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Multiplicative maps that are close to an automorphism on algebras of linear transformations

by

L. W. MARCOUX (Waterloo, ON), H. RADJAVI (Waterloo, ON) and
A. R. SOUROUR (Victoria, BC)

Abstract. Let $\mathcal{H}$ be a complex, separable Hilbert space of finite or infinite dimension, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. It is shown that if $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a multiplicative map (not assumed linear) and if φ is sufficiently close to a linear automorphism of $\mathcal{B}(\mathcal{H})$ in some uniform sense, then it is actually an automorphism; as such, there is an invertible operator S in $\mathcal{B}(\mathcal{H})$ such that $\varphi(A) = S^{-1}AS$ for all A in $\mathcal{B}(\mathcal{H})$. When $\mathcal{H}$ is finite-dimensional, similar results are obtained with the mere assumption that there exists a linear functional f on $\mathcal{B}(\mathcal{H})$ so that $f \circ \varphi$ is close to $f \circ \mu$ for some automorphism μ of $\mathcal{B}(\mathcal{H})$.

1. Introduction. Let $\mathbb{F}$ be a field and $n \geq 1$ be an integer. A map $\varphi : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ is said to be *multiplicative* if $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathbb{M}_n(\mathbb{F})$. By an *automorphism* of $\mathbb{M}_n(\mathbb{F})$, we shall mean a linear and multiplicative bijection of $\mathbb{M}_n(\mathbb{F})$ onto itself, that is, an algebraic automorphism. An example of a multiplicative map is the map $\varphi(A) := S^{-1}AS$, $A \in \mathbb{M}_n(\mathbb{F})$, where S is a fixed invertible matrix. This map is also linear. Since the definition of multiplicativity of a map does not require the map to be linear, another simple example can be derived from any given ring homomorphism $\theta : \mathbb{F} \to \mathbb{F}$ by defining $\theta^{(n)} : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ via the formula $\theta^{(n)}([a_{i,j}]) = [\theta(a_{i,j})]$. It is most interesting that, in the presence of some mild non-degeneracy conditions, a combination of the two examples above (together with the possible addition of the cofactor map) represents the general situation—see Theorem 3.1 below.

The definition of a multiplicative map can be extended to the setting where $\mathcal{S}$ is any multiplicative semigroup and $\varphi : \mathcal{S} \to \mathcal{S}$ satisfies $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{S}$. Such maps have been studied for more than half a century. In 1969, Jodeit and Lam [8] characterized multiplicative maps $\varphi : \mathbb{M}_n(\mathcal{P}) \to \mathbb{M}_n(\mathcal{P})$ when $\mathcal{P}$ is assumed to be a principal ideal domain.

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Their methods relied heavily upon the fact that $\mathcal{P}$ is commutative. In 2008, Šemrl [11] characterized multiplicative maps $\varphi : \mathbb{M}_n(\mathcal{D}) \to \mathbb{M}_n(\mathcal{D})$ in the case where $\mathcal{D}$ is a division ring. (See also [10].)

Pursuing a different line of inquiry, Cheung, Fallat and Li [3] have studied multiplicative maps $\psi : \mathcal{S} \to \mathcal{S}$ on semigroups $\mathcal{S}$ of $n \times n$ matrices which preserve certain sets of matrices such as rank k (idempotent) matrices, hermitian or normal matrices, etc., as well as multiplicative maps that preserve properties such as spectrum, spectral radius and others.

A special case where the semigroup $\mathcal{S} = \mathcal{A}_n(\mathbb{C}) \subseteq \mathbb{M}_n(\mathbb{C})$ is either the set of invertible matrices, the set of unitary matrices or a multiplicative semigroup containing all non-invertible matrices had earlier been studied by Hochwald [7], who showed that if $n \geq 2$ and $\psi : \mathcal{A}_n \to \mathbb{M}_n(\mathbb{C})$ is a spectrum-preserving multiplicative map, then there exists $S \in \mathbb{M}_n(\mathbb{C})$ invertible so that $\psi(T) = S^{-1}TS$ for all $T \in \mathcal{A}_n$. The reader is also directed to the results of Molnár [9], who described those continuous multiplicative maps on the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a separable, complex Hilbert space which preserve rank or corank. Other results on multiplicative maps on sets of matrices may be found in the paper of Guralnick, Li and Rodman [6].

In the present paper we shall first focus our attention on the case where $\mathcal{S} = \mathbb{M}_n(\mathbb{F})$ is the algebra of $n \times n$ matrices over a field $\mathbb{F}$ (and we shall concentrate mostly on the field $\mathbb{C}$ of complex numbers). Our purpose here is two-fold:

(a) Under the assumption that there exists a non-zero linear functional $f \in \mathbb{M}_n(\mathbb{F})^*$ such that
$$f(\varphi(A)) = f(A) \quad \text{for all } A \text{ in } \mathbb{M}_n(\mathbb{F}),$$
we show the existence of an invertible matrix S such that $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n(\mathbb{F})$ as above.

(b) Under the assumption that $\mathbb{F} = \mathbb{C}$ and φ is only *sufficiently close* to the identity map, i.e. there exists a functional $f \in \mathbb{M}_n(\mathbb{C})^*$ such that $|f(\varphi(A)) - f(A)|$ is sufficiently small for all $A \in \mathbb{M}_n(\mathbb{C})$, we arrive at the same conclusion. The notions of sufficiently close and sufficiently small will be made precise below.

Next, in the setting where $\mathcal{H}$ is an infinite-dimensional, complex, separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators acting on $\mathcal{H}$, we show that if $0 < \varepsilon < 1/4$, $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is multiplicative and
$$\|\varphi(A) - A\| < \varepsilon\|A\| \quad \text{for all } 0 \neq A \in \mathcal{B}(\mathcal{H}),$$
then φ is an inner automorphism of $\mathcal{B}(\mathcal{H})$ induced by an invertible operator whose distance to the identity is controlled by ε.

In both the finite-dimensional and the infinite-dimensional settings, we extend these results to the case where φ is *sufficiently close* to an automorphism of $\mathbb{M}_n(\mathbb{C})$ (resp. $\mathcal{B}(\mathcal{H})$) to conclude that φ must itself be an automorphism.

2. Multiplicative maps on $\mathbb{M}_n(\mathbb{F})$ that behave like an automorphism at a functional. Let $n \geq 1$ and $\mathbb{F}$ be a field. If $S \in \mathbb{M}_n(\mathbb{F})$ is invertible, we denote by Ad_S the map $\mathrm{Ad}_S(T) = S^{-1}TS$ for all $T \in \mathbb{M}_n(\mathbb{F})$. It is clearly an automorphism of $\mathbb{M}_n(\mathbb{F})$, and as such, it is multiplicative.

As previously mentioned, the characterization of multiplicative maps on $\mathbb{M}_n(\mathcal{P})$ where $\mathcal{P}$ is a PID was first determined by Jodeit and Lam [8]. A version of this that involves evaluation at a single functional has been obtained by W.-S. Cheung [2] in the case where $\mathcal{P} = \mathbb{C}$ is the field of complex numbers. The version presented here is a simple consequence of Theorem 4.2 of that paper:

THEOREM 2.1 (Cheung). *Suppose that $n \geq 1$ and that $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is a multiplicative map. If $0 \neq f \in \mathbb{M}_n(\mathbb{C})^*$ is a linear functional that satisfies*

$$f(\varphi(A)) = f(A) \quad \textit{for all } A \in \mathbb{M}_n(\mathbb{C}),$$

then there exists $S \in \mathbb{M}_n(\mathbb{C})$ invertible such that $\varphi = \mathrm{Ad}_S$; in particular, φ is an automorphism.

Cheung's result above produces the following simple consequence: if a multiplicative map $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ behaves like an automorphism at a functional, then φ *is* an automorphism.

