BSDEs with random terminal time and semilinear elliptic PDEs in divergence form

by

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Abstract. We study connections between Sobolev space solutions of the Dirichlet problem for semilinear second order elliptic equations in divergence form and solutions of backward stochastic differential equations with random terminal time.

1. Introduction. In Peng [9] and Darling and Pardoux [4] (see also the expository paper by Pardoux [8] and the references given therein) connections between solutions of backward stochastic differential equations (BSDEs) with random terminal time and viscosity solutions of the Dirichlet problem for semilinear second order elliptic equations in non-divergence form are investigated. In the present paper we are interested in finding connections between Sobolev space solutions of the semilinear Dirichlet problem and solutions of BSDEs with random terminal time in the case when the forward driving process is a diffusion generated by a uniformly elliptic divergence form operator with measurable coefficients. We emphasize that it is not a semimartingale generally.

To be more precise, let us fix a bounded domain (non-empty, open, connected set) $D$ in $\mathbb{R}^d$ and consider the differential operator $A$ of the form

$$
A = \frac{1}{2} \sum_{i,j=1}^{d} \partial^2_{ij}(x) \partial_i \partial_j + \sum_{i=1}^{d} b^i(x) \partial_i
$$

where $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : D \to \mathbb{R}^d$ are measurable functions such that for some $0 < \lambda \leq \Lambda$,

$$
\lambda |\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad a^{ij}(x) = a^{ji}(x), \quad |b(x)| \leq \Lambda
$$

for $x \in D$ and $\xi \in \mathbb{R}^d$. By putting $a^{ij} = \delta^i_j$, $b^i = 0$ outside $D$ we can and will
assume that $a, b$ are defined and satisfy (1.2) in all $\mathbb{R}^d$. Let $p(\cdot, \cdot, \cdot)$ denote a weak fundamental solution for $A$ and let $\mathbb{X} = \{(X, P_x); x \in \mathbb{R}^d\}$ be a Markov process for which $p$ is a transition density function, that is,

$$P_x(X_0 = x) = 1, \quad P_x(X_t \in \Gamma) = \int_{\Gamma} p(t, x, y) dy, \quad t \geq 0,$$

for any Borel $\Gamma \subset \mathbb{R}^d$ (see [16]). From [10, Theorem 3.4] it follows that for each starting point $x \in \mathbb{R}^d$ the canonical process $X$ admits a unique decomposition of the form

$$X_t - X_0 = M_t + A_t, \quad t \geq 0, \quad P_x\text{-a.s.},$$

where $M$ is a continuous martingale additive functional (in the strict sense) of $\mathbb{X}$ and $A$ is a continuous additive functional (in the strict sense) of $\mathbb{X}$ of zero quadratic variation. Moreover, the co-variation process of $M = (M^1, \ldots, M^d)$ is given by

$$\langle M^i, M^j \rangle_t = \int_0^t a^{ij}(X_s) ds, \quad t \geq 0.$$  

Suppose that $D$ is regular for $A$. For given $\varphi : \partial D \rightarrow \mathbb{R}$ and $f : D \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ consider the semilinear Dirichlet problem

$$Au = -F_u, \quad u|_{\partial D} = \varphi,$$

where

$$F_u(x) = f(x, u(x), (\sigma \nabla u)(x)), \quad x \in D,$$

and $\sigma$ is the symmetric square root of $a$. In this paper we show that under some natural assumptions on $\varphi$ and $f$ there exists a bounded continuous solution $u$ of (1.5) in the Sobolev space $W^1_0$ and for each $x \in D$ there also exists a unique solution $(Y^x, Z^x)$ of the BSDE

$$Y^x_t = \varphi(X^x_\tau) + \int_{t \wedge \tau}^\tau f(X^x_s, Y^x_s, Z^x_s) ds$$

$$- \int_{t \wedge \tau}^\tau \langle Z^x_s, \sigma^{-1}(X^x_s) dM_s^i \rangle, \quad t \geq 0, \quad P_x\text{-a.s.},$$

where

$$\tau = \inf\{t \geq 0 : |X_t| \notin D\}.$$  

Moreover, for each $x \in D$ we have

$$(Y^x_t, Z^x_t) = (u(X_{t \wedge \tau}), (\sigma \nabla u)(X_{t \wedge \tau})), \quad t \geq 0, \quad P_x\text{-a.s.}$$

From (1.8) it follows in particular that $Y^x_0 = u(x)$ $P_x$-a.s. for $x \in D$ and $Z^x_0 = (\sigma \nabla u)(x)$ $P_x$-a.s. for a.e. $x \in D$, which yields a stochastic representation of $u$ and its gradient.
For regular $a$ (i.e. if $A$ is a non-divergence form operator and $X$ is an Itô process under $P_x$), a representation similar to (1.8) was proved in Darling and Pardoux [4] under assumptions on $\varphi$ and $f$ similar to those in the present paper, and in Peng [9] under more restrictive assumptions. Let us also mention that the Cauchy problem for the semilinear parabolic equation $D_t u + Au = F_u$ with $A$ given by (1.1) is investigated in Bally, Pardoux and Stoica [1], Rozkosz [11] and Stoica [15], and the mixed boundary problem in Lejay [7]. Let us stress, however, that in [1] a representation of the form (1.8) is obtained for quasi-every starting point $x \in D$, whereas in [7, 15] only for a.e. $x \in D$.

