $\mathbb{R}_+$, where $(1/c)B = \{(1/c)x;\, x \in B\}$. If a random variable X has a distribution μ, then cX is distributed as $D_c(\mu)$. In [2], the authors showed that

$$S_{D_c(\mu)}(z) = \frac{1}{c} S_\mu(z) \quad \text{and} \quad \Sigma_{D_c(\mu)}(z) = \frac{1}{c} \Sigma_\mu(z).$$

2.2. Infinite divisibility for free additive convolution. A probability measure μ is *freely infinitely divisible* (or $\boxplus$-*infinitely divisible*) if for any $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{P}$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n\,\text{times}}.$$

We denote the class of all $\boxplus$-infinitely divisible distributions by $I^{\boxplus}$.

Remark 2.2. We can define another infinite divisibility by replacing $\boxplus$ by $\boxtimes$ or $\uplus$. But for boolean convolution, all probability measures are $\uplus$-infinitely divisible. So we shall not discuss $\uplus$-divisibility any longer.

The next proposition characterizes the $\boxplus$-infinitely divisible laws [22, Theorem 3.7.2].

Proposition 2.3. *The following are equivalent:*

(1) $\mu \in I^{\boxplus}$.
(2) $\mathcal{R}_\mu$ *has an analytic extension defined on $\mathbb{C}^-$ with values in $\mathbb{C}^- \cup \mathbb{R}$.*
(3) *There exist unique $b_\mu \in \mathbb{R}$ and finite measure ν_μ such that*

$$\mathcal{R}_\mu(z) = b_\mu + \int_{\mathbb{R}} \frac{z}{1 - tz}\, \nu_\mu(dt), \qquad z \in \mathbb{C}^-.$$

The above expression is called the $\boxplus$-*Lévy–Khintchine representation*, or simply the *Lévy–Khintchine representation.*

EXAMPLE 2.4. The typical examples of $\boxplus$-infinitely divisible distribution are Wigner's semi-circle law, Dirac's delta distribution, and the free Poisson distribution π_t with parameter $t \geq 0$ and density

$$\pi_t(dx) = \max(0,(1-t))\delta_0(dx) + \frac{1}{2\pi x}\sqrt{4t-(x-1-t)^2}\,\mathbb{1}_{[(1-\sqrt{t})^2,(1+\sqrt{t})^2]}(x)\,dx. \tag{2.1}$$

The Lévy measure ν_μ and b_μ for the semi-circle law are δ_0 and 0, and the free Poisson law π_t has $b_\mu = t$ and $\nu_\mu = t\delta_1$. We write π for π_1.

The following functional equation of the R- and S-transforms can be found in, for instance, [15] or [16, Lemma 2]:

PROPOSITION 2.5. *Assume that $\mu \in \mathcal{P}_+$. For some sufficiently small $\varepsilon > 0$, we have a region D_ε which includes $\{-it;\ 0 < t < \varepsilon\}$ such that*

$$z = R_\mu(zS_\mu(z)) \quad \textit{for } z \in D_\varepsilon. \tag{2.2}$$

3. New limit theorems. In this section, we prove a new limit theorem related to both free additive and multiplicative convolutions. We also discuss a similar result with $\boxplus$ replaced by $\uplus$. It was proved in [14] by Młotkowski that for the free Poisson law π, we have

$$D_n((\pi^{\boxtimes(n-1)})^{\uplus n}) \xrightarrow{n\to\infty} \nu_0 \quad \text{in distribution,}$$

where the pth moment of ν_0 is $p^p/p!$. We find that a theorem of this type holds more generally if we replace π by any probability distribution with finite second moment.

3.1. Expansion of the S-transform and Σ-transform. We prove expansions for the S-transform and Σ-transform under the second moment condition. For the $\mathcal{R}$-transform, a Taylor type expansion was proved by Benaych-Georges [3]. For each region A in $\mathbb{C}$, we write $z \xrightarrow{z\in A} 0$ whenever $z \to 0$ with $z \in A$.