PROPOSITION 2.2. *Suppose that $n \geq 1$ and $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is a multiplicative map. Let $\gamma : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be an automorphism. If there exists $0 \neq f \in \mathbb{M}_n(\mathbb{C})^*$ such that*

$$f(\varphi(A)) = f(\gamma(A)) \quad \textit{for all } A \in \mathbb{M}_n(\mathbb{F}),$$

then there exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ so that $\varphi = \mathrm{Ad}_S$.

Proof. The map $\varphi_1 := \gamma^{-1} \circ \varphi$ is multiplicative and $f_1 = f \circ \gamma$ is linear and non-zero. Also,

$$f_1(\varphi_1(A)) = f_1(A) \quad \text{for all } A \in \mathbb{M}_n(\mathbb{C}).$$

By Theorem 2.1, $\varphi_1 = \mathrm{Ad}_R$ for some invertible matrix R. But every automorphism γ of $\mathbb{M}_n(\mathbb{C})$ is of the form $\gamma = \mathrm{Ad}_T$ for some invertible matrix T, so $\varphi = \gamma \circ \varphi_1 = \mathrm{Ad}_T \circ \mathrm{Ad}_R = \mathrm{Ad}_S$, where $S = RT$. ■

As an apéritif for the approximate version of Cheung's theorem to come, we offer an alternative proof of Theorem 2.1 which holds for arbitrary fields. (We point out that Cheung mentions that his proof should also work for multiplicative maps on $\mathbb{M}_n(\mathbb{R})$.) As should be expected, the proof below shares some elements with Cheung's original proof.

By $\mathrm{SL}_n(\mathbb{F})$ we denote the special linear group $\{T \in \mathbb{M}_n(\mathbb{F}) : \det T = 1\}$, and by $\mathrm{GL}_n(\mathbb{F})$ we denote the group $\{T \in \mathbb{M}_n(\mathbb{F}) : \det T \neq 0\}$ of invertible matrices.

Theorem 2.3. *Let $\mathbb{F}$ be a field and let $\varphi : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ be a multiplicative map such that φ does not annihilate all of the rank one matrices* (*or equivalently, φ does not annihilate* ***any*** *rank one matrix*).

(a) *If $\varphi(0) \neq 0$, then $\varphi(\mathrm{SL}_n(\mathbb{F})) = \{I_n\}$.*
(b) *If $\varphi(0) = 0$, then there exists an invertible matrix S and a non-zero endomorphism $\theta : \mathbb{F} \to \mathbb{F}$ such that*

$$\varphi([a_{i,j}]) = S^{-1}[\theta(a_{i,j})]S \quad \text{for every } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{F}).$$

Proof. Suppose that $F_0 \in \mathbb{M}_n(\mathbb{F})$ is a fixed rank one matrix which is annihilated by φ. From the multiplicativity of φ and the fact that any rank one matrix in $\mathbb{M}_n(\mathbb{F})$ can be factored as a product of three matrices, one of which is F_0, it is not hard to see that $\varphi(F_0) = 0$ implies that $\varphi(F) = 0$ for all rank one matrices $F \in \mathbb{M}_n(\mathbb{F})$.

(a) If $\varphi(0) \neq 0$, then $\varphi(0)$ is an idempotent that commutes with the image of φ, and without loss of generality, we may assume that $\varphi(0) = I_k \oplus 0$ for some $1 \leq k \leq n$. Then $\varphi(A) = I_k \oplus \psi(A)$ for some multiplicative map $\psi : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_{n-k}(\mathbb{F})$.

Now ψ maps the set $\{D_1, \ldots, D_{2^n}\}$ of all diagonal involutions in $\mathbb{M}_n(\mathbb{F})$ into a set of commuting involutions which are then simultaneously diagonalizable. Denote by $R \in \mathbb{M}_{n-k}(\mathbb{F})$ the invertible element which simultaneously diagonalizes the set $\{\psi(D_1), \ldots, \psi(D_{2^n})\}$. Since $\mathbb{M}_{n-k}(\mathbb{F})$ has at most 2^{n-k} distinct diagonal involution elements, it follows that $\mathrm{Ad}_R\, \psi(D_1), \ldots, \mathrm{Ad}_R\, \psi(D_{2^n})$ cannot all be distinct, and hence $\psi(D_1), \ldots, \psi(D_{2^n})$ are not all distinct. Thus the (group-theoretic) kernel $\mathcal{K}$ of $\varphi|\mathrm{GL}_n(\mathbb{F})$ contains a diagonal involution other than I_n. Now $\mathcal{K}$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$ and contains a non-scalar operator, so by a theorem of [4], it includes all of $\mathrm{SL}_n(\mathbb{F})$.

(b) Suppose now that $\varphi(0)=0$. Consider the matrix units E_{ij}, $1 \leq i, j \leq n$, so that E_{ij} has a 1 in the (i, j)-position and 0's elsewhere. As we have seen, the assumption that φ does not annihilate all rank one operators implies that it does not annihilate any, and so $\varphi(E_{ij}) \neq 0$ for any $1 \leq i, j \leq n$. The images of the (mutually orthogonal) rank one idempotents $E_{11}, E_{22}, \ldots, E_{nn}$ under φ are again mutually orthogonal non-zero idempotents, which forces them each to have rank one. As such, after composing with an inner automorphism if necessary and incorporating this inner automorphism into the definition of φ, we may assume that $\varphi(E_{ii}) = E_{ii}$ for all $1 \leq i \leq n$.

We claim that after composing φ with a second inner automorphism, we may assume that $\varphi(E_{ij}) = E_{ij}$, $1 \leq i, j \leq n$. Indeed, the equation $E_{ii}E_{ij} = E_{ij} = E_{ij}E_{jj}$ implies that $\varphi(E_{ij}) = c_{ij}E_{ij}$ for non-zero scalars c_{ij}. After composing φ with the map of conjugating by $\mathrm{diag}(1, c_{12} \ldots, c_{1n})$ and its inverse, we get a new map (which we relabel as φ again) satisfying $\varphi(E_{1i}) = E_{1i}$, $1 \leq i \leq n$.

From $E_{1i}E_{i1} = E_{11}$ we then deduce that $\varphi(E_{i1}) = E_{i1}$. Finally $E_{ij} = E_{i1}E_{1j}$ implies that $\varphi(E_{ij}) = E_{ij}$ for all $1 \leq i, j \leq n$.

For every scalar α, the matrix $\varphi(\alpha I_n)$ commutes with the image of φ, which we now know to include every E_{ij}, and so $\varphi(\alpha I_n)$ is a scalar matrix $\theta(\alpha)I_n$. The map $\alpha \mapsto \theta(\alpha)$ is clearly multiplicative. As 0 and 1 are the only idempotent members of $\mathbb{F}$, we have $\theta(1) = 0$ or 1, but the former alternative implies that $\varphi(I_n) = 0$ whence φ is identically zero, which contradicts our assumptions. Thus $\theta(1) = 1$. Also $\theta(0) = 0$ or 1, but the latter alternative implies that $\varphi(0) = I_n$, contradicting our supposition that $\varphi(0) = 0$. Thus $\theta(0) = 0$.

With the above reductions, we claim that for $A = [a_{i,j}]$ we have $\varphi(A) = \theta^{(n)}(A) := [\theta(a_{i,j})]$. To prove this, assume that $A = [a_{i,j}]$ and $[b_{i,j}] = B = \varphi(A)$. Then for all i and j we have

$$b_{ij}E_{ij} = E_{ii}BE_{jj} = \varphi(E_{ii}AE_{jj}) = \varphi(a_{ij}E_{ij}) = \varphi(a_{ij}I_n)\varphi(E_{ij}) = \theta(a_{ij})E_{ij}.$$

It remains to prove that θ is additive. By applying the multiplicative map $\varphi = \theta^{(n)}$ to the equation

$$(E_{11} + E_{22} + xE_{12})(E_{11} + E_{22} + yE_{12}) = E_{11} + E_{22} + (x + y)E_{12},$$

and by then comparing the coefficients of E_{12} from either side of the resulting equation we find that $\theta(x + y) = \theta(x) + \theta(y)$, and we are done. ∎

We point out that in our original version of the paper, we had only claimed that θ is multiplicative with $\theta(0) = 0$ and $\theta(1) = 1$. We thank the referee for pointing out that θ is also additive.