As for proofs, we would like to point out only one aspect. Roughly speaking, the main difficulty in proving (1.5)–(1.8) for every (not quasi- or almost every) starting point is that we have to show $P_x$-integrability of the quadratic variation of the martingale part of (1.6) (or the martingale part of $u(X \wedge \tau)$ in its decomposition of the form (1.3)). From (1.4), (1.6) it is therefore clear that to the requirement that the solution $u$ of (1.5) belongs to the space $C([0, \infty); \mathbb{R}^d) \cap W^1_2$ a condition on the integrability of $\nabla u$ must be added. Therefore in the present paper we consider for each $x \in D$ a Sobolev space $\mathcal{W}_\alpha(x)$ with weight, the weight being the $\alpha$-Green function for $A$ in $D$ with suitably chosen $\alpha \geq 0$ (see Section 2). Using stochastic calculus we derive a sort of “energy estimates” in the space $\mathcal{W}_\alpha(x)$. The crucial fact is that these estimates do not involve any regularity assumptions about $a$ and allow one to prove existence of a unique continuous solution $u \in W^1_2$ of (1.5) such that $u \in \mathcal{W}_\alpha(x)$ for every $x \in D$. This is enough for our purposes, because from the definition of $\mathcal{W}_\alpha(x)$ it follows in particular that $E_x \int_0^\tau |\nabla u(X_t)|^2 dt < \infty$ if $u \in \mathcal{W}_\alpha(x)$. As far as we know, the idea of using the spaces $\mathcal{W}_\alpha(x)$ for investigation of the problem (1.5) is new. Notice, however, that an idea similar in spirit has appeared in [11].

**Notation.** $\Omega = C([0, \infty); \mathbb{R}^d)$ is the space of continuous $\mathbb{R}^d$-valued functions on $[0, \infty)$ equipped with the topology of uniform convergence on compact sets. $X$ is the canonical process on $\Omega$, $X^t = X_{t \wedge \tau}$, $t \geq 0$. By $E_x$ (resp. $E^m_x$) we denote expectation with respect to $P_x$ (resp. $P^m_x$), by $\mathcal{L}[Y \mid P_x]$ (resp. $\mathcal{L}[Y \mid P^m_x]$) the law of the process $Y$ under $P_x$ (resp. $P^m_x$), and by “$\Rightarrow$” convergence in law.

$D_i = \partial_i/\partial x^i$ is the partial derivative in the distribution sense, and $\nabla = (D_1, \ldots, D_d)$. By $C(G)$ we denote the set of all continuous functions on $G$, and by $C^{k,\beta}(\overline{D})$ (resp. $C^{k,\beta}(D)$) the space of functions whose $k$th order partial derivatives are uniformly Hölder continuous (resp. locally Hölder continuous) with exponent $\beta$ in $D$. Furthermore, $L_p$ (resp. $L_\infty$) is the Banach space of measurable functions on $D$ that are $p$-integrable (resp. essentially bounded); $W^1_2$ is the usual Sobolev space of all elements $u$ of $L_2$ having derivatives $D_i u$ in $L_2$, and $\tilde{W}^1_2$ is the closure of the space of infinitely dif-
differentiable functions with compact supports lying in $D$ with respect to the $W^1_2$-norm $\| \cdot \|_{W^1_2}$. By $\| \cdot \|_p$ we denote the norm in $L^p$; $(\cdot , \cdot )_2$ denotes the scalar product in $L^2$ whereas $(\cdot , \cdot )$ is the usual scalar product in $\mathbb{R}^d$; $|D|$ is the Lebesgue measure of $D$. The abbreviation “a.e.” means “almost everywhere with respect to the Lebesgue measure”.

2. Uniqueness and a priori estimates. Let $\{ \mathcal{F}_t \}_{t \geq 0}$ denote the minimum completed admissible filtration for $\mathbb{X}$.

Definition. Let $\varphi : \partial D \to \mathbb{R}$ and $f : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions and let $x \in D$. We say that a pair $\{(Y^x_t, Z^x_t); t \geq 0\}$ of $\{ \mathcal{F}_t \}_{t \geq 0}$-progressively measurable processes is a solution of the BSDE $(\varphi, f)$ associated with $(X, P_x)$ if $Y^x$ is continuous, $\int_0^t |Z^x_t|^2 \, dt < \infty$ $P_x$-a.s. and (1.6) is satisfied with the martingale additive functional $M$ of the decomposition (1.3).