LEMMA 3.1 (see [1]). *Let $\rho \in \mathcal{P}_+$ have the moment of order p, that is, for $k = 0, 1, \ldots, p$,*

$$m_k(\rho) := \int_{\mathbb{R}_+} x^k\,\rho(dx) < \infty.$$

Then its moment generating function $\Psi_\rho(z)$ has a Taylor expansion

$$\Psi_\rho(z) = \sum_{k=1}^{p-1} m_k(\rho)z^k + O(z^p), \qquad z \xrightarrow{z\in i\mathbb{C}^+} 0.$$

LEMMA 3.2. *Let $\rho \in \mathcal{P}_+$ have the moment of order $p \geq 2$ and $\rho \neq \delta_0$. Then:*

(1) *$\Psi_\rho(z)$ is univalent in $i\mathbb{C}^+$.*

(2) *The inverse function* $\varPsi_\rho^{-1}: \varPsi_\rho(i\mathbb{C}^+) \to i\mathbb{C}^+$ *of* $\varPsi_\rho$ *admits a Taylor type expansion of order* 2,

$$\varPsi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z - \frac{m_2(\rho)}{(m_1(\rho))^3} z^2 + o(z^2), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0.$$

(3) $\mathfrak{D}_\rho := \varPsi_\rho(i\mathbb{C}^+)$ *is a region contained in the disc with diameter* $\big(\rho(\{0\}) - 1, 0\big)$. *In addition,* $\varPsi_\rho(i\mathbb{C}^+) \cap \mathbb{R} = \big(\rho(\{0\}) - 1, 0\big)$,

$$\lim_{t\uparrow 0} \varPsi_\rho^{-1}(t) = 0 \quad \textit{and} \quad \lim_{t\downarrow\rho(\{0\})-1} \varPsi_\rho^{-1}(t) = \infty.$$

Proof. (1) and (3) are proved in Bercovici and Voiculescu [6, Proposition 6.2].

(2) Step 1. We shall first prove that

$$\varPsi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z + o(z), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0.$$

Take any continuous path $\{z(t)\}_{t\in(0,1]}$ in $\mathfrak{D}_\rho$ such that $\lim_{t\downarrow 0} z(t) = 0$. By (1), we can choose a unique continuous path $\{\omega(t)\}_{t\in(0,1]}$ with $\lim_{t\downarrow 0}\omega(t) = 0$ and $\varPsi_\rho(\omega(t)) = z(t)$. Then

$$\lim_{t\downarrow 0} \frac{\varPsi_\rho^{-1}(z(t))}{z(t)} = \lim_{t\downarrow 0} \frac{\omega(t)}{\varPsi_\rho(\omega(t))} = \lim_{t\downarrow 0} \frac{1}{\varPsi_\rho(\omega(t))/\omega(t)} = \frac{1}{m_1}.$$

As the path $z(t)$ is arbitrary, it follows that

$$\varPsi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z + o(z), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0.$$

Step 2. Using Step 1, we shall give a Taylor type expansion of order 2 as $z \xrightarrow{z\in\mathfrak{D}_\rho} 0$. Indeed,

$$\begin{aligned}\frac{\varPsi_\rho^{-1}(\varPsi_\rho(z)) - \frac{1}{m_1(\rho)}\varPsi_\rho(z)}{\varPsi_\rho(z)^2} &= \frac{z - \frac{1}{m_1(\rho)}(m_1(\rho)z + m_2(\rho)z^2 + O(z^3))}{(m_1(\rho)z + m_2(\rho)z^2 + O(z^3))^2} \\ &= \frac{\left(\frac{m_2(\rho)}{m_1(\rho)} z^2 + O(z^3)\right)}{(m_1(\rho))^2 z^2 + O(z^3)} \to \frac{m_2(\rho)}{(m_1(\rho))^3} \quad \text{as } z \xrightarrow{z\in\mathfrak{D}_\rho} 0.\end{aligned}$$

As a result,

$$\varPsi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} z + \frac{m_2(\rho)}{(m_1(\rho))^3} z^2 + o(z^2), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0. \blacksquare$$

3.2. Limit theorems. Here we shall state the main theorem.

Theorem 3.3. *Assume that* $\rho \in \mathcal{P}_+$ *has the second moment and put* $\gamma = \mathrm{Var}(\rho)/(m_1(\rho))^2$.