THEOREM 2.4. *Let $\varphi : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ be a multiplicative map for which there exists a non-zero linear functional $f \in \mathbb{M}_n(\mathbb{F})^*$ satisfying*

$$f(\varphi(A)) = f(A) \quad \text{for all } A \in \mathbb{M}_n(\mathbb{F}).$$

Then there exists an invertible matrix S such that $\varphi(A) = S^{-1}AS$.

Furthermore, if $C \in \mathbb{M}_n(\mathbb{F})$ is chosen so that f is given by $f(A) = \mathrm{tr}(AC)$, then S commutes with C.

Proof. First observe φ cannot annihilate all rank one matrices. If it did, then so would f. But f is linear, and so we would conclude that $f = 0$, a contradiction.

It follows that $\varphi(0) = 0$. Indeed, otherwise, by Theorem 2.3 we have $\varphi(\mathrm{SL}_n(\mathbb{F})) = \{I_n\}$. Thus, if $A \in \mathrm{SL}_n(\mathbb{F})$, then $\mathrm{tr}(AC) = f(A) = f(\varphi(A)) =$

$\mathrm{tr}(C)$. In particular, if N is nilpotent, then $I + N \in \mathrm{SL}_n(\mathbb{F})$ and so $\mathrm{tr}(NC) = 0$. Since the nilpotents span $\mathrm{sl}_n(\mathbb{F}) := \{R \in \mathbb{M}_n(\mathbb{F}) : \mathrm{tr}(F) = 0\}$, we get $C = cI_n$ for a scalar c.

If $n \geq 3$, then for every $m \leq n-2$ (including $m = 0$), there is a matrix A in $\mathrm{SL}_n(\mathbb{F})$ of the form $A = I_m \oplus U$, where U has zero diagonal. (For example, letting $\{E_{ij} : 1 \leq i, j \leq n\}$ denote the matrix units of $\mathbb{M}_n(\mathbb{F})$ as before, we may take $U = E_{m+1,m+2} + E_{m+2,m+3} + \cdots + E_{n-1,n} + (-1)^{n-m+1} E_{n,m+1}$.) Thus $mc = \mathrm{tr}(AC) = \mathrm{tr}(C) = nc$. This easily implies that $c = 0$, a contradiction.

If $n = 2$, then the assumption that $\varphi(0) \neq 0$ implies that (without loss of generality) we may write $\varphi(0) = I_2$ or $\varphi(0) = I_1 \oplus 0$ and so we have either $\varphi(A) = I_2$ for every A or $\varphi(A) = 1 \oplus \psi(A)$ for all A, where $\psi : \mathbb{M}_2(\mathbb{F}) \to \mathbb{F}$ is multiplicative. In the former case, $\mathrm{tr}(AC) = \mathrm{tr}(C)$ for every A, implying again that $C = 0$. In the latter case, we have

$$c = \mathrm{tr}(E_{11}C) = f(\varphi(0)) = f(0) = 0.$$

We are now in a position to apply Theorem 2.3 to obtain an invertible matrix S and a homomorphism θ satisfying the conditions of that theorem. For every i and j, we have

$$\mathrm{tr}(E_{ij}SCS^{-1}) = \mathrm{tr}(S^{-1}(E_{ij}SC)) = \mathrm{tr}(\varphi(E_{ij})C) = \mathrm{tr}(E_{ij}C).$$

Therefore $SCS^{-1} = C$, or equivalently, $SC = CS$.

Finally, for $a \in \mathbb{F}$,

$$\begin{aligned} f(aE_{ij}) = \mathrm{tr}(aE_{ij}C) &= \mathrm{tr}(\varphi(aE_{ij}C)) = f(\varphi(aE_{ij})) \\ &= f(S^{-1}[\theta^{(n)}(aE_{ij})]S) = f(\theta(a)S^{-1}E_{ij}S) \\ &= \theta(a)\,\mathrm{tr}(S^{-1}E_{ij}SC) = \theta(a)\,\mathrm{tr}(E_{ij}C) = \theta(a)f(E_{ij}). \end{aligned}$$

Since $0 \neq f$, this implies that $\theta(a) = a$ for every $a \in \mathbb{F}$. ∎

3. Multiplicative maps on $\mathbb{M}_n(\mathbb{C})$ that can be approximated by automorphisms. Our present goal is to obtain an approximate version of Theorem 2.4 in the case where $\mathbb{F} = \mathbb{C}$. The proofs will require us to understand the Jodeit–Lam [8] classification of multiplicative maps on $\mathbb{M}_n(\mathbb{C})$.

Let $\mathbb{F}$ be a field and $n \geq 1$. We define the *cofactor map*

$$\Gamma : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F}), \qquad [a_{i,j}] \mapsto [b_{i,j}],$$

where $b_{i,j}$ is the (i,j)-cofactor of the matrix $A = [a_{i,j}]$.

A multiplicative map $\beta : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ is said to be *degenerate* if $\beta(A) = 0$ for all singular matrices A.

THEOREM 3.1 (Jodeit–Lam). *Let $\mathbb{F}$ be a field and let $\varphi : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ be a multiplicative map. Then one of the following holds:*

(a) *There exists a ring homomorphism* $\theta : \mathbb{F} \to \mathbb{F}$ *and an invertible matrix* $S \in \mathbb{M}_n(\mathbb{F})$ *so that*

$$\varphi([a_{i,j}]) = S^{-1}[\theta(a_{i,j})]S \quad \textit{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{F}).$$

(b) *There exists a ring homomorphism* $\theta : \mathbb{F} \to \mathbb{F}$ *and an invertible matrix* $S \in \mathbb{M}_n(\mathbb{F})$ *so that*

$$\varphi([a_{i,j}]) = S^{-1}\Gamma([\theta(a_{i,j})])S \quad \textit{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{F}),$$

where Γ *is the cofactor map from above.*

(c) *There exists a degenerate map* $\beta : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ *and an idempotent* $E = E^2 \in \mathbb{M}_n(\mathbb{F})$ *so that*

$$\varphi([a_{i,j}]) = \beta([a_{i,j}]) + E \quad \textit{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{F}).$$

LEMMA 3.2. *Let* $n \geq 2$ *and* $f \in \mathbb{M}_n(\mathbb{C})^*$ *satisfy* $f(I_n) = 1$. *Suppose that* $0 \neq \varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ *is multiplicative. If*

$$|f(\varphi(A)) - f(A)| < \|A\| \quad \textit{for all } 0 \neq A \in \mathbb{M}_n(\mathbb{C}),$$

then $\varphi(0) = 0$ *and* $\varphi(I_n) = I_n$.

Proof. Let $P = \varphi(0)$, so that $P^2 = \varphi(0^2) = \varphi(0) = P$. Furthermore, $P\varphi(A) = P = \varphi(A)P$ for all $A \in \mathbb{M}_n(\mathbb{C})$. Similarly, if $Q = \varphi(I_n)$, then $Q^2 = Q$ and $Q\varphi(A) = \varphi(A) = \varphi(A)Q$ for all $A \in \mathbb{M}_n(\mathbb{C})$.

In particular, $QP = P = QP$, so that $Q \geq P$. (Note that Q and P are not orthogonal projections: $Q \geq P$ means precisely that $PQ = P = QP$, so that $\operatorname{ran} P \subseteq \operatorname{ran} Q$ and $\ker P \supseteq \ker Q$.)

