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BSDEs with random terminal time and semilinear elliptic PDEs in divergence form

by

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Abstract. We study connections between Sobolev space solutions of the Dirichlet problem for semilinear second order elliptic equations in divergence form and solutions of backward stochastic differential equations with random terminal time.

1. Introduction. In Peng [9] and Darling and Pardoux [4] (see also the expository paper by Pardoux [8] and the references given therein) connections between solutions of backward stochastic differential equations (BSDEs) with random terminal time and viscosity solutions of the Dirichlet problem for semilinear second order elliptic equations in non-divergence form are investigated. In the present paper we are interested in finding connections between Sobolev space solutions of the semilinear Dirichlet problem and solutions of BSDEs with random terminal time in the case when the forward driving process is a diffusion generated by a uniformly elliptic divergence form operator with measurable coefficients. We emphasize that it is not a semimartingale generally.

To be more precise, let us fix a bounded domain (non-empty, open, connected set) D in $\mathbb{R}^d$ and consider the differential operator A of the form

$$A = \frac{1}{2}\sum_{i,j=1}^{d} D_j(a^{ij}(x)D_i) + \sum_{i=1}^{d} b^i(x)D_i \tag{1.1}$$

where $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : D \to \mathbb{R}^d$ are measurable functions such that for some $0 < \lambda \le \Lambda$,

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad a^{ij}(x) = a^{ji}(x), \quad |b(x)| \le \Lambda \tag{1.2}$$

for $x \in D$ and $\xi \in \mathbb{R}^d$. By putting $a^{ij} = \delta^i_j$, $b^i = 0$ outside D we can and will

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assume that a, b are defined and satisfy (1.2) in all $\mathbb{R}^d$. Let $p(\cdot,\cdot,\cdot)$ denote a weak fundamental solution for A and let $\mathbb{X} = \{(X, P_x);\, x \in \mathbb{R}^d\}$ be a Markov process for which p is a transition density function, that is,

$$P_x(X_0 = x) = 1, \qquad P_x(X_t \in \Gamma) = \int_\Gamma p(t,x,y)\,dy, \quad t \geq 0,$$

for any Borel $\Gamma \subset \mathbb{R}^d$ (see [16]). From [10, Theorem 3.4] it follows that for each starting point $x \in \mathbb{R}^d$ the canonical process X admits a unique decomposition of the form

$$X_t - X_0 = M_t + A_t, \qquad t \geq 0,\ P_x\text{-a.s.}, \tag{1.3}$$

where M is a continuous martingale additive functional (in the strict sense) of $\mathbb{X}$ and A is a continuous additive functional (in the strict sense) of $\mathbb{X}$ of zero quadratic variation. Moreover, the co-variation process of $M = (M^1, \ldots, M^d)$ is given by

$$\langle M^i, M^j\rangle_t = \int_0^t a^{ij}(X_s)\,ds, \qquad t \geq 0. \tag{1.4}$$

Suppose that D is regular for A. For given $\varphi : \partial D \to \mathbb{R}$ and $f : D\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ consider the semilinear Dirichlet problem

$$Au = -F_u, \quad u|_{\partial D} = \varphi, \tag{1.5}$$

where

$$F_u(x) = f(x, u(x), (\sigma\nabla u)(x)), \qquad x \in D,$$

and σ is the symmetric square root of a. In this paper we show that under some natural assumptions on φ and f there exists a bounded continuous solution u of (1.5) in the Sobolev space W_2^1 and for each $x \in D$ there also exists a unique solution (Y^x, Z^x) of the BSDE

$$\begin{aligned} Y_t^x = \varphi(X_\tau) &+ \int_{t\wedge\tau}^{\tau} f(X_s, Y_s^x, Z_s^x)\,ds \\ &- \int_{t\wedge\tau}^{\tau} \langle Z_s^x, \sigma^{-1}(X_s)\,dM_s\rangle, \qquad t \geq 0,\ P_x\text{-a.s.}, \end{aligned} \tag{1.6}$$

where

$$\tau = \inf\{t \geq 0 : |X_t| \notin D\}. \tag{1.7}$$

Moreover, for each $x \in D$ we have

$$(Y_t^x, Z_t^x) = (u(X_{t\wedge\tau}), (\sigma\nabla u)(X_{t\wedge\tau})), \qquad t \geq 0,\ P_x\text{-a.s.} \tag{1.8}$$

From (1.8) it follows in particular that $Y_0^x = u(x)$ P_x-a.s. for $x \in D$ and $Z_0^x = (\sigma\nabla u)(x)$ P_x-a.s. for a.e. $x \in D$, which yields a stochastic representation of u and its gradient.

For regular a (i.e. if A is a non-divergence form operator and X is an Itô process under P_x), a representation similar to (1.8) was proved in Darling and Pardoux [4] under assumptions on φ and f similar to those in the present paper, and in Peng [9] under more restrictive assumptions. Let us also mention that the Cauchy problem for the semilinear parabolic equation $D_t u + Au = F_u$ with A given by (1.1) is investigated in Bally, Pardoux and Stoica [1], Rozkosz [11] and Stoica [15], and the mixed boundary problem in Lejay [7]. Let us stress, however, that in [1] a representation of the form (1.8) is obtained for quasi-every starting point $x \in D$, whereas in [7, 15] only for a.e. $x \in D$.

As for proofs, we would like to point out only one aspect. Roughly speaking, the main difficulty in proving (1.5)–(1.8) for every (and not quasi- or almost every) starting point is that we have to show P_x-integrability of the quadratic variation of the martingale part of (1.6) (or the martingale part of $u(X_{\cdot\wedge\tau})$ in its decomposition of the form (1.3)). From (1.4), (1.6) it is therefore clear that to the requirement that the solution u of (1.5) belongs to the space $C([0,\infty);\mathbb{R}^d) \cap W_2^1$ a condition on the integrability of ∇u must be added. Therefore in the present paper we consider for each $x \in D$ a Sobolev space $\mathcal{W}_\alpha(x)$ with weight, the weight being the α-Green function for A in D with suitably chosen $\alpha \geq 0$ (see Section 2). Using stochastic calculus we derive a sort of "energy estimates" in the space $\mathcal{W}_\alpha(x)$. The crucial fact is that these estimates do not involve any regularity assumptions about a and allow one to prove existence of a unique continuous solution $u \in W_2^1$ of (1.5) such that $u \in \mathcal{W}_\alpha(x)$ for every $x \in D$. This is enough for our purposes, because from the definition of $\mathcal{W}_\alpha(x)$ it follows in particular that $E_x \int_0^\tau |\nabla u(X_t)|^2\,dt < \infty$ if $u \in \mathcal{W}_\alpha(x)$. As far as we know, the idea of using the spaces $\mathcal{W}_\alpha(x)$ for investigation of the problem (1.5) is new. Notice, however, that an idea similar in spirit has appeared in [11].

NOTATION. $\Omega = C([0,\infty);\mathbb{R}^d)$ is the space of continuous $\mathbb{R}^d$-valued functions on $[0,\infty)$ equipped with the topology of uniform convergence on compact sets. X is the canonical process on Ω, $X_t^\tau = X_{t\wedge\tau}$, $t \geq 0$. By E_x (resp. E_x^n) we denote expectation with respect to P_x (resp. P_x^n), by $\mathcal{L}[Y \mid P_x]$ (resp. $\mathcal{L}[Y \mid P_x^n]$) the law of the process Y under P_x (resp. P_x^n), and by "$\Rightarrow$" convergence in law.

$D_i = \partial/\partial x^i$ is the partial derivative in the distribution sense, and $\nabla = (D_1,\dots,D_d)$. By $C(G)$ we denote the set of all continuous functions on G, and by $C^{k,\beta}(\overline{D})$ (resp. $C^{k,\beta}(D)$) the space of functions whose kth order partial derivatives are uniformly Hölder continuous (resp. locally Hölder continuous) with exponent β in D. Furthermore, $\mathbb{L}_p$ (resp. $\mathbb{L}_\infty$) is the Banach space of measurable functions on D that are p-integrable (resp. essentially bounded); W_2^1 is the usual Sobolev space of all elements u of $\mathbb{L}_2$ having derivatives $D_i u$ in $\mathbb{L}_2$, and $\mathring{W}_2^1$ is the closure of the space of infinitely differentiable functions with compact supports lying in D with respect to the W_2^1-norm $\|\cdot\|_{W_2^1}$. By $\|\cdot\|_p$ we denote the norm in $\mathbb{L}_p$; $(\cdot,\cdot)_2$ denotes the scalar product in $\mathbb{L}_2$ whereas $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^d$; $|D|$ is the Lebesgue measure of D. The abbreviation "a.e." means "almost everywhere with respect to the Lebesgue measure".

2. Uniqueness and a priori estimates. Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the minimum completed admissible filtration for $\mathbb{X}$.

DEFINITION. Let $\varphi : \partial D \to \mathbb{R}$ and $f : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions and let $x \in D$. We say that a pair $\{(Y_t^x, Z_t^x); t \geq 0\}$ of $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable processes is a solution of the BSDE (φ, f) associated with (X, P_x) if Y^x is continuous, $\int_0^\tau |Z_t^x|^2\,dt < \infty$ P_x-a.s. and (1.6) is satisfied with the martingale additive functional M of the decomposition (1.3).

In fact we will consider solutions with better integrability properties, namely solutions in the class $\mathcal{H}_{\alpha;x}^{1+d}$ of all $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable processes ξ with values in $\mathbb{R}^{1+d}$ having a finite norm

$$\|\!|\xi|\!\|_{\alpha;x}^2 = E_x \int_0^\tau e^{\alpha t}|\xi_t|^2\,dt$$

for some $\alpha \geq 0$.