(1) *There exist* $s_0 > 0$ *and* $s_1 < 0$ *such that the* S*-transform of* ρ *is*

$$S_\rho(z) = s_0 + s_1 z + o(z), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0,$$

and there exists a probability measure $\mathfrak{y}_\gamma \in \mathcal{P}_+$ such that

$$D_{s_0^n/n}((\rho^{\boxtimes n})^{\boxplus n}) \to \mathfrak{y}_\gamma \quad \text{in distribution.}$$

In addition, the S-transform of the limit distribution $\mathfrak{y}_\gamma$ is

$$S_{\mathfrak{y}_\gamma}(z) = \exp(-\gamma z).$$

(2) *There exist $\sigma_0 > 0$ and $\sigma_1 < 0$ such that the Σ-transform of ρ is*

$$\Sigma_\rho(z) = \sigma_0 + \sigma_1 z + o(z), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0,$$

and there exists a probability measure $\mathfrak{s}_\gamma \in \mathcal{P}_+$ such that

$$D_{s_0^n/n}((\rho^{\boxtimes n-1})^{\uplus n}) \to \mathfrak{s}_\gamma \quad \text{in distribution.}$$

In addition, the Σ-transform of the limit distribution $\mathfrak{s}_\gamma$ is

$$\Sigma_{\mathfrak{s}_\gamma}(z) = \exp(-\gamma z).$$

Proof. Using Lemma 3.2, we have

$$S_\rho(z) = \frac{z+1}{z}\Psi_\rho^{-1}(z) = \frac{1}{m_1(\rho)} - \frac{\mathrm{Var}(\rho)}{(m_1(\rho))^3} z + o(z), \qquad z \xrightarrow{z\in\mathfrak{D}_\rho} 0.$$

Let $s_0 = 1/m_1(\rho)$ and $s_1 = -\mathrm{Var}(\rho)/(m_1(\rho))^3$. By Proposition 2.1, we obtain

$$\begin{aligned}
S_{D_{s_0^n/n}((\rho^{\boxtimes n})^{\boxplus n})}(z) &= \frac{n}{s_0^n} S_{(\rho^{\boxtimes n})^{\boxplus n}}(z) = \frac{1}{s_0^n} S_{\rho^{\boxtimes n}}\left(\frac{z}{n}\right) \\
&= \frac{1}{s_0^n}\left(S_\rho\left(\frac{z}{n}\right)\right)^n = \frac{1}{s_0^n}\left(s_0 + s_1\frac{z}{n} + o\left(\frac{1}{n}\right)\right)^n \\
&= \left(1 + \frac{s_1 z}{s_0 n} + o\left(\frac{1}{n}\right)\right)^n \\
&\to \exp\left(\frac{s_1}{s_0} z\right) = \exp(-\gamma z) \qquad \text{as } n \to \infty.
\end{aligned}$$

From [5, Lemma 7.1] and [6, Theorem 6.13(ii)], there exists a free multiplicative infinitely divisible measure $\mathfrak{y}_\gamma$ such that $S_{\mathfrak{y}_\gamma}(z) = \exp(-\gamma z)$. The proof for (2) is the same as for the free additive case. ∎

We can exchange the order of free multiplicative and freely additive (or boolean additive) convolutions. The difference is in the scaling speed.

COROLLARY 3.4. *Under the same setting as in Theorem 3.3, we have*

$$D_{s_0^n/n^n}((\rho^{\boxplus n})^{\boxtimes n}) \to \mathfrak{y}_\gamma, \qquad D_{s_0^n/n^n}((\rho^{\uplus n-1})^{\boxtimes n}) \to \mathfrak{s}_\gamma,$$

as $n \to \infty$.

Proof. As in the proof of Theorem 3.3, this can be proved by using Proposition 2.1 and Lemma 3.2. ∎

4. Lambert W-function and infinite divisibility of the limit distribution

4.1. On the limit distribution of the free case. When we calculate the R-transform or the moment generating function, Lambert's W-function, which satisfies the functional equation

$$z = W(z)\exp(W(z)),$$

plays an important role. This function has been studied for a long period and several nice properties of it as a real and complex function are known. For more details on the Lambert W-function, see, for instance, [8]. Let $W_0(z)$ be the principal branch of the Lambert W-function.

By Proposition 2.5 and the S-transform of $\mathfrak{y}_\gamma$, we have

$$R_{\mathfrak{y}_\gamma}(ze^{-\gamma z}) = z, \qquad 1/z \in \Gamma_{\alpha,\beta}.$$

This functional equation suggests that the R-transform is given by using Lambert's W-function.

Theorem 4.1.