Consider the decomposition $\mathbb{C}^n = \operatorname{ran} P \dotplus \operatorname{ran}(Q - P) \dotplus \operatorname{ran}(I_n - Q)$. Relative to this decomposition, we may write

$$\varphi(A) = \begin{bmatrix} I & 0 & 0 \\ 0 & \chi(A) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for each $A \in \mathbb{M}_n(\mathbb{C})$. Here $\chi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a multiplicative map, where $k = \operatorname{rank}(Q - P)$.

Now $\chi(0) = 0$ and $\chi(I_n) = I_k$. Consider $\mathcal{K} := \ker \chi|_{\mathrm{GL}_n(\mathbb{C})}$, so that $K \lhd \mathrm{GL}_n(\mathbb{C})$. It is easy to verify that all diagonal involutions in $\mathbb{M}_n(\mathbb{C})$ are mapped to commuting involutions in $\mathbb{M}_k(\mathbb{C})$ by χ. Suppose that $P = \varphi(0) \neq 0$ or that $Q = \varphi(I_n) \neq I_n$. Then $k < n$. We shall use this to arrive at a contradiction.

Since $\mathbb{M}_n(\mathbb{C})$ has exactly 2^n diagonal involutions, and since $\mathbb{M}_k(\mathbb{C})$ has at most 2^k commuting involutions (as these may be simultaneously diagonalized by an appropriate choice of basis for $\mathbb{C}^k$), it follows that $\mathcal{K}$ contains some diagonal involutions other than I_n. By the result of [4] cited earlier, $\mathcal{K} \supseteq \mathrm{SL}_n(\mathbb{C})$. In particular, if $\alpha \in \mathbb{C}$ and $\alpha^n = 1$, then $\alpha I_n \in \mathrm{SL}_n(\mathbb{C}) \subseteq \mathcal{K}$.

Let $\Omega := \{\beta \in \mathbb{C} : \beta I_n \in \mathcal{K}\}$. We shall demonstrate that $\Omega = \{1\}$, thereby arriving at the desired contradiction. Indeed, if $\beta \in \Omega$, then for all $j \geq 1$,

$$|f(\varphi(\beta^j I_n)) - f(\beta^j I_n)| = |f(\varphi(\beta I_n)^j) - f((\beta I_n)^j)| = |f(Q^j) - \beta^j| < |\beta|^j.$$

It follows that

$$|\beta^{-j} f(Q) - 1| < 1 \quad \text{for all } j \geq 1.$$

In particular, we see that $f(Q) \neq 0$.

But $\mathcal{K}$ is a group, and so $\beta \in \Omega$ implies that $\beta^{-1} \in \Omega$, from which we see that

$$|\beta^j f(Q) - 1| < 1 \quad \text{for all } j \geq 1.$$

Suppose $\Omega \not\subseteq \mathbb{T}$. Choose $\beta \in \Omega$ with $|\beta| > 1$. Then we can find a sequence $(j_m)_m$ of non-zero integers so that $\lim_m |\beta^m| = \infty$, which clearly leads to a contradiction of the inequality $|\beta^j f(Q) - 1| < 1$ for all $j \geq 1$. Hence $\Omega \subseteq \mathbb{T}$.

Now suppose that $\beta \in \Omega \setminus \{1\}$. Then for some $j_0 \geq 1$, $\mathrm{Re}(\beta^{j_0} f(Q)) < 0$, which again contradicts the inequality $|\beta^j f(Q) - 1| < 1$ for all $j \geq 1$. Thus $\Omega = \{1\}$, as was sufficient to complete the proof. ∎

Let $C \in \mathbb{M}_n(\mathbb{C})$. Let $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$ denote the *singular numbers* of C, i.e. the eigenvalues of $(C^*C)^{1/2}$ repeated according to multiplicity. Recall that the the *trace norm* of C is $\|C\|_1 := \sum_{k=1}^n \gamma_k$, and that there exist orthonormal bases $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ for $\mathbb{C}^n$ so that $C = \sum_{k=1}^n \gamma_k f_k \otimes e_k^*$, where, for $x, y \in \mathbb{C}^n$, $y \otimes x^*(z) = \langle z, x\rangle y$ for all $z \in \mathbb{C}^n$.

LEMMA 3.3. *Let $n \geq 2$ be an integer, $f \in \mathbb{M}_n(\mathbb{C})^*$ be a norm one linear functional and $0 \neq \varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a multiplicative map. If*

$$|f(\varphi(A)) - f(A)| < (1 - 1/n)\|A\| \quad \textit{for all } A \in \mathbb{M}_n(\mathbb{C}),$$

then there exists a rank one operator $B \in \mathbb{M}_n(\mathbb{C})$ so that $\varphi(B) \neq 0$.

Proof. We remind the reader that the multiplicativity of φ implies in the context of this lemma that $\varphi(F) \neq 0$ for all rank one operators F in $\mathbb{M}_n(\mathbb{C})$.

The proof consists of examining the cases presented in Theorem 3.1.

(a) Suppose that there exists a ring homomorphism $\theta : \mathbb{C} \to \mathbb{C}$ and an invertible matrix $S \in \mathbb{M}_n(\mathbb{F})$ so that $\varphi([a_{i,j}]) = S^{-1}[\theta(a_{i,j})]S$. If $\theta(1) = 0$, then $\theta(z) = 0$ for all $z \in \mathbb{C}$, and hence $\varphi = 0$, a contradiction. Since $\theta(1)$ must be an idempotent in $\mathbb{C}$, the only other possibility is that $\theta(1) = 1$. Let $B = \mathrm{diag}(1, 0, 0, \ldots, 0)$. Then B has rank one and $\varphi(B) = S^{-1}BS \neq 0$.

(b) Suppose that there exists a ring homomorphism $\theta : \mathbb{F} \to \mathbb{F}$ and an invertible matrix $S \in \mathbb{M}_n(\mathbb{F})$ so that

$$\varphi([a_{i,j}]) = S^{-1}\Gamma([\theta(a_{i,j})])S \quad \text{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{F}),$$

where Γ is the cofactor map. We first show that the hypothesis on f implies that $n = 2$. Indeed, suppose that there exists a ring homomorphism $\theta : \mathbb{C} \to \mathbb{C}$ and an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ so that

$$\varphi([a_{i,j}]) = S^{-1}\Gamma([\theta(a_{i,j})])S \quad \text{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C}).$$

Let $z \in \mathbb{C}$. As before, we may assume that $\theta(1) = 1$. Then

$$\begin{aligned}\varphi(zA) = \varphi(zI_n)\varphi(A) &= S^{-1}\Gamma(\theta(zI_n))S\varphi(A)\\ &= S^{-1}(\theta(z))^{n-1}I_nS\varphi(A) = (\theta(z))^{n-1}\varphi(A).\end{aligned}$$

In particular, if $z = m > 1$ is an integer then $\varphi(mA) = m^{n-1}\varphi(A)$, so by our hypotheses,

$$|m^{n-1}f(\varphi(A)) - mf(A)| = |f(\varphi(mA)) - f(mA)| < (1 - 1/n)m\|A\|.$$

If $n \geq 3$, then dividing by m and taking limits as m increases to infinity shows that $f(\varphi(A)) = 0$ for all $A \in \mathbb{M}_n(\mathbb{C})$. But then

$$|f(A)| = |f(\varphi(A)) - f(A)| < (1 - 1/n)\|A\| \quad \text{for all } A \in \mathbb{M}_n(\mathbb{C}),$$

contradicting the fact that f has norm one.

If $n = 2$, then letting $B = \operatorname{diag}(1, 0)$ shows that $\varphi(B) = S^{-1}\operatorname{diag}(0, 1)S \neq 0$, and obviously B has rank one.

(c) Suppose that there exists a degenerate map $\beta : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ and an idempotent $E = E^2 \in \mathbb{M}_n(\mathbb{C})$ so that

$$\varphi([a_{i,j}]) = \beta([a_{i,j}]) + E \quad \text{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C}).$$

If $E \neq 0$, then, since every rank one operator has determinant zero, we see that $\varphi(B) = E \neq 0$ for every rank one operator and we are done.