We will assume that D is a bounded domain in $\mathbb{R}^d$ which is *regular* for A, that is, $P_x(\overline{\tau} = 0) = 1$, $x \in \partial D$, where $\overline{\tau} = \inf\{t > 0 : |X_t| \notin D\}$. We will need the following assumptions on φ and f:

(i) $\varphi : \partial D \to \mathbb{R}$ is continuous and can be extended by continuity to $\overline{D}$ so that the extended function (still denoted by φ) belongs to W_2^1;

(ii) $f : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the Carathéodory condition, that is, $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in D$ and $f(\cdot, y, z)$ is measurable for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$;

(iii) $(y_1 - y_2)\cdot(f(x,y_1,z) - f(x,y_2,z)) \leq \mu|y_1 - y_2|^2$ for some $\mu \in \mathbb{R}$ (the monotonicity condition);

(iv) $|f(x,y,z_1) - f(x,y,z_2)| \leq L|z_1 - z_2|$ for some $L \geq 0$;

(v) $|f(x,y,0)| \leq g(x) + K|y|$ for some $K \geq 0$ and non-negative $g \in \mathbb{L}_p$ with $p > 2 \vee d$ if $2\mu + L^2 \leq 0$, and $g \in \mathbb{L}_\infty$ in the general case.

For μ, L we will assume that

(vi′) $2\mu + L^2 \leq 0$ if $b \neq 0$ (non-symmetric case), or

(vi″) $\sup_{a\in\mathcal{A}(\lambda,\Lambda)} \sup_{x\in D} E_x^{(a)} e^{\alpha\tau} < \infty$ for some $\alpha > 2\mu + L^2$ if $b = 0$ (symmetric case). Here $E_x^{(a)}$ denotes expectation with respect to the measure $P_x^{(a)}$ corresponding to operator (1.1) with $b = 0$ and $\mathcal{A}(\lambda,\Lambda)$ is the set of all measurable $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying (1.2).

REMARK 2.1. Since there exists $c = c(\lambda, \Lambda, d)$ such that

$$p(t,x,y) \le \frac{c}{t^{d/2}} \exp\Big(-\frac{|y-x|^2}{ct}\Big) \tag{2.1}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$ (see, e.g., [16, Section I.1]), the proof of [3, Proposition 1.18] shows that there is $C > 0$ depending only on λ, Λ, d such that $\sup_{a\in\mathcal{A}(\lambda,\Lambda)} \sup_{x\in D} E_x e^{\alpha\tau} < \infty$ for any $\alpha < \alpha_0 \equiv C|D|^{-2/d}$. Therefore (vi$''$) is satisfied if $2\mu + L^2 < \alpha_0$.

PROPOSITION 2.2. *If* (i)–(iv) *are satisfied and* $E_x e^{\alpha\tau} < \infty$, *where* $\alpha = 2\mu + L^2$, *then the BSDE* (φ, f) *has at most one solution in the space* $\mathcal{H}^{1+d}_{\alpha;x}$.

Proof. See, e.g., the proof of [4, Proposition 3.2]. ∎

The remainder of this section is devoted to some a priori estimates on solutions of (1.6). These estimates will be needed in the next two sections.

PROPOSITION 2.3. *Assume* (i)–(iv) *and that* $\|f(\cdot,0,0)\|_{\alpha;x} < \infty$ *for some* $\alpha \ge 0$. *If* $(Y^x, Z^x) \in \mathcal{H}^{1+d}_{0;x}$ *is a solution of the BSDE* (φ, f) *such that*

$$E_x \sup_{0\le t\le\tau} e^{\alpha t}|Y^x_t|^2 < \infty, \tag{2.2}$$

then $(Y^x, Z^x) \in \mathcal{H}^{1+d}_{\alpha;x}$ *and for any* $\delta, \varepsilon > 0$,

$$\begin{aligned} E_x\Big\{e^{\alpha(t\wedge\tau)}|Y^x_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{\tau} e^{\alpha s}((\alpha-\alpha_1)|Y^x_s|^2 + \varepsilon(1+\varepsilon)^{-1}|Z^x_s|^2)\,ds\Big\} \\ \le E_x\Big\{e^{\alpha\tau}|\varphi(X_\tau)|^2 + \delta^{-1}\int_{t\wedge\tau}^{\tau} e^{\alpha s}|f(X_s,0,0)|^2\,ds\Big\} \end{aligned} \tag{2.3}$$

for all $t \ge 0$, *where* $\alpha_1 = \delta + 2\mu + (1+\varepsilon)L^2$.

Proof. See, e.g., the proof of [4, Corollary 4.4.1]. ∎

For given $x \in D$ let $\mathcal{L}_\alpha(x)$ denote the space of functions having a finite norm

$$\|u\|^2_{\alpha;x} = E_x\int_0^\tau e^{\alpha t}|u(X_t)|^2\,dt = |\!|\!|u(X)|\!|\!|^2_{\alpha;x}$$

and let $\mathcal{W}_\alpha(x)$ denote the space of all elements of $\mathcal{L}_\alpha(x)$ having generalized derivatives $D_i u$ in $\mathcal{L}_\alpha(x)$ and equipped with the norm

$$\|u\|^2_{\mathcal{W}_\alpha(x)} = \|u\|^2_{\alpha;x} + \|\nabla u\|^2_{\alpha;x}\,.$$

Notice that by Fubini's theorem, $\|u\|^2_{\alpha;x} = \int_0^\infty T_t u^2(x)\,dt = U_\alpha u^2(x)$, where $\{T_t, t \ge 0\}$ is the semigroup of positive linear operators on $\mathbb{L}_\infty$ defined by $T_t f(x) = E_x \mathbf{1}_{\{t<\tau\}} e^{\alpha t} f(X_t)$ and U_α is the potential operator for $\{T_t\}$. By (2.1) there is $C = C(\lambda, \Lambda, d)$ such that $T_t f(x) \le Ce^{\alpha t}P_t(\mathbf{1}_D f)(x)$, where $\{P_t, t \ge 0\}$ is the transition semigroup of a standard d-dimensional Wiener process. Hence, by [3, Theorem 3.10], for each $t > 0$, T_t is also a bounded operator from $\mathbb{L}_1$ to $\mathbb{L}_\infty$. Therefore, if $\sup_{x\in D} U_\alpha 1(x) < \infty$, then there is a measurable density $U_\alpha(\cdot,\cdot)$ such that

$$\|u\|^2_{\alpha;x} = \int_D |u(y)|^2 U_\alpha(x,y)\,dy$$

(see, e.g., the proof of Corollary to Theorem 3.18 in [3]). Thus $\mathcal{L}_\alpha(x)$ is in fact an $\mathbb{L}_2$-space with the weight $U_\alpha(x,\cdot)$. The function U_α will be called the *α-Green function* for A in D. In the case where $\alpha = 0$ the α-Green function reduces to the usual Green function which we will denote by G_D. Notice also that if $b = 0$ then the operators T_t are symmetric, and consequently $U_\alpha(x,y) = U_\alpha(y,x)$, $x,y \in D$.

Suppose u is a solution of (1.5) such that $(Y,Z) = (u(X^\tau),(\sigma\nabla u)(X^\tau)) \in \mathcal{H}^{1+d}_{0,x}$ is a solution of (1.6) satisfying (2.2). Then from (2.3) with $t = 0$ it follows that

$$\begin{aligned} |u(x)|^2 + (\alpha-\alpha_1)\|u\|^2_{\alpha;x} + \varepsilon(1+\varepsilon)^{-1}\|\sigma\nabla u\|^2_{\alpha;x} \\ \le E_x e^{\alpha\tau}|\varphi(X_\tau)|^2 + \delta^{-1}\|f(\cdot,0,0)\|^2_{\alpha;x}. \end{aligned} \tag{2.4}$$

Consider now the problem (1.5) with $f(x,y,z) = F(x)$, that is, the linear problem. Suppose u is a solution of it such that (1.8) is a solution to (1.6) in the space $\mathcal{H}^{1+d}_{0,x}$. By Itô's formula, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} |Y^x_0|^2 + \int_0^{\tau\wedge n} |Z^x_s|^2\,ds + 2\int_0^{\tau\wedge n} \langle Y^x_s Z^x_s, \sigma^{-1}(X_s)\,dM_s\rangle \\ = |Y^x_{\tau\wedge n}|^2 + 2\int_0^{\tau\wedge n} F(X_s)Y^x_s\,ds. \end{aligned}$$

Taking expectations and letting $n\to\infty$ gives

$$E_x\Big\{|Y^x_0|^2 + \int_0^\tau |Z^x_t|^2\,ds\Big\} \le E_x\Big\{|Y^x_\tau|^2 + 2\int_0^\tau F(X_t)Y^x_t\,dt\Big\},$$

and so

$$|u(x)|^2 + \|\nabla u\|^2_{0;x} \le E_x|\varphi(X_\tau)|^2 + \|F\|^2_{0;x} + \|u\|^2_{0;x}. \tag{2.5}$$

By (1.6),

$$|u(x)| \le E_x|\varphi(X_\tau)| + E_x\int_0^\tau |F(X_t)|\,dt. \tag{2.6}$$

Obviously $\|u\|_{0;x}^2 \le \|u\|_\infty^2 E_x\tau$, so from (2.5), (2.6) we see that

$$\|\nabla u\|_{0;x}^2 \le (1+2E_x\tau)\|\varphi\|_\infty^2 + \|F\|_{0;x}^2 + 2E_x\tau \sup_{x\in D}\Big(E_x\int_0^\tau |F(X_t)|\,dt\Big)^2. \tag{2.7}$$

3. Linear equations. It is well known that in case $d>1$ for any $\varphi\in W_2^1$ and $F\in\mathbb{L}_2$ the problem

$$Au = -F, \quad u|_{\partial D} = \varphi \tag{3.1}$$

has a unique weak solution in W_2^1, that is, there exists a unique $u\in W_2^1$ such that $u-\varphi\in \mathring{W}_2^1$ and $(a\nabla u,\nabla\eta)_2 - 2(\nabla u,\eta b)_2 = 2(F,\eta)_2$ for all $\eta\in\mathring{W}_2^1$ (see, e.g., [5, Theorem 8.3]). If, in addition, φ satisfies (i) and $F\in\mathbb{L}_p$ with $p>d/2$ then u is continuous on $\overline{D}$ ([5, Theorem 8.31]). In the case where $d=1$ the problem (3.1) has an even more regular weak solution for $F\in\mathbb{L}_p$ with $p=1$ (the substitution $v=au'/2$ reduces $Au=-F$ to the ordinary linear differential equation $v'+2ba^{-1}v=-F$, so one can easily write down an explicit formula for the solution). Our purpose is to show that the condition $F\in\mathbb{L}_p$ can be replaced by

$$\sup_{x\in D}\|F\|_{0;x} < \infty \tag{3.2}$$

and, what is more important, that under (3.2) the pair $(u(X^\tau),(\sigma\nabla u)(X^\tau))$ is a solution of the BSDE (φ,F). To prove this, we will need some auxiliary results.