In fact we will consider solutions with better integrability properties, namely solutions in the class $\mathcal{H}_{\alpha; x}^{1+d}$ of all $\{ \mathcal{F}_t \}_{t \geq 0}$-progressively measurable processes $\xi$ with values in $\mathbb{R}^{1+d}$ having a finite norm

$$\| \xi \|_{\alpha; x}^2 = E_x \left[ \int_0^\tau e^{\alpha t} |\xi_t|^2 \, dt \right]$$

for some $\alpha \geq 0$.

We will assume that $D$ is a bounded domain in $\mathbb{R}^d$ which is regular for $A$, that is, $P_x(\tau = 0) = 1$, $x \in \partial D$, where $\tau = \inf\{ t > 0 : |X_t| \notin D \}$. We will need the following assumptions on $\varphi$ and $f$:

(i) $\varphi : \partial D \to \mathbb{R}$ is continuous and can be extended by continuity to $\bar{D}$ so that the extended function (still denoted by $\varphi$) belongs to $W^1_2$;

(ii) $f : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the Carathéodory condition, that is, $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in D$ and $f(\cdot, y, z)$ is measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$;

(iii) $(y_1 - y_2) \cdot (f(x, y_1, z) - f(x, y_2, z)) \leq \mu |y_1 - y_2|^2$ for some $\mu \in \mathbb{R}$ (the monotonicity condition);

(iv) $|f(x, y, z_1) - f(x, y, z_2)| \leq L |z_1 - z_2|$ for some $L \geq 0$;

(v) $|f(x, y, 0)| \leq g(x) + K |y|$ for some $K \geq 0$ and non-negative $g \in L^p$ with $p > 2 \vee d$ if $2 \mu + L^2 \leq 0$, and $g \in L^\infty$ in the general case.

For $\mu, L$ we will assume that

(vi') $2 \mu + L^2 \leq 0$ if $b \neq 0$ (non-symmetric case), or

(vi'') $\sup_{a \in A(\lambda, \Lambda)} \sup_{x \in D} E_x^{(a)} e^{\alpha \tau} < \infty$ for some $\alpha > 2 \mu + L^2$ if $b = 0$ (symmetric case). Here $E_x^{(a)}$ denotes expectation with respect to the measure $P_x^{(a)}$ corresponding to operator (1.1) with $b = 0$ and $A(\lambda, \Lambda)$ is the set of all measurable $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying (1.2).
Remark 2.1. Since there exists $c = c(\lambda, \Lambda, d)$ such that
\begin{equation}
 p(t, x, y) \leq \frac{c}{t^{d/2}} \exp\left(-\frac{|y - x|^2}{ct}\right)
\end{equation}
for all $t > 0$ and $x, y \in \mathbb{R}^d$ (see, e.g., [16, Section I.1]), the proof of [3, Proposition 1.18] shows that there is $C > 0$ depending only on $\lambda, \Lambda, d$ such that $\sup_{\alpha \in (\Lambda, \lambda)} \sup_{x \in D} E_x e^{\alpha x} < \infty$ for any $\alpha < \alpha_0 \equiv C |D|^{-2/d}$. Therefore (vi') is satisfied if $2\mu + L^2 < \alpha_0$.

**Proposition 2.2.** If (i)-(iv) are satisfied and $E_x e^{\alpha x} < \infty$, where $\alpha = 2\mu + L^2$, then the BSDE $(\varphi, f)$ has at most one solution in the space $\mathcal{H}_{\alpha;x}^{1+d}$.

**Proof.** See, e.g., the proof of [4, Proposition 3.2].

The remainder of this section is devoted to some a priori estimates on solutions of (1.6). These estimates will be needed in the next two sections.

**Proposition 2.3.** Assume (i)-(iv) and that $\|f(\cdot, 0, 0)\|_{\alpha;x} < \infty$ for some $\alpha \geq 0$. If $(Y^x, Z^x) \in \mathcal{H}_{\alpha;x}^{1+d}$ is a solution of the BSDE $(\varphi, f)$ such that
\begin{equation}
 E_x \sup_{0 \leq t \leq \tau} e^{\alpha t} |Y^x_t|^2 < \infty,
\end{equation}
then $(Y^x, Z^x) \in \mathcal{H}_{\alpha;x}^{1+d}$ and for any $\delta, \varepsilon > 0$,
\begin{equation}
 E_x \left\{ e^{\alpha (t \wedge \tau)} |Y^x_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^\tau e^{\alpha s} ((\alpha - \alpha_1) |Y^x_s|^2 + \varepsilon(1 + \varepsilon)^{-1} |Z^x_s|^2) \, ds \right\}
\end{equation}
\begin{equation}
\leq E_x \left\{ e^{\alpha \tau} |\varphi(X_\tau)|^2 + \delta^{-1} \int_{t \wedge \tau}^\tau e^{\alpha s} |f(X_s, 0, 0)|^2 \, ds \right\}
\end{equation}
for all $t \geq 0$, where $\alpha_1 = \delta + 2\mu + (1 + \varepsilon)L^2$.