If $E = 0$, then $\varphi(A) = 0$ for all singular matrices A. Hence

$$|f(A)| = |f(\varphi(A)) - f(A)| < (1 - 1/n)\|A\| \quad \text{for all } A \text{ with } \det A = 0.$$

Choose $C \in \mathbb{M}_n(\mathbb{C})$ such that $\|C\|_1 = 1$ and $f(X) = \operatorname{tr}(XC)$ for all $X \in \mathbb{M}_n(\mathbb{C})$. Write $C = \sum_{k=1}^n \gamma_k f_k \otimes e_k^*$, where $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ are orthonormal bases for $\mathbb{C}^n$, and where $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$ are the singular numbers of C. Let $A = \sum_{k=1}^{n-1} e_k \otimes f_k^*$. Then $\|A\| = 1$, $\operatorname{rank} A \leq n - 1$ and so $\det A = 0$, and

$$|f(A)| = |\operatorname{tr}(CA)| = \sum_{k=1}^{n-1} \gamma_k < 1 - \frac{1}{n}.$$

Since $\|C\|_1 = 1$, we know that $\sum_{k=1}^n \gamma_k = 1$, so $\gamma_n > 1/n$, a contradiction, since then $1 = \sum_{k=1}^n \gamma_k \geq n\gamma_n > 1$. This shows that the case where $E = 0$ does not occur under our hypotheses. ■

The above shows that if $n \geq 3$ and $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is multiplicative, and

$$|f(\varphi(A)) - f(A)| < (1 - 1/n)\|A\| \quad \text{for all } A \in \mathbb{M}_n(\mathbb{C}),$$

then either $\varphi([a_{i,j}]) = S^{-1}([\theta(a_{i,j})])S$ for all $A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C})$ or $\varphi(A) = \beta(A) + E$, where $\beta : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is degenerate and $E = E^2 \neq 0$.

If $n = 2$, then it is also possible that

$$\varphi([a_{i,j}]) = S^{-1}\Gamma([\theta(a_{i,j})])S \quad \text{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C}).$$

But then

$$\Gamma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so that case (b) reduces to case (a) for 2×2 matrices.

LEMMA 3.4. *Let $\alpha \in \mathbb{C}$ and suppose that $\theta : \mathbb{C} \to \mathbb{C}$ is a multiplicative map satisfying*

$$|\alpha\theta(z) - z| < |z| \quad \textit{for all } 0 \neq z \in \mathbb{C}.$$

Then $|\alpha - 1| < 1$ and $\theta(z) = z$ for all $z \in \mathbb{C}$.

Proof. The inequality in the hypothesis implies that $\theta(z) \neq 0$ if $0 \neq z \in \mathbb{C}$. Similarly, $\alpha \neq 0$.

Next, with $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, consider the map

$$\lambda : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto \theta(z)/z.$$

Clearly λ is a group homomorphism, and

$$|\alpha\lambda(z) - 1| < 1 \quad \text{for all } z \in \mathbb{C}^*.$$

Hence $\lambda(\mathbb{C}^*)$ is a bounded subgroup of $\mathbb{C}^*$, which implies that $\lambda(\mathbb{C}^*) \subseteq \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

But then $|\lambda(z) - 1/\alpha| < |1/\alpha|$ for all $z \in \mathbb{C}^*$ forces the subgroup $\lambda(\mathbb{C}^*)$ of $\mathbb{T}$ to lie in a fixed open half-plane (determined by a line through the origin). This is only possible if $\lambda(\mathbb{C}^*) = \{1\}$, which implies that $\theta(z) = z$ for all $z \in \mathbb{C}^*$. Setting $z = 1$ we see that $|\alpha - 1| < 1$.

Finally, $\theta(0) = \theta(2 \cdot 0) = \theta(2)\theta(0) = 2 \cdot \theta(0)$ and so $\theta(0) = 0$. ■

We are now ready to prove our main finite-dimensional result.

THEOREM 3.5. *Let $n \geq 2$ be an integer and $f \in \mathbb{M}_n(\mathbb{C})^*$ be a norm one linear functional on $\mathbb{M}_n(\mathbb{C})$ with $f(I_n) = 1$. Suppose that $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is a multiplicative map for which*

$$|f(\varphi(A)) - f(A)| < (1 - 1/n)\|A\| \quad \textit{for all } 0 \neq A \in \mathbb{M}_n(\mathbb{C}).$$

Then there exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ such that $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n(\mathbb{C})$. In particular, φ is linear.

Proof. By Lemma 3.2, we may assume that $\varphi(0) = 0$ and $\varphi(I_n) = I_n$. Furthermore, by Lemma 3.3, there exists a rank one operator B so that $\varphi(B) \neq 0$.

By Theorem 2.3, there exists an invertible matrix S and a multiplicative map $\theta : \mathbb{C} \to \mathbb{C}$ with $\theta(0) = 0$ and $\theta(1) = 1$ such that

$$\varphi([a_{i,j}]) = S^{-1}[\theta(a_{i,j})]S \quad \text{for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C}).$$

Now $I_n \in \mathrm{SL}_n(\mathbb{C})$ and $f(I_n) = 1$. For all $z \in \mathbb{C}$,

$$|f(\varphi(zI_n)) - f(zI_n)| < (1 - 1/n)\|zI_n\|,$$

so that

$$|f(\theta(z)\, I_n) - z| = |\theta(z) - z| < (1 - 1/n)|z|.$$

We apply Lemma 3.4 to conclude that $\theta(z) = z$ for all $z \in \mathbb{C}$, completing the proof. ∎

As a simple consequence of Theorem 3.5, we obtain the following analogue of Proposition 2.2:

Corollary 3.6. *Let $n \geq 2$, $f \in \mathbb{M}_n(\mathbb{C})^*$ be a norm one linear functional satisfying $f(I_n) = 1$, and let $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a multiplicative map. Suppose that $\mu : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is an algebra automorphism of $\mathbb{M}_n(\mathbb{C})$. If*

$$|f(\varphi(A)) - f(\mu(A))| < (1 - 1/n)\|\mu(A)\| \quad \textit{for all } 0 \neq A \in \mathbb{M}_n(\mathbb{C}),$$

then there exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ such that $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n(\mathbb{C})$.

Proof. Let $\tau = \varphi \circ \mu^{-1}$, so that τ is a multiplicative map on $\mathbb{M}_n(\mathbb{C})$. Now

$$|f(\tau(B)) - f(B)| < (1 - 1/n)\|B\| \quad \text{for all } B = \mu(A),\ A \in \mathbb{M}_n(\mathbb{C}),$$

and hence for all $B \in \mathbb{M}_n(\mathbb{C})$.

By Theorem 3.5, there exists T invertible so that

$$\tau(B) = T^{-1}BT \quad \text{for all } B \in \mathbb{M}_n(\mathbb{C}).$$

As is well-known, there exists $X \in \mathbb{M}_n(\mathbb{C})$ invertible so that $\mu(A) = X^{-1}AX$ for all $A \in \mathbb{M}_n(\mathbb{C})$, and thus $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n(\mathbb{C})$, where $S = XT$. ∎

Example 3.7. The constant of $1 - 1/n$ appearing in the statement of Theorem 3.5 is the best possible.

Suppose that $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is given by

$$A \mapsto \begin{cases} 0 & \text{if } A \text{ is not invertible,} \\ A & \text{if } A \text{ is invertible.} \end{cases}$$

Let $f = \frac{1}{n}\mathrm{tr}$ denote the normalized trace functional on $\mathbb{M}_n(\mathbb{C})$. Then

$$|f(\varphi(A)) - f(A)| = \begin{cases} |f(A)| & \text{if } A \text{ is not invertible,} \\ 0 & \text{if } A \text{ is invertible.} \end{cases}$$

Next, observe that

$$|f(A)| = \left|\frac{1}{n}\mathrm{tr}(A)\right| \leq \frac{1}{n}\|A\|_1.$$

But if A is not invertible, then $A = U|A| = U\mathrm{diag}(s_1, \ldots, s_n)$, where $s_1 \geq \cdots \geq s_n$ are the singular numbers of A, and $s_n = 0$ by virtue of the fact that A is not invertible. Furthermore, $s_1 = \|A\|$.