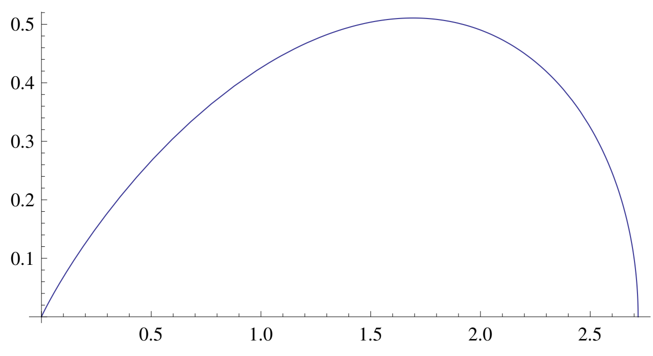
(1) *The $\mathcal{R}$- and R-transforms of the probability measure $\mathfrak{y}_\gamma$ are*

$$\mathcal{R}_{\mathfrak{y}_\gamma}(z) = \frac{-W_0(-\gamma z)}{\gamma z}, \qquad R_{\mathfrak{y}_\gamma}(z) = -\frac{1}{\gamma}W_0(-\gamma z).$$

(2) *$\mathfrak{y}_\gamma$ is both $\boxplus$-infinitely divisible and $\boxtimes$-infinitely divisible.*
(3) *The free cumulant sequence of $\mathfrak{y}_\gamma$ is $\{(\gamma n)^{n-1}/n!\}_{n\in\mathbb{N}}$.*
(4) *The Lévy measure $\nu_{\mathfrak{y}_\gamma}$ of $\mathfrak{y}_\gamma$ is*

$$\nu_{\mathfrak{y}_\gamma}(ds) = \frac{1}{\gamma\pi} s f^{-1}(\gamma/s) 1_{[0,\gamma e]}(s)\, ds,$$

where $f(u) = u\csc u \exp(-u\cot u)$. For $\gamma = 1$, the shape of the density of $\nu_{\mathfrak{y}_1}$ is shown in the graph below.



(5) *We have*

$$\mathfrak{y}_\gamma^{\boxplus t} = D_t(\mathfrak{y}_\gamma^{\boxtimes 1/t}), \qquad \mathfrak{y}_\gamma^{\boxtimes t} = D_t(\mathfrak{y}_\gamma^{\boxplus 1/t}).$$

In the proof of this theorem we apply the following prposition (for instance, see [8, Section 4] and [13, Theorem 3.1]):

PROPOSITION 4.2.

(1) *The principal branch of* $W_0(z)$ *has an analytic extension on the set* $\mathbb{C} \setminus (-\infty, -1/e]$ *and it maps* $\mathbb{C}^-$ *into* $\mathbb{C}^- \cup \mathbb{R}$.
(2) *For any* $z \in \mathbb{C}^+$, *we have the integral representation*

$$\frac{W_0(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{(1 - u\cot u)^2 + u^2}{z + u\csc u \exp(-u\cot u)}\, du.$$

Proof of Theorem 4.1. (1) The $\boxtimes$-infinite divisibility is trivial from the form of the S-transform and the facts in [5]. By Proposition 2.5,

$$R_{\mathfrak{y}_\gamma}(ze^{-\gamma z}) = z.$$

Then the R-transform is given by using Lambert's W-function as follows:

$$R_{\mathfrak{y}_\gamma}(z) = -\frac{1}{\gamma} W_0(-\gamma z),$$

and hence

$$\mathcal{R}_{\mathfrak{y}_\gamma}(z) = \frac{W_0(-\gamma z)}{-\gamma z}.$$

(2) By Propositions 2.3 and 4.2(1), $\mathcal{R}_{\mathfrak{y}_\gamma}$ has an analytic extension defined on $\mathbb{C}^-$ with values in $\mathbb{C}^- \cup \mathbb{R}$, which means that $\mathfrak{y}_\gamma$ is $\boxplus$-infinitely divisible.

(3) The Taylor type expansion of $-W_0(-z)$ at the origin is obtained from equation (3.1) of [8, p. 339].

(4) We put $g(u) = (1 - u\cot(u))^2 + u^2$. Noting that

$$g(u) = \frac{uf'(u)}{f(u)}, \tag{4.1}$$

we obtain

$$\begin{aligned}\mathcal{R}_{\mathfrak{y}_\gamma}(z) &= \frac{1}{\pi} \int_0^\pi \frac{g(u)}{-\gamma z + f(u)}\, du \\ &= \frac{1}{\pi} \int_0^\pi \frac{g(u)/f(u)}{1 - \gamma z/f(u)}\, du = \frac{1}{\gamma\pi} \int_0^{\gamma e} \frac{f^{-1}(\gamma/s)}{1 - sz}\, ds,\end{aligned}$$

where we have changed the variables as $s = \gamma/f(u)$. Hence

$$\begin{aligned}\mathcal{R}_{\mathfrak{y}_\gamma}(z) &= \frac{1}{\gamma\pi} \int_0^{\gamma e} \left(\frac{sz}{1 - sz} + 1\right) f^{-1}(\gamma/s)\, ds \\ &= \frac{1}{\gamma} + \frac{1}{\gamma\pi} \int_0^{\gamma e} \frac{z}{1 - sz} s f^{-1}(\gamma/s)\, ds.\end{aligned}$$