PROPOSITION 3.1. *If* $\varphi\in C(\partial D)$ *and* F *satisfies* (3.2) *then* $u:\overline{D}\to\mathbb{R}$ *defined by*

$$u(x) = E_x\Big\{\varphi(X_\tau) + \int_0^\tau F(X_t)\,dt\Big\}, \quad x\in\overline{D},$$

is continuous.

Proof. Since D is regular, $P_x(\tau=\overline{\tau})=1$ for $x\in\overline{D}$. Hence, by standard arguments, $\overline{D}\ni x\mapsto E_x\varphi(X_\tau)$ is continuous (see, e.g., the proof of [3, Theorem 1.23]). Therefore we only need to show that if $\{x_n\}\subset\overline{D}$, $x\in\partial D$ and $x_n\to x$ then $E_{x_n}\int_0^\tau F(X_t)\,dt \to E_x\int_0^\tau F(X_t)\,dt = 0$, and for this purpose it suffices to show that $E_{x_n}\tau\to 0$, because by Schwarz's inequality $|E_{x_n}\int_0^\tau F(X_t)\,dt| \le \|F\|_{0,x}(E_{x_n}\tau)^{1/2}$. By [3, Proposition 1.19], for each $t>0$ the function $x\mapsto P_x(\tau>t)$ is upper semicontinuous in $\mathbb{R}^d$, so $\limsup_{n\to\infty}P_{x_n}(\tau>t)\le P_x(\tau>t)=0$. On the other hand, since $P_x(\tau>t)\le e^{-\alpha_0 t/2}\sup_{x\in D}E_x e^{\alpha_0\tau/2}$ for $t\ge 0$, it follows from Remark 2.1 that the functions $t\mapsto P_x(\tau>t)$ are bounded by an integrable function uniformly in $x \in \overline{D}$. Therefore applying Fatou's lemma gives

$$\limsup_{n\to\infty} E_{x_n}\tau = \limsup_{n\to\infty} \int_0^\infty P_{x_n}(\tau > t)\,dt \le \int_0^\infty \limsup_{n\to\infty} P_{x_n}(\tau > t)\,dt = 0,$$

and the proof is complete. ∎

LEMMA 3.2. *Let $x \in D$. If $u_n \to u$ in $\mathbb{L}_2$ and $\{u_n\}$ is bounded in $\mathcal{L}_0(x)$ then $u \in \mathcal{L}_0(x)$ and $\|u\|_{0;x} \le \limsup_{n\to\infty} \|u_n\|_{0;x}$. If, in addition, $\{u_n\}$ is a Cauchy sequence in $\mathcal{L}_0(x)$ then $u_n \to u$ in $\mathcal{L}_0(x)$.*

Proof. Let $\{V_m\}$ be a sequence of open sets such that $x \notin \overline{V}_m \subset V_{m+1} \subset \overline{V}_{m+1} \subset D$ for $m \in \mathbb{N}$ and $\bigcup_{m=1}^\infty V_m = D$. It is known that $G_D(x,\cdot)$ is continuous on $D \setminus \{x\}$ (see [14]). Therefore $G_D(x,\cdot)$ is bounded on V_m, and consequently $\|\mathbf{1}_{V_m}(u_n - u)\|_{0;x}^2 = \int_{V_m} |u_n(y) - u(y)|^2\, G_D(x,y)\,dy \to 0$, that is, $\mathbf{1}_{V_m} u_n \to \mathbf{1}_{V_m} u$ in $\mathcal{L}_0(x)$. On the other hand, $\{u_n\}$ is bounded in $\mathcal{L}_0(x)$ and therefore weakly relatively compact. Suppose $u_{n(k)} \to v$ weakly in $\mathcal{L}_0(x)$ for some subsequence $\{n(k)\}$. Then $\mathbf{1}_{V_m} u_{n(k)} \to \mathbf{1}_{V_m} v$ weakly in $\mathcal{L}_0(x)$ as well. It follows that $\|\mathbf{1}_{V_m} u\|_{0;x} = \|\mathbf{1}_{V_m} v\|_{0;x}$. Since $\mathbf{1}_{V_m} u^2 \uparrow \mathbf{1}_{D\setminus\{x\}} u^2$ and $\mathbf{1}_{V_m} v^2 \uparrow \mathbf{1}_{D\setminus\{x\}} v^2$ pointwise, using the monotone convergence theorem yields $\|u\|_{0;x} = \|v\|_{0,x}$. Let $(\cdot,\cdot)_{0;x}$ denote the scalar product in $\mathcal{L}_0(x)$. Since $(u_{n(k)}, v)_{0;x} \to (v,v)_{0;x}$, it follows that $|(u_{n(k)}, v)_{0;x}| \to (v,v)_{0;x}$, hence

$$\begin{aligned}\|v\|_{0;x}^2 = \limsup_{k\to\infty} |(u_{n(k)}, v)_{0;x}| &\le \limsup_{k\to\infty} \|u_{n(k)}\|_{0;x} \cdot \|v\|_{0;x} \\ &\le \limsup_{n\to\infty} \|u_n\|_{0;x} \cdot \|v\|_{0;x},\end{aligned}$$

which proves the first assertion. The second one follows from the fact that if $u_n \to v$ in $\mathcal{L}_0(x)$ then $\|\mathbf{1}_{V_m}(u - v)\|_{0;x} = 0$ for $m \in \mathbb{N}$. ∎

For $n \in \mathbb{N}$ let $a_n : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b_n : \mathbb{R}^d \to \mathbb{R}^d$ be measurable functions satisfying (1.2) and let $\varphi_n \in W_2^1$ and $F_n \in \mathbb{L}_2$. Set

$$A^n = \frac{1}{2} \sum_{i,j=1}^d D_j(a_n^{ij}(x) D_i) + \sum_{i=1}^d b_n^i(x) D_i \tag{3.3}$$

and let $u_n \in W_2^1$ be a weak solution of the problem

$$A^n u_n = -F_n, \quad u_n|_{\partial D} = \varphi_n. \tag{3.4}$$

In the rest of this section we will assume that $a_n^{ij} \to a^{ij}$ and $b_n^i \to b^i$ a.e. for $i,j = 1,\dots,d$.

The following proposition is undoubtedly known. We provide a proof, because we could not find an appropriate reference.

PROPOSITION 3.3. *If $\varphi_n \to \varphi$ in W_2^1 and $F_n \to F$ in $\mathbb{L}_2$ then $u_n \to u$ in W_2^1, where u is a unique weak solution of the problem* (3.1).

Proof. First we show that there is $C = C(\lambda, \Lambda)$ such that

$$\|u\|_2 \le C\|F\|_2 \tag{3.5}$$

for some $C = C(\lambda, \Lambda)$. We can certainly assume that F is bounded, for if $\{f_k\}$ is a sequence of bounded functions such that $f_k \to F$ in $\mathbb{L}_2$ and u_k is a weak solution of (3.1) with F replaced by f_k, $k \in \mathbb{N}$, then $\|u\|_2 = \lim_{k\to\infty} \|u_k\|_2$ by [5, Corollary 8.7]. Let $\widehat{A} = \sum_{i,j=1}^d D_j\big(\frac{1}{2}a^{ij}(x)D_i - b^j(x)\big)$ be the formal adjoint of A. By [5, Theorem 8.6] there exists a unique weak solution $v \in \mathring{W}_2^1$ of the problem $\widehat{A}v = -1$, $v|_{\partial D} = 0$. Since, by [5, Theorem 8.15], u, v are bounded, it follows that $uv \in \mathring{W}_2^1$, and hence $-2^{-1}(a\nabla u, \nabla(uv))_2 + (\nabla u, uvb)_2 = -(F, uv)_2$. Therefore

$$(a\nabla u, v\nabla u)_2 + (a\nabla u, u\nabla v)_2 - 2(\nabla u, uvb)_2 = 2(F, uv)_2.$$

On the other hand, $-2^{-1}(a\nabla v, \nabla u^2)_2 + (vb, \nabla u^2)_2 = -(u^2, 1)_2$, and consequently

$$\|u\|_2^2 = (a\nabla u, u\nabla v)_2 - 2(\nabla u, uvb)_2.$$

By the above,

$$\|u\|_2^2 + (a\nabla u, v\nabla u)_2 = 2(F, uv)_2, \tag{3.6}$$

and so $\|u\|_2 \le \|v\|_\infty \|F\|_2$, which proves (3.5), because from [5, Theorem 8.16] it follows that v is bounded by a constant depending only on λ, Λ.