Hence $\|A\|_1 = \sum_{k=1}^{n-1} s_k \leq (n-1)\|A\|$, so that $|f(A)| \leq (1 - 1/n)\|A\|$.

Despite this, it is clear that there does not exist any invertible matrix S such that $\varphi(T) = S^{-1}TS$ for all $T \in \mathbb{M}_n(\mathbb{C})$.

Having said that this constant is the best possible (in general), we should point out that in fact the estimate can be improved slightly if we know a bit more about the functional f. Of course, every non-zero functional f on $\mathbb{M}_n(\mathbb{C})$ corresponds to a matrix $Q \in \mathbb{M}_n(\mathbb{C})$ via the map

$$f(A) = \mathrm{tr}(AQ) \quad \text{ for all } A \in \mathbb{M}_n(\mathbb{C}).$$

Associated to f, therefore, is the rank of the matrix Q. The trace functional used above has full rank equal to n. If we know that a given functional g has rank less than n, the constant can be made smaller. We leave the verification of this, as well as the calculation of the constant as a function of the rank of Q, to the reader.

In the absence of a functional f, we can considerably improve the constant in our estimate.

Proposition 3.8. *Let $n \geq 2$ and let $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a multiplicative map and*

$$\|\varphi(A) - A\| < \|A\| \quad \textit{for all } 0 \neq A \in \mathbb{M}_n(\mathbb{C}).$$

Then $\varphi(A) = S^{-1}AS$ for some invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$. Furthermore, $\|S\|\,\|S^{-1}\| < 2$.

Proof. Since $\|\varphi(A) - A\| < \|A\|$ for all $0 \neq A \in \mathbb{M}_n(\mathbb{C})$, φ cannot annihilate any rank one operator. Moreover, by Lemma 3.2, $\varphi(0) = 0$.

We may now apply Theorem 2.3(b) to conclude that there exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ and a multiplicative map $\theta : \mathbb{C} \to \mathbb{C}$ satisfying $\theta(0) = 0$ and $\theta(1) = 1$ so that

$$\varphi([a_{i,j}]) = S^{-1}[\theta(a_{i,j})]S \quad \text{ for all } A = [a_{i,j}] \in \mathbb{M}_n(\mathbb{C}).$$

For $z \in \mathbb{C}^*$, we have $\varphi(zI_n) = \theta(z)I_n$ and

$$\|\varphi(zI_n) - zI_n\| < \|zI_n\|.$$

This in turn implies that $|\theta(z) - z| < |z|$ for all $z \in \mathbb{C}^*$.

By Lemma 3.4, $\theta(z) = z$ for all $z \in \mathbb{C}$ and so $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n(\mathbb{C})$.

As for the condition number of S, first note that without loss of generality, we may assume that $\|S\| = 1$. Choose $x, y \in \mathbb{C}^n$ with $\|x\| = \|y\| = 1$ so that $\|S^{-1}x\| = \|S^{-1}\|$ and $\|S^*y\| = \|S^*\| = 1$. Let $R = x \otimes y^*$. Then

$$\|S^{-1}RS - R\| = \|S^{-1}x \otimes (S^*y)^* - x \otimes y^*\| < \|x \otimes y^*\|$$

and so $\|S^{-1}\|\,\|S\| = \|S^{-1}x \otimes (S^*y)^*\| < 2$. ∎

Example 3.9. We note that the map

$$\varphi(A) \mapsto \begin{cases} 0 & \text{if } A \text{ is not invertible,} \\ A & \text{if } A \text{ is invertible,} \end{cases}$$

from Example 3.7 satisfies $\|\varphi(A) - A\| \le \|A\|$ for all $A \in \mathbb{M}_n(\mathbb{C})$. Hence 1 is the best possible constant one can obtain in Proposition 3.8.

4. Multiplicative maps on $\mathcal{B}(\mathcal{H})$ that can be approximated by automorphisms. We conclude by exhibiting a surprising application of these results to multiplicative maps on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on an infinite-dimensional, separable, complex Hilbert space $\mathcal{H}$.

Recall that a commuting family $\mathcal{E}$ of idempotents in $\mathcal{B}(\mathcal{H})$ is called a *Boolean algebra* if $\{0, I\} \in \mathcal{E}$ and if $E, F \in \mathcal{E}$ implies that $E \wedge F := EF$ and $E \vee F := (E + F - EF)$ belong to $\mathcal{E}$. We note that $E \wedge F$ is the largest idempotent dominated by both E and F, i.e. $E(E \wedge F) = E \wedge F = (E \wedge F)E$ and $F(E \wedge F) = (E \wedge F) = (E \wedge F)F$, while $E \vee F$ is the smallest idempotent dominating both E and F, in the sense that $(E \vee F)E = E = E(E \vee F)$ and $(E \vee F)F = F = F(E \vee F)$.

Theorem 4.1. *Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space and $\delta \le 1/4$. Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a multiplicative map and suppose that*

$$\|\varphi(A) - A\| < \delta\|A\| \quad \textit{for all } 0 \neq A \in \mathcal{B}(\mathcal{H}).$$

Then there exists $S \in \mathcal{B}(\mathcal{H})$ invertible so that

$$\varphi(A) = S^{-1}AS \quad \textit{for all } A \in \mathcal{B}(\mathcal{H}).$$

Proof. First let us fix an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$. Consider the Boolean algebra $\mathcal{B}$ of all projections which are diagonal relative to this basis, and let $\mathcal{B}_0$ denote the set of finite-rank projections in $\mathcal{B}$. For each $P \in \mathcal{B}$, $E_P := \varphi(P)$ is an idempotent and the fact that $\|E_P - P\| < \delta$ implies that $\|E_P\| < 1+\delta$. That is, $\mathcal{E} := \{E_P = \varphi(P) : P \in \mathcal{B}\}$ is bounded. Furthermore, if $P, Q \in \mathcal{B}$, then $E_P E_Q = \varphi(P)\varphi(Q) = \varphi(PQ) = E_{PQ} \in \mathcal{E}$.

As for $E_P \vee E_Q$, let $R = P \vee Q = P + Q - PQ \in \mathcal{B}$. Let $E_R = \varphi(R)$. Then $E_R E_P = \varphi(R)\varphi(P) = \varphi(P) = E_P = E_P E_R$, and similarly $E_R E_Q =$

$E_Q = E_Q E_R$. Hence $E_R \geq E_P \vee E_Q$. But

$$\begin{aligned}\|E_R - (E_P \vee E_Q)\| &\leq \|E_R - R\| + \|R - (E_P \vee E_Q)\| \\ &= \|E_R - R\| + \|(P + Q - PQ) - (E_P + E_Q - E_P E_Q)\| \\ &\leq \|E_R - R\| + \|(P - E_P)\| + \|(Q - E_Q)\| + \|PQ - E_P E_Q\| \\ &< \delta\|R\| + \delta\|P\| + \delta\|Q\| + \|PQ - E_{PQ}\| \\ &< 3\delta + \delta\|PQ\| = 4\delta \leq 1,\end{aligned}$$

which implies that $E_P \vee E_Q = E_R \in \mathcal{E}$. It follows that $\mathcal{E}$ is a bounded, (clearly abelian, since $\mathcal{B}$ is abelian) Boolean algebra of idempotents.