Therefore we obtain the Lévy measure

$$\nu_{\mathfrak{y}_\gamma}(ds) = \frac{sf^{-1}(\gamma/s)}{\gamma\pi}\, ds.$$

(5) This is a direct consequence of Proposition 2.1. ■

Remark 4.3. Here we consider the limit distribution with parameter $\gamma = 1$; for example, this is the case if ρ is the free Poisson distribution with parameter 1. We write $\mathfrak{y}$ instead of $\mathfrak{y}_1$. There exists a probability measure ρ such that

$$\mathcal{R}_\rho(z) = \frac{\mathcal{R}_{\mathfrak{y}}(z) - 1}{z}. \tag{4.2}$$

Indeed, if we consider the shifted free cumulant sequence $\{k_n(\rho)\}_{n\in\mathbb{N}} = \{(n+1)^n/(n+1)!\}_{n\in\mathbb{N}}$, which is the sequence of coefficients of the Taylor expansion of R_ρ at 0, then it becomes the moment sequence of a probability measure. This means that the measure ρ is a free compound Poisson distribution with compound measure σ, whose moments are $m_n(\sigma) = (n+1)^n/(n+1)!$. From (4.2), we have

$$z\mathcal{R}_{\mathfrak{y}}(zM_\rho(z)) = zM_\rho(z). \tag{4.3}$$

By putting $P(z) = zM_\rho(z)$ and using the Lagrange inversion formula, (4.3) implies that

$$n\text{th coefficient of } \{P(z)\} = \frac{1}{n} \times ((n-1)\text{st coefficient of } \mathcal{R}_\rho(z)).$$

Hence we obtain the moments of ρ as

$$m_n(\rho) = (2n+1)^{n-1}/n!.$$

4.2. On the limit distribution in the boolean case. Let $\mathfrak{s} := \mathfrak{s}_1$ denote a probability measure with moment sequence $\{n^n/n!\}_{n\geq 0}$, the positivity of which is ensured by [14]. Then its moment generating function $M_{\mathfrak{s}}(z)$ can be given by

$$M_{\mathfrak{s}}(z) = \sum_{n=0}^{\infty} \frac{n^n}{n!} z^n = \frac{1}{1-\eta(z)} \tag{1}$$

where

$$\eta(z) = -W_0(-z), \qquad z \in \mathbb{C} \setminus [1/e, \infty).$$

Remark 4.4. The following useful facts on the function η can be found in [9, Sect. 2]: The map

$$\theta \mapsto \frac{\sin\theta}{\theta} \exp(\theta \cot\theta)$$

is a bijection of $(0,\pi)$ onto $(0,e)$, and if we define $\eta^+,\eta^- : [1/e,\infty)\to\mathbb{C}$ by

$$\eta^{\pm}\left(\frac{\theta}{\sin\theta}\exp(-\theta\cot\theta)\right)=\theta\cot\theta\pm i\theta,\qquad 0\le\theta<\pi,$$

then

$$\eta^{\pm}(x)=\lim_{y\downarrow 0}\eta(x+iy),\qquad x\in[1/e,\infty).$$

From (1), the Cauchy transform of the measure $\mathfrak{s}$ is

$$G_{\mathfrak{s}}(\zeta)=\frac{1}{\zeta}\,\frac{1}{1-\eta(1/\zeta)}\qquad\text{for } \zeta\in\mathbb{C}\setminus[0,e].$$