Now, write $w_n = u_n - \varphi_n - u + \varphi$. Since $w_n \in \mathring{W}_2^1$,

$$\begin{aligned}
\lambda\|\nabla w_n\|_2^2 &\le (a_n\nabla w_n, \nabla w_n)_2 \\
&= (a_n\nabla u_n, \nabla w_n)_2 - (a_n\nabla\varphi_n, \nabla w_n)_2 \\
&\quad + ((a - a_n)\nabla(u - \varphi), \nabla w_n)_2 - (a\nabla u, \nabla w_n)_2 + (a\nabla\varphi, \nabla w_n)_2 \\
&= 2(F_n - F, w_n)_2 + ((a - a_n)\nabla u, \nabla w_n)_2 + (a_n\nabla(\varphi - \varphi_n), \nabla w_n)_2 \\
&\quad + 2(\nabla w_n, w_n b_n)_2 + 2(\nabla(\varphi_n - \varphi), w_n b)_2 + 2(\nabla u, w_n(b_n - b))_2.
\end{aligned}$$

From this and (3.5) the proposition follows. ∎

Let $\mathbb{X}^n = \{(X, P_x^n);\ x \in \mathbb{R}^d\}$ be a Markov process corresponding to the operator A^n and let M^n denote the martingale additive functional of $\mathbb{X}^n$ of the decomposition of X of the form (1.3).

LEMMA 3.4. *$\mathcal{L}[(X, M^n)\,|\,P_x^n] \Rightarrow \mathcal{L}[(X, M)\,|\,P_x]$ in $C([0,\infty);\mathbb{R}^{2d})$ for each $x \in D$.*

Proof. Since $P_x^n \Rightarrow P_x$ in $C([0,\infty);\mathbb{R}^d)$ (see, e.g., [16]), it suffices to repeat, with some obvious changes, the proof of [11, Lemma 3.2]. ∎

In the proof of the following lemma we combine ideas from the proofs of [3, Theorem A.1] and [17, Lemma 11.1.2].

LEMMA 3.5. *Let $x \in D$. There exists a sequence of regular domains $\{D_m\}$ such that $\{x\} \subset D_m \subset \overline{D}_m \subset D$ for $m \in \mathbb{N}$, $\bigcup_{m=1}^{\infty} D_m = D$ and $\tau_m = \inf\{t \geq 0 : |X_t| \notin D_m\}$ is continuous at ω for P_x-almost all $\omega \in \Omega$.*

Proof. Set $\varepsilon_0 = \operatorname{dist}(x, \partial D)$. For $\varepsilon \in (0, \varepsilon_0)$ let $D_\varepsilon = \{y \in D : \operatorname{dist}(y, \partial D) > \varepsilon\}$ and let E_ε denote the connected component of D_ε containing x. In the proof of [3, Theorem A.1] it is shown that the sets E_ε are regular and $\bigcup_{0<\varepsilon<\varepsilon_0} E_\varepsilon = D$, so it is clear that the lemma will be proved once we show that except for at most countably many ε's from $(0, \varepsilon_0)$, $\sigma_\varepsilon = \inf\{t \geq 0 : |X_t| \notin E_\varepsilon\}$ is continuous for P_x-almost all ω in Ω. Let $\sigma_\varepsilon^+ = \inf\{t \geq 0 : |X_t| \notin \overline{E}_\varepsilon\}$. Since $\sigma_\varepsilon^+ = \lim_{\delta \downarrow \varepsilon} \sigma_\delta$ and $\varepsilon \mapsto E_x \exp(-\sigma_\varepsilon)$ is a non-negative non-increasing function,

$$E_x \exp(-\sigma_\varepsilon^+) = \lim_{\delta \downarrow \varepsilon} E_x \exp(-\sigma_\delta) = E_x \exp(-\sigma_\varepsilon)$$

for all but a countable number of ε's. Since $\sigma_\varepsilon^+ \geq \sigma_\varepsilon$ this shows that $\sigma_\varepsilon^+ = \sigma_\varepsilon$ for all but a countable number of ε's, which is the desired conclusion because σ_ε is lower semicontinuous and σ_ε^+ is upper semicontinuous. ∎

LEMMA 3.6. *For given $x \in D$ let $\{D_m\}$ be the sequence of domains of Lemma 3.5 and let $\{\tau_m\}$ be the sequence of the corresponding stopping times. Assume $F_n, F : D \to \mathbb{R}$ and $G_n = (G_n^1, \ldots, G_n^d), G = (G^1, \ldots, G^d) : \mathbb{R}^d \to \mathbb{R}$ are measurable functions such that $F_n \to F$ and $G_n^i \to G^i$, $i = 1, \ldots, d$, in $\mathbb{L}_2$. Then for any $m, N \in \mathbb{N}$ and $\delta > 0$,*

$$\mathcal{L}\Big[\Big(X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} F_n(X_t)\,dt, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} \langle G_n(X_t),\, dM_t^n\rangle\Big) \,\Big|\, P_x^n\Big]$$
$$\Rightarrow \mathcal{L}\Big[\Big(X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} F(X_t)\,dt, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} \langle G(X_t), dM_t\rangle\Big) \,\Big|\, P_x\Big]$$

in $C([0,\infty); \mathbb{R}^{d+2})$ as $n \to \infty$.

Proof. By Lemmas 3.4 and 3.5,

$$\mathcal{L}[((N \wedge \tau_m) \vee \delta, X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}, M^n) \mid P_x^n] \Rightarrow \mathcal{L}[((N \wedge \tau_m) \vee \delta, X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}, M) \mid P_x]$$

in $\mathbb{R} \times C([0,\infty); \mathbb{R}^{2d})$ as $n \to \infty$. On the other hand, by (2.1), for any $h \in \mathbb{L}_1$,

$$E_x^n \int_\delta^{(N\wedge\tau_m)\vee\delta} |h(X_t)|\,dt \leq C(\lambda, \Lambda) \int_\delta^{N\vee\delta} t^{-d/2}\,dt \int_{D_m} |h(y)|\,dy$$
$$\leq C(\lambda, \Lambda, d, N, \delta)\|h\|_1,$$

and similar estimates hold if we replace E_x^n by E_x. Therefore the lemma follows by the same method as in the proof of [12, Theorem 1]. ■

We are now ready to prove our main result on the Dirichlet problem for linear equations.

THEOREM 3.7. *If φ satisfies* (i) *and $F \in \mathbb{L}_2$ satisfies* (3.2) *then there exists a unique weak solution $u \in W_2^1 \cap C(\overline{D})$ of the problem* (3.1). *Moreover, for each $x \in D$ the pair* (1.8) *is a solution, in $\mathcal{H}_{0;x}^{1+d}$, of the BSDE (φ, F) associated with (X, P_x).*

Proof. We first prove the theorem under the additional assumption that F is bounded. Let $j(x) = c\exp(-1/(1-|x|^2))$ if $|x| < 1$ and $j(x) = 0$ if $|x| \geq 1$, where c is chosen so that $\int_{\mathbb{R}^d} j(x)\,dx = 1$, and let $j_n(x) = n^d j(nx)$. Set

$$a_n^{ij} = a^{ij} * j_n, \qquad b_n^i = b^i * j_n, \qquad \varphi_n = \varphi * j_n, \qquad F_n = F * j_n \tag{3.7}$$

(we first extend φ to a continuous function on $\mathbb{R}^d$ and F to be zero outside D) and let $u_n \in C^2(D) \cap C(\overline{D})$ be a (unique) solution to the problem (3.4). Then for any $n, m, N \in \mathbb{N}$,

$$\begin{aligned} u_n(X_{t\wedge N\wedge\tau_m}) = u_n(X_{N\wedge\tau_m}) &+ \int_{t\wedge N\wedge\tau_m}^{N\wedge\tau_m} F_n(X_s)\,ds \\ &- \int_{t\wedge N\wedge\tau_m}^{N\wedge\tau_m} \langle\nabla u_n(X_s), dM_s^n\rangle, \qquad t \geq 0,\ P_x^n\text{-a.s.} \end{aligned} \tag{3.8}$$

By Proposition 3.3, $u_n \to u$ in W_2^1, where $u \in W_2^1$ is a unique solution of the problem (3.1). On the other hand, by (2.6), $\{u_n\}$ is bounded on D and hence, by De Giorgi's continuity theorem (see [5, Theorem 8.24]), equicontinuous in each compact subset of D. Therefore there is a bounded continuous (on D) version of u (still denoted by u) such that $u_n \to u$ uniformly on compact subsets of D. As a consequence, by Lemma 3.6 and the continuous mapping theorem (see [2, Theorem 5.5]),

$$\begin{aligned} &\mathcal{L}\Big[\Big(u_n(X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}), \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} F_n(X_t)\,dt, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} \langle\nabla u_n(X_t), dM_t^n\rangle\Big) \,\Big|\, P_x^n\Big] \\ &\quad\Rightarrow \mathcal{L}\Big[\Big(u(X_{(\cdot\wedge N\wedge\tau_m)\vee\delta}), \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} F(X_t)\,dt, \int_\delta^{(\cdot\wedge N\wedge\tau_m)\vee\delta} \langle\nabla u(X_t), dM_t\rangle\Big) \,\Big|\, P_x\Big] \end{aligned}$$

in $C([0,\infty); \mathbb{R}^3)$. By the above and (3.8), using once again the continuous mapping theorem, we get

$$u(X_{(t\wedge N\wedge\tau_m)\vee\delta}) = u(X_{(N\wedge\tau_m)\vee\delta}) + \int\limits_{(t\wedge N\wedge\tau_m)\vee\delta}^{(N\wedge\tau_m)\vee\delta} F(X_s)\,ds$$
$$- \int\limits_{(t\wedge N\wedge\tau_m)\vee\delta}^{(N\wedge\tau_m)\vee\delta} \langle\nabla u(X_s), dM_s\rangle, \quad t\geq 0,\ P_x\text{-a.s.}$$