By Lemma XV.6.2 of [5], there exists an invertible operator $X \in \mathcal{B}(\mathcal{H})$ such that $\|X\|\,\|X^{-1}\| \leq 1 + 2(1 + \delta) < 4$ and $D_P := X^{-1} E_P X$ is a self-adjoint projection for each $P \in \mathcal{B}$. Thus $\mathcal{D} := \{X^{-1}\varphi(P)X : X \in \mathcal{B}\}$ is a Boolean algebra of projections. Note that $\operatorname{rank} D_P = \operatorname{rank} P$ for all $P \in \mathcal{B}$. Indeed, $\|E_P - P\| < 1$ implies that $\operatorname{rank} E_P = \operatorname{rank} P$ for all $P \in \mathcal{B}$, while the map $\mathrm{Ad}_X : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\mathrm{Ad}_X(T) = X^{-1}TX$ is clearly rank-preserving as well. Hence $\tau := \mathrm{Ad}_X \circ \varphi : \mathcal{B} \to \mathcal{D}$ is a rank-preserving map.

Let $\mathcal{E}_0 = \{E_P : P \in \mathcal{B}_0\}$. We claim that $I = \sup \mathcal{E}_0$. That the supremum on the right-hand side of this equation exists (note that we do not yet claim that it lies in $\mathcal{E}$) follows from the fact that $D_\infty := \sup\{\tau(P) = D_P : P \in \mathcal{B}_0\}$ exists in $\mathcal{B}(\mathcal{H})$ by virtue of being the supremum of a family of commuting projections. Now $E_\infty := \mathrm{Ad}_{X^{-1}}(D_\infty)$ is easily seen to be the supremum of $\mathcal{E}_0$, and clearly $E_\infty \neq 0$.

Suppose that $E_\infty \neq I$, and consider the decomposition $\mathcal{H} = \operatorname{ran} E_\infty \oplus \operatorname{ran} E_\infty^\perp$. With respect to this decomposition we may write

$$E_\infty = \begin{bmatrix} I & E_{1,2} \\ 0 & 0 \end{bmatrix}.$$

Let $0 < \varepsilon < \delta$ and let $x \in \operatorname{ran} E_\infty^\perp$ be a norm one vector. Choose $P_x \in \mathcal{B}_0$ so that $\|x - P_x x\| < \varepsilon$. Since $E_\infty = \sup \mathcal{E}_0$, $E_\infty E_{P_x} = E_{P_x} = E_{P_x} E_\infty$, which implies that $\operatorname{ran} E_{P_x} \subseteq \operatorname{ran} E_\infty$. Moreover,

$$\|E_{P_x} x - x\| \leq \|(E_{P_x} - P_x)x\| + \|P_x x - x\| < \delta\|P_x\|\,\|x\| + \varepsilon < \delta + \varepsilon < 1/2.$$

But $\operatorname{ran} E_{P_x} \subseteq \operatorname{ran} E_\infty$ implies that

$$E_{P_x} = \begin{bmatrix} Z_1 & Z_2 \\ 0 & 0 \end{bmatrix}$$

with respect to the above decomposition of $\mathcal{H}$, and hence

$$\|E_{P_x} x - x\| = \left\| \begin{bmatrix} Z_1 - I & Z_2 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \right\| = \left\| \begin{bmatrix} Z_2 x \\ -x \end{bmatrix} \right\| < \frac{1}{2},$$

contradicting the fact that $\|x\| = 1$. (We remark that here we have conflated the vector $x \in \operatorname{ran} E_\infty^\perp$ with the vector $\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right]$ in $\mathcal{H} = \operatorname{ran} E_\infty \oplus \operatorname{ran} E_\infty^\perp$.)

Thus $E_\infty = I$. From this it follows that $\sup\{\tau(P) : P \in \mathcal{B}_0\} = X^{-1} E_\infty X = I$ as well.

For each $n \geq 1$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{e_1, \ldots, e_n\}$. If $D_n = \tau(P_n)$, then from the above, $\operatorname{rank} D_n = n$ and $P_n \geq P_{n-1}$ implies that $D_n \geq D_{n-1}$ for all $n \geq 2$, so that by choosing a basis $\{f_n\}$ for $(D_n - D_{n-1})\mathcal{H}$, we get an isometry $U : \mathcal{H} \to \mathcal{H}$ defined by $Ue_n = f_n$, $n \geq 1$. (Note that $D_k - D_{k-1}$ is orthogonal to $D_n - D_{n-1}$ when $k \neq n$ since $P_k - P_{k-1}$ is orthogonal to $P_n - P_{n-1}$.) But the strong limit of the sequence $(P_n)_n$ is equal to I. If $P \in \mathcal{B}_0$, then there exists $n \geq 1$ so that $P_n \geq P$, whence $D_n \geq D_P$. It follows that $\sup\{D_n : n \geq 1\} \geq \sup\{D_P : P \in \mathcal{B}_0\} = I$, and thus $\{f_n\}_{n=1}^\infty$ is actually an orthonormal basis for $\mathcal{H}$. But then U is a unitary operator, and $U^* D_n U = P_n$ for all $n \geq 1$.

Define $\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ via $\rho(T) = (\operatorname{Ad}_U \circ \operatorname{Ad}_X \circ \varphi)(T)$, and observe that ρ is multiplicative.

Note also that $\rho(P_n) = P_n$ for each $n \geq 1$, and in fact, if $Q_n = P_n - P_{n-1}$ for all $n \geq 1$ (where $P_0 := 0$), then $\rho(Q_n) = \rho(Q_n P_n) = \rho(Q_n) P_n$, while $0 = \rho(0) = \rho(Q_n P_{n-1}) = \rho(Q_n) P_{n-1}$, which implies that $\rho(Q_n) = P_n - P_{n-1} = Q_n$.

Next, denote by $E_{i,j}$ the rank one matrix unit $e_j \otimes e_i^*$. Then $E_{i,j} = Q_i E_{i,j} Q_j$ implies that $\rho(E_{i,j}) = Q_i \rho(E_{i,j}) Q_j$. Choose $\alpha_{i,j} \in \mathbb{C}$ so that $\rho(E_{i,j}) = \alpha_{i,j} E_{i,j}$. Now,

$$\begin{aligned} \|\rho(E_{i,j})\| &\leq \|\operatorname{Ad}_U\| \, \|\operatorname{Ad}_X\| \, \|\varphi(E_{i,j})\| \leq \|X\| \, \|X^{-1}\| (1+\delta) \|E_{i,j}\| \\ &= \|X\| \, \|X^{-1}\| (1+\delta). \end{aligned}$$

We also have

$$Q_i = \rho(Q_i) = \rho(E_{i,j} E_{j,i}) = \rho(E_{i,j}) \rho(E_{j,i}) = \alpha_{i,j} \alpha_{j,i} Q_i.$$

Combining the two facts above, we see that

- $|\alpha_{i,j}| \leq \|X\| \, \|X^{-1}\| (1+\delta)$ for all $i, j \geq 1$ and so $\{|\alpha_{i,j}| : i, j \geq 1\}$ is bounded above, and
- $\alpha_{j,i} = \alpha_{i,j}^{-1}$ for all i and j, which implies that $\{|\alpha_{i,j}| : i, j \geq 1\}$ is also bounded below by a positive constant.

Using this, it is not too hard to see that we can choose a diagonal similarity Y so that

$$Y^{-1} \rho(E_{i,i+1}) Y = E_{i,i+1}$$

for each $i \geq 1$, and thus $Y^{-1} \rho(E_{i,j}) Y = E_{i,j}$ for all $i, j \geq 1$. Let $\beta = \operatorname{Ad}_Y \circ \rho$.