Now we apply the Stieltjes inversion formula to obtain the density function $\varphi_{\mathfrak{s}}(t)$ of the measure $\mathfrak{s}$: For $t\in[0,e]$,

$$\begin{aligned}\varphi_{\mathfrak{s}}(t)&=-\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}(G_{\mathfrak{s}}(t+i\varepsilon))\\&=-\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}\left(\frac{1}{t+i\varepsilon}\,\frac{1}{1-\eta\left(\frac{1}{t+i\varepsilon}\right)}\right)\\&=-\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}\left(\frac{t-i\varepsilon}{t^2+\varepsilon^2}\,\frac{1}{1-\eta\left(\frac{t-i\varepsilon}{t^2+\varepsilon^2}\right)}\right)\\&=-\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{t}\,\frac{1}{1-\eta^-(1/t)}\right),\end{aligned}$$

where the function η^- is defined as in remark above. Here we change the variables

$$\frac{1}{t}=\frac{\theta}{\sin\theta}\exp(-\theta\cot\theta);$$

then it follows that

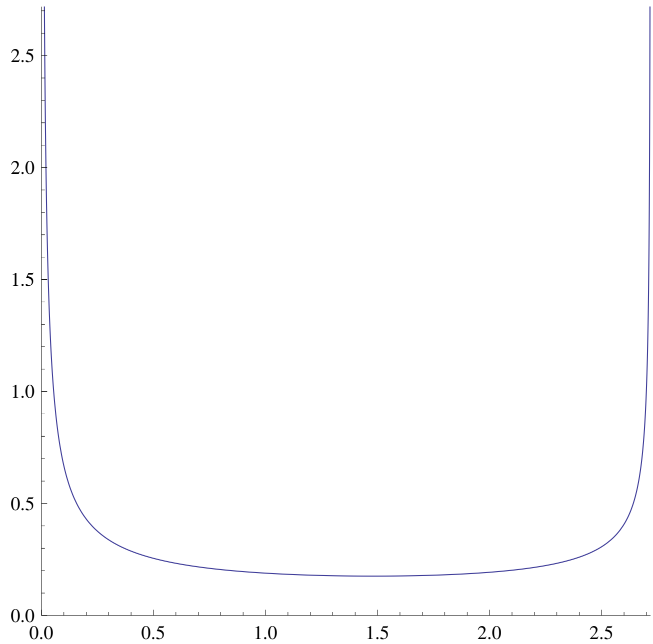
$$\begin{aligned}\varphi_{\mathfrak{s}}(t)&=-\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{t}\,\frac{1}{1-(\theta\cot\theta-i\theta)}\right)\\&=\frac{1}{\pi}\,\frac{1}{t}\,\frac{\theta}{(1-\theta\cot\theta)^2+\theta^2}\\&=\frac{1}{\pi}\left(\frac{\theta}{\sin\theta}\exp(-\theta\cot\theta)\right)\left(\frac{\theta}{(1-\theta\cot\theta)^2+\theta^2}\right)\\&=\frac{1}{\pi}\,\frac{\theta^2\exp(-\theta\cot\theta)}{\sin\theta((1-\theta\cot\theta)^2+\theta^2)}.\end{aligned}$$

Thus we obtain the following proposition:

PROPOSITION 4.5. *The probability density function $\varphi_{\mathfrak{s}}$ of the measure $\mathfrak{s}$ can be given in implicit (parametric) form as*

$$\varphi_{\mathfrak{s}}\left(\frac{\sin v}{v}\exp(v\cot v)\right)=\frac{1}{\pi}\,\frac{v^2\exp(-v\cot v)}{\sin v((1-v\cot v)^2+v^2)},\qquad 0<v<\pi.$$

Remark 4.6. (1) The shape of the density function of $\varphi_{\mathfrak{s}}$ is as in the graph below, in particular it is non-unimodal.



(2) The function $(1 - v\cot v)^2 + v^2$ also appears in the integral representation of $W_0(z)/z$ as mentioned in Proposition 4.2:

$$\frac{W_0(z)}{z} = \frac{1}{\pi}\int_0^\pi \frac{(1 - v\cot v)^2 + v^2}{z + \frac{v}{\sin v}\exp(-v\cot v)}\,dv.$$

Thus using $f(v) = v\csc v\exp(-v\cot v)$ and (4.1) again, the parametric form of the density function can be rewritten as

$$\varphi_{\mathfrak{s}}\left(\frac{1}{f(v)}\right) = \frac{1}{\pi}\frac{(f(v))^2}{f'(v)}.$$

Concluding remark. Dykema and Haagerup [9] found the limit distribution of the DT-operator $\mathrm{DT}(1, \delta_0)$. Its moment sequence is $m_n = n^n/(n+1)!$. It is similar to the moment sequences of the distribution $\mathfrak{s}$ and of ρ in Remark 4.3. A natural question arises: how do we obtain these distributions via random matrix models?

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