Letting $\delta\downarrow 0$ and then $N\uparrow\infty$ we see that

$$u(X_{t\wedge\tau_m}) = u(X_{\tau_m}) + \int\limits_{t\wedge\tau_m}^{\tau_m} F(X_s)\,ds - \int\limits_{t\wedge\tau_m}^{\tau_m} \langle\nabla u(X_s), dM_s\rangle \quad P_x\text{-a.s.}$$

for $t\geq 0$. Since $\tau_m\uparrow\tau$ P_x-a.s. and F is bounded, $\int_{t\wedge\tau_m}^{\tau_m} F(X_s)\,ds \to \int_{t\wedge\tau}^{\tau} F(X_s)\,ds$ P_x-a.s. Furthermore, by (2.5) and boundedness of $\{u_n\}$, $\{\nabla u_n\}$ is bounded in $\mathcal{L}_0(x)$ for each $x\in D$. From this and Lemma 3.2 we conclude that $\nabla u\in\mathcal{L}_0(x)$ for $x\in D$, hence that $\int_{t\wedge\tau_m}^{\tau_m}\langle\nabla u(X_s), dM_s\rangle \to \int_{t\wedge\tau}^{\tau}\langle\nabla u(X_s), dM_s\rangle$ P_x-a.s. Finally, by Proposition 3.1, u is continuous in $\overline{D}$ and $u=\varphi$ on ∂D. Hence $u(X_{t\wedge\tau_m})\to u(X_{t\wedge\tau})$ and $u(X_{\tau_m})\to\varphi(X_\tau)$ P_x-a.s. Thus, for each $x\in D$,

$$u(X_{t\wedge\tau}) = \varphi(X_\tau) + \int\limits_{t\wedge\tau}^{\tau} F(X_s)\,ds - \int\limits_{t\wedge\tau}^{\tau}\langle\nabla u(X_s), dM_s\rangle \quad P_x\text{-a.s.}$$

for each $t\geq 0$. Since the processes on the left and right-hand side of the above equality are continuous, this proves the proposition for bounded F.

To prove the general case, we set $F_n = (-n)\vee F\wedge n$. By what has already been proved, the pair $(u_n(X^\tau), (\sigma\nabla u_n)(X^\tau))$, where $u_n\in W_2^1\cap C(\overline{D})$ is a solution of the problem $Au_n = -F_n$, $u_n|_{\partial D}=\varphi$, solves the BSDE (φ, F_n). In particular,

$$u_n(X_{t\wedge\tau_m}) = \varphi(X_\tau) + \int\limits_{t\wedge\tau_m}^{\tau} F_n(X_s)\,ds \tag{3.9}$$
$$- \int\limits_{t\wedge\tau_m}^{\tau}\langle\nabla u_n(X_s), dM_s\rangle, \quad t\geq 0,\ P_x\text{-a.s.}$$

Clearly $|F_n - F|\to 0$ and $|F_n-F|\leq F$. Therefore, since F satisfies (3.2), the Lebesgue dominated convergence theorem shows that

$$\lim_{n\to\infty}\|F_n-F\|^2_{0;x} = \lim_{n\to\infty}\int\limits_D |F_n-F|^2(y)G_D(x,y)\,dy = 0 \tag{3.10}$$

for $x\in D$. Similarly, since $F\in\mathbb{L}_2$, we have $F_n\to F$ in $\mathbb{L}_2$, and consequently, by Proposition 3.3, $u_n\to u$ in W_2^1, where u is a unique solution of (3.1). Moreover, for each $x\in D$,

$$\nabla u_n \to \nabla u \quad \text{in } \mathcal{L}_0(x). \tag{3.11}$$

Indeed, from (2.7) it follows that for any $n, m \in \mathbb{N}$,

$$\|\nabla(u_n - u_m)\|_{0;x}^2 \le \|F_n - F_m\|_{0;x}^2 + 2(E_x\tau) \sup_{x\in D} \Big(\int_0^\tau |F_n - F_m|(X_t)\, dt\Big)^2.$$

By (3.10), $\lim_{n,m\to\infty} \|F_n - F_m\|_{0;x}^2 = 0$. At the same time, since $|F_n - F_m| \le |F|$ and $|F_n - F_m| = 0$ if $|F| \le N$ and $n, m \ge N$, for any $n, m \ge N$ we have

$$\begin{aligned}(3.12)\qquad \sup_{x\in D} E_x \int_0^\tau |F_n - F_m|(X_t)\, dt &\le \sup_{x\in D} E_x \int_0^\tau \mathbf{1}_{\{|F|>N\}} |F|(X_t)\, dt \\ &\le \sup_{x\in D} \Big(E_x \int_0^\tau \mathbf{1}_{\{|F|>N\}}(X_t)\, dt \cdot E_x \int_0^\tau |F(X_t)|^2\, dt\Big)^{1/2} \\ &\le N^{-1} \sup_{x\in D} \|F\|_{0;x}^2.\end{aligned}$$

Therefore $\{\nabla u_n\}$ is a Cauchy sequence in $\mathcal{L}_0(x)$. On the other hand, for any V_m defined in the proof of Lemma 3.2 we have

$$\begin{aligned}&\limsup_{n\to\infty} \|\mathbf{1}_{V_m} \nabla(u_n - u)\|_{0;x}^2 \\ &\qquad \le \sup_{y\in V_m} G_D(x, y) \cdot \limsup_{n\to\infty} \int_{V_m} |\nabla(u_n - u)|^2(y)\, dy = 0,\end{aligned}$$

so (3.11) follows. We now observe that by De Giorgi's continuity theorem (see [5, Theorem 8.24]), for any compact subset K of D and any $x, y \in K$ and $N \in \mathbb{N}$ we have

$$\sup_{n\le N} |u_n(x) - u_n(y)| \le C_1 \sup_{n\le N} (\|u_n\|_2 + \lambda^{-1} \|F_n\|_d) |x - y|^\alpha$$

where $C_1 = C_1(\lambda, \Lambda, d, \operatorname{dist}(K, \partial D))$ and $\alpha = \alpha(\lambda, \Lambda, d) > 0$. Moreover,

$$\begin{aligned}&\sup_{n>N} |u_n(x) - u_n(y)| \\ &\quad \le |u_N(x) - u_N(y)| + \sup_{n>N} (|u_n(x) - u_N(x)| + |u_n(y) - u_N(y)|) \\ &\quad \le |u_N(x) - u_N(y)| + 2 \sup_{n>N} \sup_{x\in K} E_x \int_0^\tau |F_n - F_N|(X_t)\, dt.\end{aligned}$$

Analysis similar to that in the proof of (3.12) shows that the second term on the right-hand side of the last inequality is bounded by $2N^{-1} \sup_{x\in K} \|F\|_{0;x}^2$. Since $\|u_n\|_2 \le C_2(\|\varphi\|_{W_2^1} + \|F\|_2)$ for some C_2 not depending on n (see [5, Corollary 8.7] or the proof of Proposition 3.3), putting together the above estimates yields

$$\sup_{n\in\mathbb{N}}|u_n(x)-u_n(y)|\le 2C_1(C_2(\|\varphi\|_{W_2^1}+\|F\|_2)+\lambda^{-1}N|D|^d)|x-y|^\alpha \\ +2N^{-1}\sup_{x\in K}\|F\|_{0;x}^2.$$

This proves that $\{u_n\}$ is equicontinuous in K. By (2.6) and (3.2) it is also bounded on any compact subset of D, and hence, by the Ascoli–Arzelà theorem, converges to u uniformly on any such subset. Using this and (3.10), (3.11), and letting $n\to\infty$ in (3.9) gives

$$u(X_{t\wedge\tau_m})=\varphi(X_\tau)+\int_{t\wedge\tau_m}^{\tau}F(X_s)\,ds-\int_{t\wedge\tau_m}^{\tau}\langle\nabla u(X_t),dM_s\rangle,\quad t\ge 0,$$

P_x-a.s. Hence, letting $m\to\infty$ we obtain (3.9), because $\tau_m\uparrow\tau$ P_x-a.s. and u is continuous on $\overline{D}$. This proves the theorem. ∎

4. Semilinear PDEs and BSDEs. In this section we prove our main theorem on connections between weak solutions of the problem (1.5) and solutions of BSDEs of the form (1.6). We begin with results on uniqueness, existence and regularity of solutions of (1.5).

Before proving the first theorem we observe that if $\sup_x E_xe^{\alpha\tau}<\infty$ then $U_\alpha 1$ is a bounded weak solution of the problem $(\alpha+A)U_\alpha 1=-1$, $U_\alpha 1|_{\partial D}=0$. This follows from the fact that for bounded f the function G_Df is a weak solution of the problem $AG_Df=-f$, $G_Df|_{\partial D}=0$ and $G_D(\alpha U_\alpha 1)=U_\alpha 1-G_D1$, because using the Markov property and integration by parts gives

$$E_x\int_0^\tau\alpha\Big(E_{X_t}\int_0^\tau e^{\alpha s}\,ds\Big)dt=E_x\int_0^\tau E_x((e^{\alpha\tau}-1)\,|\,\mathcal{F}_t)\alpha e^{-\alpha t}\,dt \\ =E_x(e^{\alpha\tau}-1).$$

THEOREM 4.1. *Assume that $\varphi\in W_2^1$ and f satisfies* (ii)–(v). *If either $\mu+L^2\le 0$, or $b=0$ and $\sup_{x\in D}E_xe^{\alpha\tau}<\infty$ for some $\alpha\ge 2\mu+L^2$, then the problem* (1.5) *cannot have more than one weak solution in W_2^1.*