Then β is a multiplicative map and $\beta(E_{i,j}) = E_{i,j}$ for all $i, j \geq 1$. If $z \in \mathbb{C}$, then $\beta(zQ_i) = \beta(zE_{i,i}) = \beta(Q_i(zE_{i,i})Q_i) = \beta(Q_i)\beta(zE_{i,j})\beta(Q_i) =$

$Q_i\beta(zE_{i,j})Q_i = \theta_i(z)Q_i$ for some $\theta_i(z) \in \mathbb{C}$. It is easy to see that each $\theta_i : \mathbb{C} \to \mathbb{C}$ is a multiplicative map, $i \geq 1$. Furthermore,

$$\beta(zQ_i) = \mathrm{Ad}_Y \circ \mathrm{Ad}_U \circ \mathrm{Ad}_X \circ \varphi(zQ_i)$$

is clearly similar to $\varphi(zQ_i)$, and hence $\sigma(\varphi(zQ_i)) = \sigma(\beta(zQ_i)) = \{0, \theta_i(z)\}$. But then the trace of $\varphi(zQ_i)$ equals $\theta_i(z)$.

Since $\|\varphi(zQ_i) - zQ_i\| < \delta\|zQ_i\| = \delta|z| < |z|$, and since the trace functional $\mathrm{tr}(\cdot)$ is a norm one functional on the set of trace-class operators (which includes finite rank operators), we find that

$$\begin{aligned} |\theta_i(z) - z| &= |\mathrm{tr}(\varphi(zQ_i)) - \mathrm{tr}(zQ_i)| = |\mathrm{tr}(\varphi(zQ_i) - zQ_i)| \\ &< |z| \quad \text{for all } 0 \neq z \in \mathbb{C}. \end{aligned}$$

By Lemma 3.4, $\theta_i(z) = z$ for all $z \in \mathbb{C}$ and all $i \geq 1$.

Finally, let $A = [a_{i,j}] \in \mathcal{B}(\mathcal{H})$. Then $a_{i,j}E_{i,j} = Q_iAQ_j$, so

$$\begin{aligned} Q_i\beta(A)Q_j &= \beta(Q_iAQ_j) = \beta(a_{i,j}E_{i,j}) = \beta(a_{i,j}Q_i)\beta(E_{i,j}) \\ &= \theta_i(a_{i,j})Q_iE_{i,j} = a_{i,j}E_{i,j}. \end{aligned}$$

In other words, $\beta(A) = A$ for all $A \in \mathcal{B}(\mathcal{H})$.

But then $\varphi(A) = \mathrm{Ad}_{X^{-1}} \circ \mathrm{Ad}_{U^{-1}} \circ \mathrm{Ad}_{Y^{-1}} \circ \beta(A) = \mathrm{Ad}_{X^{-1}} \circ \mathrm{Ad}_{U^{-1}} \circ \mathrm{Ad}_{Y^{-1}}(A) = \mathrm{Ad}_S(A)$, where $S = X^{-1}U^{-1}Y^{-1}$. ∎

LEMMA 4.2. *Let $\mathcal{H}$ be a complex Hilbert space and fix $\varepsilon > 0$. If $\|S^{-1}AS - A\| < \varepsilon\|A\|$ for all $0 \neq A \in \mathcal{B}(\mathcal{H})$, then there exists $\kappa \in \mathbb{C}$ so that $\|\kappa S - I\| \leq \varepsilon(1 + \varepsilon)$.*

Proof. We can scale S by κ so that $1 \in \partial(\sigma(\kappa S))$, the boundary of the spectrum of κS, in which case 1 is an approximate eigenvalue of κS. We now assume that this has been done and we replace the original S by κS.

Let $\delta > 0$ and choose unit vectors $x, y \in \mathcal{H}$, so that

- $\|Sx - x\| < \frac{\delta}{\|S\|\,\|S^{-1}\|}$, and
- $\|S\| - \|Sy\| < \frac{\delta}{\|S\|\,\|S^{-1}\|}$.

Let $A = y \otimes x^*$, so that $\|A\| = 1$ and $\operatorname{rank} A = 1$. Then

$$\begin{aligned} \|Sy\| - \|y\| \leq \|Sy - y\| &\leq \|Sy - SAS^{-1}x\| + \|SAS^{-1}x - Ax\| \\ &\leq \|S\|\,\|A\|\,\|x - S^{-1}x\| + \varepsilon\|A\|\,\|x\| \\ &\leq \|S\|\,\|A\|\,\|S^{-1}\|\,\|Sx - x\| + \varepsilon < \delta + \varepsilon. \end{aligned}$$

This implies that $\|S\| < 1 + 2\delta + \varepsilon$.

Now for any operator B with $\|B\| = 1$,

$$\|SB - BS\| \leq \|SBS^{-1} - B\|\,\|S\| < \varepsilon(1 + \varepsilon + 2\delta).$$

Given a unit vector $z \in \mathcal{H}$, let $B_z := z \otimes x^*$ so that $\|B_z\| = 1$ and $B_z x = z$. Then

$$\begin{aligned}\|Sz - z\| &= \|SB_z x - B_z S_x x\| + \|B_z\|\,\|Sx - x\| < \varepsilon(1+\varepsilon+2\delta)\|x\| + \|B_z\|\delta \\ &< \varepsilon(1+\varepsilon+2\delta) + \delta.\end{aligned}$$

The result now follows by taking limits as δ tends to zero. ∎

Corollary 4.3. *Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space, $0 < \varepsilon < 1/4$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a multiplicative map. Suppose that*

$$\|\varphi(A) - A\| < \varepsilon\|A\| \quad \text{for all } 0 \neq A \in \mathcal{B}(\mathcal{H}).$$

Then there exists $T \in \mathcal{B}(\mathcal{H})$ invertible so that $\|T - I\| \leq \varepsilon(1+\varepsilon)$ and

$$\varphi(A) = T^{-1}AT \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

Proof. By Theorem 4.1, there exists $S \in \mathcal{B}(\mathcal{H})$ invertible so that $\varphi(A) = S^{-1}AS$ for all $A \in \mathcal{B}(\mathcal{H})$. By Lemma 4.2, there exists $\kappa \in \mathbb{C}$ so that

$$\|\kappa S - I\| \leq \varepsilon(1+\varepsilon).$$

Let $T = \kappa S$. ∎

The following is the infinite-dimensional analogue of Proposition 2.2 and Corollary 3.6.

Corollary 4.4. *Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space and $0 < \varepsilon < 1/4$. Let $\mu : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear automorphism and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a multiplicative map satisfying*

$$\|\varphi(A) - \mu(A)\| < \varepsilon\|\mu(A)\| \quad \text{for all } 0 \neq A \in \mathcal{B}(\mathcal{H}).$$

Then there exists $R \in \mathcal{B}(\mathcal{H})$ invertible so that

$$\varphi(A) = R^{-1}AR \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

Proof. As before, we set $\tau = \varphi \circ \mu^{-1}$, so that τ is multiplicative on $\mathcal{B}(\mathcal{H})$. Then

$$\|\tau(\mu(A)) - \mu(A)\| < \varepsilon\|\mu(A)\| \quad \text{for all } 0 \neq A \in \mathcal{B}(\mathcal{H}).$$

By Corollary 4.3, there exists $T \in \mathcal{B}(\mathcal{H})$ so that

$$\tau(B) = \mathrm{Ad}_T(B) = T^{-1}BT \quad \text{for all } B \in \mathcal{B}(\mathcal{H}).$$

Hence $\varphi = \tau \circ \mu$ is again an automorphism of $\mathcal{B}(\mathcal{H})$. But every automorphism of $\mathcal{B}(\mathcal{H})$ is spatial (see, for example, [1]), and so we can find $R \in \mathcal{B}(\mathcal{H})$ so that $\varphi(A) = \mathrm{Ad}_R(A) = R^{-1}AR$ for all $A \in \mathcal{B}(\mathcal{H})$. ∎

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L. W. Marcoux, H. Radjavi
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
E-mail: LWMarcoux@uwaterloo.ca
hradjavi@uwaterloo.ca

A. R. Sourour
Department of Mathematics and Statistics
University of Victoria
Victoria, BC, Canada V8W 3P4
E-mail: sourour@math.uvic.ca

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