Proof. Suppose that $u_1,u_2\in W_2^1$ are weak solutions of (1.5). Then $u=u_1-u_2\in\mathring{W}_2^1$ and u solves the problem (3.1) with $F=F_{u_1}-F_{u_2}$, $\varphi=0$. Let $\{F_n\}$ be a sequence of bounded measurable functions such that $F_n\to F$ in $\mathbb{L}_2$, and for $n\in\mathbb{N}$ let $w_n\in\mathring{W}_2^1$ be a weak solution of the problem $Aw_n=-F_n$, $w_n|_{\partial D}=0$. Assume that $b=0$. As in the proof of (3.6) one can show that

$$\|w_n\|_2^2+\alpha(w_n,w_nU_\alpha 1)_2+(a\nabla w_n,U_\alpha 1\nabla w_n)_2=2(F_n,w_nU_\alpha 1)_2.$$

By Proposition 3.3, $w_n \to u$ in W^1_2, so letting $n \to \infty$ we obtain

$$\|u\|_2^2 + \alpha(u, uU_\alpha 1)_2 + (a\nabla u, U_\alpha 1\nabla u)_2 = 2(F, uU_\alpha 1)_2. \tag{4.1}$$

On the other hand, by (iii), (iv),

$$\begin{aligned} 2(F, uU_\alpha 1)_2 &\le 2\mu(u, uU_\alpha 1)_2 + 2L(|u|\cdot|\sigma\nabla u|, U_\alpha 1)_2 \\ &\le 2\mu(u, uU_\alpha 1)_2 + (a\nabla u, U_\alpha 1\nabla u)_2 + L^2(u, uU_\alpha 1)_2. \end{aligned}$$

Hence $\|u\|_2^2 + \alpha(u, uU_\alpha 1)_2 \le (2\mu + L^2)(u, uU_\alpha 1)_2$, and the theorem follows.

Consider now the case $b \neq 0$. Let $\widehat{A}$ be the formal adjoint of A and let $\widehat{G}_D 1(x) = \int_D G(y,x)\,dy$, $x \in D$. Since $\widehat{G}_D 1 \in \mathring{W}^1_2$ is a weak solution of the problem $\widehat{A}\widehat{G}_D 1 = -1$, $\widehat{G}_D 1|_{\partial D} = 0$, analysis similar to that in the proof of (4.1) shows that

$$\|u\|_2^2 + (a\nabla u, \widehat{G}_D 1\nabla u)_2 = 2(F, u\widehat{G}_D 1)_2. \tag{4.2}$$

Hence $\|u\|_2^2 \le (2\mu + L^2)(u, u\widehat{G}_D 1)_2$, and the proof is complete. ∎

THEOREM 4.2. *Assume* (i)–(v) *and* (vi′) *or* (vi″). *Then there exists a* (*unique*) *solution* $u \in W^1_2 \cap C(\overline{D})$ *of the problem* (1.5). *Moreover*, $u \in \mathcal{W}_{\alpha\vee 0}(x)$ *for each* $x \in D$ *and* $\sup_{x\in D}\|F_u\|_{\alpha\vee 0;x} < \infty$ *with* $\alpha = 0$ *in the first case and any* α *satisfying condition* (vi″) *in the second one.*

Proof. By Remark 2.1 we may and will assume that $\alpha > 0$. Throughout the proof $\gamma = \alpha$ if $b = 0$ and $2\mu + L^2 > 0$, and $\gamma = \alpha_0/q'$ if $2\mu + L^2 \le 0$, where α_0 is the constant from Remark 2.1 and q' is the Hölder conjugate of some $q \in (1, p/(2\vee d))$. Moreover, we write $\alpha_1 = \delta + 2\mu + (1+\varepsilon)L^2$, where $\delta, \varepsilon > 0$ are chosen so that $\alpha_1 < \gamma$, $\delta + \varepsilon L^2 < \alpha_0/2$ and $\varepsilon L^2\|G_D 1\|_\infty < 1$.

The proof will be divided into 5 steps.

Step 1. We first prove that for any square-integrable $w : D \to \mathbb{R}^d$ such that $\sup_{x\in D}\|w\|_{\mathcal{W}_\gamma(x)} < \infty$ the problem

$$Av = -\widetilde{F}_v, \quad v|_{\partial D} = \varphi, \tag{4.3}$$

where $\widetilde{F}_v(x) = \widetilde{f}(x, v(x))$, $\widetilde{f}(x,y) = f(x,y,w(x))$, $x \in D$, $y \in \mathbb{R}$, is uniquely solvable in W^1_2. For this purpose, for $n \in \mathbb{N}$ we define a_n, b_n by (3.7), and $h_n = (h_n^1, \dots, h_n^d) : D \to \mathbb{R}^d$ and $f_n : D\times\mathbb{R} \to \mathbb{R}$ by

$$h_n^i(x) = (-n) \vee w^i(x) \wedge n, \qquad i = 1, \dots, d, \qquad f_n(x,y) = f(x,y,h_n(x)).$$

Let $\{(X, P_x^n); x \in \mathbb{R}^d\}$ be a diffusion corresponding to A^n defined by (3.3). By [4, Theorem 3.4] (see also [8, Theorem 3.1]), for each $x \in D$ there is a unique solution $(Y^{x,n}, Z^{x,n})$ of the BSDE $(\varphi, \widetilde{f})$ associated with (X, P_x^n), and moreover, by [4, Lemma 6.2], $Y^{x,n} = v_n(X^\tau)$ P_x^n-a.s., where $v_n(x) = Y_0^{x,n}$ for $x \in D$. We will show that v_n is a unique solution of the problem

$$A^n v_n = -\widetilde{F}_n, \quad v_n|_{\partial D} = \varphi \tag{4.4}$$

in W^1_2, where $\widetilde{F}_n : D \to \mathbb{R}$, $\widetilde{F}_n(x) = f_n(x, v_n(x))$. Since $\alpha_1 < \gamma$, by (2.4) applied to v_n we have

$$(4.5)\qquad |v_n(x)|^2 \le \|\varphi\|^2_\infty E^n_x e^{\gamma\tau} + 2\delta^{-1}(\|g\|^2_{\gamma;x} + L^2\|w\|^2_{\gamma;x}).$$

If $2\mu + L^2 > 0$ then $\|g\|^2_{\gamma;x} \le U_\gamma(x)\|g\|_\infty$, and if $2\mu + L^2 \le 0$ then by Hölder's inequality,

$$\|g\|^2_{\gamma;x} \le (U_{\gamma q'}(x))^{1/q'}\Big(\int_D (G_D(x,y))^{(p/2q)'}\,dy\Big)^{1/(p/2q)'q}\|g\|^2_p,$$

where $(p/2q)'$ is the Hölder conjugate to $p/2q$. By [13, Corollary A.1.2] and [5, Theorem 8.16] (for $d = 1$ see remarks at the beginning of Section 3), $\sup_{x\in D}\|G_D(x,\cdot)\|_r < \infty$ for any $r > 1$ whose Hölder conjugate is greater than $(2\vee d)/2$. Hence, in both cases, $\sup_{x\in D}\|g\|^2_{\gamma;x} < \infty$. From this and (4.5) we see that $v_n \in \mathbb{L}_2$, hence $\widetilde{F}_n \in \mathbb{L}_2$. Set now $\varphi_k = \varphi * j_k$ and $\widetilde{F}_{n,k} = ((-k)\vee \widetilde{F}_n \wedge k) * j_k$ for $k \in \mathbb{N}$, where j_k is defined as in the proof of Theorem 3.7. Since D is regular, $\varphi_k \in C^\beta(\overline{D}) = C^{0,\beta}(\overline{D})$ and $\widetilde{F}_{n,k}$ are bounded and belong to $C^\beta(D)$, it follows from [5, Theorem 6.11] that there is a unique solution $v_{n,k} \in C^{2,\beta}(D)\cap C(\overline{D})$ of the problem $A^n v_{n,k} = -\widetilde{F}_{n,k}$, $v_{n,k}|_{\partial D} = \varphi_k$. By Itô's formula, $(Y^{n,k}_t, Z^{n,k}_t) = (v_{n,k}(X^\tau_t), (\sigma_n\nabla v_{n,k})(X^\tau_t))$, $t \ge 0$, is a solution of the BSDE $(\varphi_k, \widetilde{F}_{n,k})$ associated with (X, P^n_x). Therefore

$$(4.6)\qquad \begin{aligned} Y^{n,k}_t &= \varphi_k(X_\tau) + \int_{t\wedge\tau}^{\tau} \widetilde{F}_{n,k}(X_s)\,ds \\ &\quad - \int_{t\wedge\tau}^{\tau} \langle Z^{n,k}_s, \sigma^{-1}_n(X_s)dM^n_s\rangle, \qquad t\ge 0,\ P^n_x\text{-a.s.} \end{aligned}$$

Obviously $\varphi_k \to \varphi$ uniformly on $\overline{D}$ and $\widetilde{F}_{n,k} \to \widetilde{F}_n$ in $\mathbb{L}_2$ as $k \to \infty$. Hence, by Proposition 3.3, $v_{n,k} \to \widetilde{v}_n$ in W^1_2, where $\widetilde{v}_n \in W^1_2$ is a unique solution of the problem $A^n\widetilde{v}_n = -\widetilde{F}_n$, $\widetilde{v}_n|_{\partial D} = \varphi$. On the other hand, by (2.6) we have

$$|v_{n,k}(x)| \le \|\varphi\|_\infty + \|g^{1/2} * j_k\|_{0;x} + K\|v_n\|_\infty + Ln,$$

since $\|\varphi_k\|_\infty \le \|\varphi\|_\infty$ and $|\widetilde{F}_{n,k}| \le |\widetilde{F}_n| * j_k$. Consequently, $\{v_{n,k}\}_{k\in\mathbb{N}}$ is bounded in D. Using the above properties of $\{\varphi_k\}_{k\in\mathbb{N}}$, $\{\widetilde{F}_{n,k}\}_{k\in\mathbb{N}}$, $\{v_{n,k}\}_{k\in\mathbb{N}}$ and the fact that $\widetilde{F}_n \in \mathcal{L}_0(x)$ for $x \in D$, as in the proof of (3.9) we deduce from (4.6) that there is a continuous version $\overline{v}_n$ of $\widetilde{v}_n$ such that

$$\overline{v}_n(X_t) = \varphi(X_\tau) + \int_{t\wedge\tau}^{\tau} \widetilde{F}_n(X_s)\,ds - \int_{t\wedge\tau}^{\tau} \langle\nabla\overline{v}_n(X_s), dM^n_s\rangle, \qquad t\ge 0,$$

P^n_x-a.s. At the same time,

$$Y_t^{x,n} = \varphi(X_\tau) + \int_{t\wedge\tau}^{\tau} \widetilde{F}_n(X_s)\,ds - \int_{t\wedge\tau}^{\tau} \langle Z_s^{x,n}, \sigma_n^{-1}(X_s)dM_s^n\rangle, \qquad t \ge 0,$$

P_x^n-a.s. Putting

$$\xi_t = Y_t^{x,n} - \overline{v}_n(X_t), \quad N_t = \int_0^t \langle Z_s^{x,n} - (\sigma_n\nabla\overline{v}_n)(X_s), \sigma_n^{-1}(X_s)dM_s^n\rangle$$

we have $\xi_t = N_{t\wedge\tau} - N_\tau$, $t \ge 0$, and hence

$$\xi_t = \xi_{t\wedge\tau} = E_x^n(\xi_{t\wedge\tau}\,|\,\mathcal{F}_{t\wedge\tau}) = E_x^n(-N_\tau\,|\,\mathcal{F}_{t\wedge\tau}) + N_{t\wedge\tau} = 0.$$

Therefore $\overline{v}_n(X_t^\tau) = Y_t^{x,n}$, $t \ge 0$, P_x^n a.s., and consequently $\overline{v}_n(x) = \overline{v}_n(X_0^\tau) = Y_0^{x,n} = v_n(x)$.

From what has already been proved it follows that v_n is a unique, in W_2^1, weak solution of the problem (4.4), and moreover, $(v_n(X^\tau), (\sigma_n\nabla v_n)(X^\tau))$ is a solution of the BSDE $(\varphi, \widetilde{f})$ associated with (X, P_x^n). By the definition of a weak solution, $v_n - \varphi \in \mathring{W}_2^1$ and

$$\begin{aligned}(a_n\nabla(v_n-\varphi), \nabla\eta)_2 = 2(\widetilde{F}_n, \eta)_2 - (a_n\nabla\varphi, \nabla\eta)_2 + 2(\nabla(v_n-\varphi), \eta b_n)_2 \\ + 2(\nabla\varphi, \eta b_n)_2\end{aligned}$$

for any $\eta \in \mathring{W}_2^1$. Putting $\eta = v_n - \varphi$ we obtain

$$\begin{aligned}&(a_n\nabla(v_n-\varphi), \nabla(v_n-\varphi))_2 \\ &\quad\le 2(\widetilde{F}_n - \widetilde{F}_\varphi, v_n-\varphi)_2 + 2(\widetilde{F}_\varphi, v_n-\varphi)_2 + \Lambda\|\nabla\varphi\|_2\cdot\|\nabla(v_n-\varphi)\|_2 \\ &\qquad + 2\Lambda\|\nabla(v_n-\varphi)\|_2\cdot\|v_n-\varphi\|_2 + 2\Lambda\|\nabla\varphi\|_2\cdot\|v_n-\varphi\|_2 \\ &\quad\le 2\mu\|v_n-\varphi\|_2^2 + 2(\|g\|_2 + K\|\varphi\|_2 + L\|w\|_2)\cdot\|v_n-\varphi\|_2 \\ &\qquad + \lambda^{-1}\Lambda^2\|\nabla\varphi\|_2^2 + 4\lambda^{-1}\Lambda^2\|v_n-\varphi\|_2^2 + 2^{-1}\lambda\|\nabla(v_n-\varphi)\|_2^2 \\ &\qquad + 2\Lambda\|\nabla\varphi\|_2\cdot\|v_n-\varphi\|_2,\end{aligned}$$

and hence $\|\nabla(v_n-\varphi)\|_2 \le C(\|g\|_2 + \|\varphi\|_{W_2^1} + \|w\|_2 + \|v_n-\varphi\|_2)$ for some $C = C(\lambda, \Lambda, \mu, K, L)$. From this and (4.5) we see that $\{v_n-\varphi\}_{n\in\mathbb{N}}$ is bounded in $\mathring{W}_2^1$, and therefore, by Rellich's theorem (see, e.g., [5, Theorem 7.22]), precompact in $\mathbb{L}_2$. Suppose that for some subsequence (still denoted by n) $v_n \to v$ in $\mathbb{L}_2$. Then $\widetilde{F}_{v_n} \to \widetilde{F}_v$ in $\mathbb{L}_2$ by [6, Theorem 2.1]. From this and Proposition 3.3 we conclude that v is a weak solution of (4.3).

Step 2. There exists a version u of v such that u is continuous on $\overline{D}$, $\sup_{x\in D}\|u\|_{\mathcal{W}_\gamma(x)} < \infty$ and for each $x \in D$ the pair (1.8) is a solution, in $\mathcal{H}_{\gamma;x}^{1+d}$, of the BSDE $(\varphi, \widetilde{f})$ associated with (X, P_x). Indeed, by Lemma 3.2 and (4.5), $\sup_{x\in D}\|v\|_{0;x} < \infty$. Hence $\sup_{x\in D}\|\widetilde{F}_v\|_{0;x} < \infty$, and so, by Theorem 3.7, there is a unique solution $u \in W_2^1 \cap C(\overline{D})$ of (3.1) with F replaced by $\widetilde{F}_v$. Since the problem (3.1) is uniquely solvable in W_2^1, it follows that

$u = v$ a.e. Moreover, again by Theorem 3.7, the pair (1.8) solves the BSDE $(\varphi, \widetilde{f})$ in $\mathcal{H}^{1+d}_{0;x}$. Since (2.2) is satisfied and $\sup_{x\in D}\|\widetilde{f}(\cdot,0)\|_{\gamma;x} < \infty$, it follows from Proposition 2.3 that u is in fact a solution of class $\mathcal{H}^{1+d}_{\gamma;x}$ for every x and its norms in $\mathcal{H}^{1+d}_{\gamma;x}$ are bounded uniformly in $x \in D$.

Step 3. There is a weak solution $u \in W^1_2$ of (1.5). To prove this, denote by V the space W^1_2 equipped with the norm

$$\|u\|_V^2 = \begin{cases} \|u\|_2^2 + (\alpha-\alpha_1)(u, uU_\alpha 1)_2 + (a\nabla u, U_\alpha 1\nabla u)_2 & \text{if } b = 0, \\ \|u\|_2^2 + (a\nabla u, \widehat{G}_D 1\nabla u)_2 & \text{if } b \neq 0, \end{cases}$$

and let $V_\gamma = \{u \in V : \sup_{x\in D}\|u\|_{\mathcal{W}_\gamma(x)} < \infty\}$. Define the mapping $\varPhi : V_\gamma \to V$ by letting $\varPhi(\widetilde{w})$ be the solution of (4.3) with $w = \sigma\nabla\widetilde{w}$. By what has already been proved, $\varPhi$ is well defined and in fact $\varPhi : V_\gamma \to V_\gamma$. Set $\varPhi(w_i) = v_i$, $i = 1, 2$, $v = v_1 - v_2$, $w = w_1 - w_2$ and $F = f(\cdot, v_1, \sigma\nabla w_1) - f(\cdot, v_2, \sigma\nabla w_2)$. If $b = 0$ then as in the proof of Theorem 4.1 we show that (4.1) holds with v in place of u. On the other hand,

$$\begin{aligned} 2(F, vU_\alpha 1)_2 &= 2(f(\cdot, v_1, \sigma\nabla w_1) - f(\cdot, v_2, \sigma\nabla w_1), vU_\alpha 1)_2 \\ &\quad + 2(f(\cdot, v_2, \sigma\nabla w_1) - f(\cdot, v_2, \sigma\nabla w_2), vU_\alpha 1)_2 \\ &\le 2\mu(v, vU_\alpha 1)_2 + 2L(|\sigma\nabla w|, |vU_\alpha 1|)_2 \\ &\le (2\mu + (1+\varepsilon)L^2)(v, vU_\alpha 1)_2 + (1+\varepsilon)^{-1}(a\nabla w, U_\alpha 1\nabla w)_2, \end{aligned}$$

the last inequality being a consequence of an elementary inequality $2ab \le \eta a^2 + \eta^{-1}b^2$ with $\eta = (1+\varepsilon)L$. Hence

$$\begin{aligned} &\|v\|_2^2 + \alpha(v, vU_\alpha 1)_2 + (a\nabla v, U_\alpha 1\nabla v)_2 \\ &\qquad \le (2\mu + (1+\varepsilon)L^2)(v, vU_\alpha 1)_2 + (1+\varepsilon)^{-1}(a\nabla w, U_\alpha 1\nabla w)_2. \end{aligned}$$

If $b \neq 0$ then (4.2) holds with v in place of u and, since $2\mu \le -L^2$, we have

$$(F, v\widehat{G}_D 1)_2 \le \varepsilon L^2\|\widehat{G}_D 1\|_\infty\|v\|_2^2 + (1+\varepsilon)^{-1}(a\nabla w, \widehat{G}_D 1\nabla w)_2.$$

Thus, in both cases $\varPhi$ is contractive. Set $u_0 = 0$, $u_n = \varPhi(u_{n-1})$, $n \in \mathbb{N}$. Then $\{u_n\}$ is a Cauchy sequence in V and hence bounded in $\mathbb{L}_2$. In fact, it is bounded in W^1_2. Indeed, we have

$$\begin{aligned} \lambda\|\nabla(u_n - \varphi)\|_2^2 &\le (a\nabla(u_n - \varphi), \nabla(u_n - \varphi))_2 \\ &= -(a\nabla\varphi, \nabla(u_n - \varphi))_2 + 2(\nabla u_n, (u_n - \varphi)b)_2 \\ &\quad + 2(f(\cdot, u_n, \sigma\nabla u_{n-1}), u_n - \varphi)_2 \\ &\le 4^{-1}\lambda\|\nabla(u_n - \varphi)\|_2^2 + \lambda^{-1}\varLambda^2\|\nabla\varphi\|_2^2 \\ &\quad + 2^{-1}\lambda\|\nabla(u_n - \varphi)\|_2^2 + 2^{-1}\lambda\|\nabla\varphi\|_2^2 + 4\lambda^{-1}\varLambda^2\|u_n - \varphi\|_2^2 \\ &\quad + 2(g + K|u_n| + L|\sigma\nabla u_{n-1}|, |u_n - \varphi|)_2. \end{aligned}$$

Hence $\|\nabla u_n\|_2^2 \le C_1 + C_2\|\nabla u_{n-1}\|_2$ for some C_1, C_2 depending only on $\lambda, \Lambda, K, L, \|g\|_2$ and $\sup_{n\ge 0}\|u_n\|_2$, which implies boundedness of $\{u_n\}$ in W_2^1. Therefore $\{u_n\}$ is weakly relatively compact in W_2^1. Since we know that $\{u_n\}$ is Cauchy in V as well, it converges in V to some $\widetilde{u}$.

Step 4. $\widetilde{u} \in V_\gamma$. To prove this, for fixed $x \in D$ we denote by $\mathcal{V}(x)$ the space $\mathcal{W}_\gamma(x)$ equipped with a norm $\|\cdot\|_{\mathcal{V}(x)}$ equivalent to $\|\cdot\|_{\mathcal{W}_\gamma(x)}$, defined by

$$\|u\|_{\mathcal{V}(x)}^2 = (\gamma - \alpha_1)\|u\|_{\gamma;x}^2 + \|\sigma\nabla u\|_{\gamma;x}^2.$$

Define $\{u_n\}$ as in Step 3. By Step 2, $\Phi : \mathcal{V}(x) \to \mathcal{V}(x)$ and for each $n \in \mathbb{N}$ the pair $(w_{n+1}(X_\cdot^\tau), (\sigma\nabla w_{n+1})(X_\cdot^\tau))$, where $w_n = u_n - u_{n-1}$, is a solution, in $\mathcal{H}_{\gamma;x}^{1+d}$, of the BSDE $(0, F_{n+1} - F_n)$ with $F_n = f(\cdot, u_n, \sigma\nabla u_{n-1})$. By Itô's formula,

$$0 = e^{\gamma\tau}|w_{n+1}(X_\tau)|^2 = |w_{n+1}(X_0)|^2 + \gamma\int_0^\tau e^{\gamma s}|w_{n+1}(X_s)|^2\,ds$$
$$+ \int_0^\tau e^{\gamma s} w_{n+1}(X_s)\,d(w_{n+1}(X_s)) + \int_0^\tau e^{\gamma s}\,d\langle w_{n+1}(X_\cdot)\rangle_s.$$

Hence

$$E_x\int_0^\tau e^{\gamma t}\{\gamma|w_{n+1}(X_t)|^2 + (a\nabla w_{n+1}, \nabla w_{n+1})(X_t)\}\,dt$$
$$\le 2E_x\int_0^\tau e^{\gamma t}(F_{n+1} - F_n)\cdot w_{n+1}(X_t)\,dt.$$

We have

$$\begin{aligned}
2(F_{n+1} - F_n)\cdot w_{n+1} &= 2(f(\cdot, u_{n+1}, \sigma\nabla u_n) - f(\cdot, u_n, \sigma\nabla u_{n-1}))\cdot w_{n+1} \\
&= 2(f(\cdot, u_{n+1}, \sigma\nabla u_n) - f(\cdot, u_n, \sigma\nabla u_n))\cdot w_{n+1} \\
&\quad + 2(f(\cdot, u_n, \sigma\nabla u_n) - f(\cdot, u_n, \sigma\nabla u_{n-1}))\cdot w_{n+1} \\
&\le 2\mu|w_{n+1}|^2 + 2L|\sigma\nabla w_n|\cdot|w_{n+1}| \\
&\le (2\mu + (1+\varepsilon)L^2)|w_{n+1}|^2 + (1+\varepsilon)^{-1}(a\nabla w_n, \nabla w_n).
\end{aligned}$$

By the above,

$$E_x\int_0^\tau e^{\gamma t}\{(\gamma - 2\mu - (1+\varepsilon)L^2)|w_{n+1}(X_t)|^2 + (a\nabla w_{n+1}, \nabla w_{n+1})(X_t)\}\,dt$$
$$\le (1+\varepsilon)^{-1}E_x\int_0^\tau e^{\gamma t}(a\nabla w_n, \nabla w_n)(X_t)\,dt,$$

which shows that $\|w_{n+1}\|_{\mathcal{V}(x)}^2 \le (1+\varepsilon)^{-1}\|w_n\|_{\mathcal{V}(x)}^2$. Hence $\{u_n\}$ is a Cauchy sequence in $\mathcal{V}(x)$ and therefore converges to some u. In particular, $u_n \to u$

in $\mathcal{L}_0(x)$. On the other hand, since $u_n \to \widetilde{u}$ in V, $u_n \to \widetilde{u}$ in $\mathbb{L}_2$, and hence $u = \widetilde{u}$ a.e. by Lemma 3.2. Consequently, $u_n \to \widetilde{u}$ in $\mathcal{V}(x)$ and

$$\|u_1 - \widetilde{u}\|_{\mathcal{V}(x)} \le q(1-q)^{-1}\|u_1 - u_0\|_{\mathcal{V}(x)} \quad \text{with } q = (1+\varepsilon)^{-1/2}.$$

Hence $\|\widetilde{u}\|_{\mathcal{V}(x)} \le (1-q)^{-1}\|u_1\|_{\mathcal{V}(x)}$, and so $u \in V_\gamma$ since $u_1 \in V_\gamma$.

Step 5. By Step 4, $\Phi(\widetilde{u})$ is well defined. Therefore $\widetilde{u}$ is a fixed point of Φ, because $\|\widetilde{u} - \Phi(\widetilde{u})\|_V \le \|\widetilde{u} - u_n\|_V + \|\Phi(u_{n-1}) - \Phi(\widetilde{u})\|_V$ for $n \in \mathbb{N}$. Consequently, $\widetilde{u} \in W_2^1$ is a weak solution of (3.2). Since $F_{\widetilde{u}} \in V_\gamma$ it follows from Theorems 3.7 and 4.1 that $\widetilde{u}$ has a version $u \in C(\overline{D})$, which completes the proof. ∎

From the above proof it is evident that if $a \in C^{1,\beta}(\overline{D})$ for some $\beta \in (0,1)$ then in case $b = 0$ condition (vi$''$) may be replaced by $\sup_{x\in D} E_x e^{\alpha\tau} < \infty$ (in Step 1 we need not approximate a by $\{a_n\}$). We do not know whether (vi$''$) may be replaced by the latter condition in the case of arbitrary a satisfying (1.2).

THEOREM 4.3. *Assume* (i)–(vi) *and define* α *as in Theorem* 4.2. *If* $u \in W_2^1 \cap C(\overline{D})$ *is a solution of* (1.5) *then for each* $x \in D$ *the pair* (1.8) *is a unique, in the class* $\mathcal{H}^{1+d}_{\alpha\vee 0;x}$, *solution of the BSDE* (φ, f) *associated with* (X, P_x).

Proof. This follows from Theorems 3.7 and 4.2. ∎

References

[1] V. Bally, E. Pardoux and L. Stoica, *Backward stochastic differential equations associated to a symmetric Markov process*, Potential Anal. 22 (2005), 17–60.

[2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[3] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer, Berlin, 1995.

[4] R. W. R. Darling and E. Pardoux, *Backwards SDE with random terminal time and applications to semilinear elliptic PDE*, Ann. Probab. 25 (1997), 1135–1159.

[5] G. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983.

[6] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964.

[7] A. Lejay, *BSDE driven by Dirichlet process and semi-linear parabolic PDE. Application to homogenization*, Stochastic Process. Appl. 97 (2002), 1–39.

[8] É. Pardoux, *Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order*, in: Stochastic Analysis and Related Topics VI (Geilo, 1996), L. Decreusefond *et al.* (eds.), Birkhaüser, Boston, 1998, 79–127.

[9] S. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stochastics Stochastics Rep. 37 (1991), 61–74.

[10] A. Rozkosz, *On Dirichlet processes associated with second order divergence form operators*, Potential Anal. 14 (2001), 123–149.

[11] —, *Backward SDEs and Cauchy problem for semilinear equations in divergence form*, Probab. Theory Related Fields 125 (2003), 393–407.

[12] A. Rozkosz and L. Słomiński, *On existence and stability of weak solutions of multidimensional stochastic differential equations with measurable coefficients*, Stochastic Process. Appl. 37 (1991), 187–197.

[13] B. Simon, *Schrödinger semigroup*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 447–526.

[14] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier 15 (1965), no. 1, 189–258.

[15] I. L. Stoica, *A probabilistic interpretation of the divergence and BSDE's*, Stochastic Process. Appl. 103 (2003), 31–55.

[16] D.W. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, in: Séminaire de Probabilités XXII, Lecture Notes in Math. 1321, Springer, Berlin, 1988, 316–347.

[17] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, New York, 1979.

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