**Proof.** See, e.g., the proof of [4, Corollary 4.4.1].

For given $x \in D$ let $\mathcal{L}_\alpha(x)$ denote the space of functions having a finite norm
\[ \|u\|_{\alpha;x}^2 = E_x \int_0^\tau e^{\alpha t} |u(X_t)|^2 \, dt = \|u(X)\|_{\alpha;x}^2 \]
and let $\mathcal{W}_\alpha(x)$ denote the space of all elements of $\mathcal{L}_\alpha(x)$ having generalized derivatives $D_i u$ in $\mathcal{L}_\alpha(x)$ and equipped with the norm
\[ \|u\|^2_{\mathcal{W}_\alpha(x)} = \|u\|^2_{\alpha;x} + \|\nabla u\|^2_{\alpha;x}. \]

Notice that by Fubini's theorem, $\|u\|^2_{\alpha;x} = \int_0^\infty T_t u^2(x) \, dt = U_\alpha u^2(x)$, where $\{T_t, t \geq 0\}$ is the semigroup of positive linear operators on $\mathbb{L}_\infty$ defined by $T_t f(x) = E_x 1_{\{t > \tau\}} e^{\alpha t} f(X_t)$ and $U_\alpha$ is the potential operator for $\{T_t\}$. By (2.1) there is $C = C(\lambda, \Lambda, d)$ such that $T_t f(x) \leq C e^{\alpha t} P_t(1_{D}f)(x)$, where $\{P_t, t \geq 0\}$ is the transition semigroup of a standard $d$-dimensional Wiener...
process. Hence, by [3, Theorem 3.10], for each \( t > 0 \), \( T_t \) is also a bounded operator from \( L_1 \) to \( L_\infty \). Therefore, if \( \sup_{x \in D} U_\alpha 1(x) < \infty \), then there is a measurable density \( U_\alpha (\cdot, \cdot) \) such that

\[
\|u\|_{\alpha;x}^2 = \int_D |u(y)|^2 U_\alpha (x,y) \, dy
\]

(see, e.g., the proof of Corollary to Theorem 3.18 in [3]). Thus \( \mathcal{L}_\alpha (x) \) is in fact an \( L_2 \)-space with the weight \( U_\alpha (x, \cdot) \). The function \( U_\alpha \) will be called the \( \alpha \)-\textit{Green function} for \( A \) in \( D \). In the case where \( \alpha = 0 \) the \( \alpha \)-Green function reduces to the usual Green function which we will denote by \( G_D \). Notice also that if \( b = 0 \) then the operators \( T_t \) are symmetric, and consequently \( U_\alpha (x, y) = U_\alpha (y, x) \), \( x, y \in D \).

Suppose \( u \) is a solution of (1.5) such that \( (Y, Z) = (u(X^\tau), (\sigma \nabla u)(X^\tau)) \in \mathcal{H}^{1+d}_{0,x} \) is a solution of (1.6) satisfying (2.2). Then from (2.3) with \( t = 0 \) it follows that

\[
\begin{align*}
|u(x)|^2 + (\alpha - \alpha_1)\|u\|_{\alpha;x}^2 + \varepsilon (1 + \varepsilon)^{-1}\|\sigma \nabla u\|_{\alpha;x}^2 & \leq E_x e^{\alpha \tau} |\varphi (X_\tau)|^2 + \delta^{-1}\|f(\cdot, 0, 0)\|_{\alpha;x}^2.
\end{align*}
\]

Consider now the problem (1.5) with \( f(x, y, z) = F(x) \), that is, the linear problem. Suppose \( u \) is a solution of it such that (1.8) is a solution to (1.6) in the space \( \mathcal{H}^{1+d}_{0,x} \). By Itô’s formula, for any \( n \in \mathbb{N} \) we have

\[
|Y_0^x|^2 + \int_0^{\tau \wedge n} |Z_s^x|^2 \, ds + 2 \int_0^{\tau \wedge n} \langle Y_s^x Z_s^x, \sigma^{-1}(X_s) \rangle \, dM_s = |Y_{\tau \wedge n}^x|^2 + 2 \int_0^{\tau \wedge n} F(X_s) Y_s^x \, ds.
\]

Taking expectations and letting \( n \to \infty \) gives

\[
E_x \left\{ |Y_0^x|^2 + \int_0^\tau |Z_t^x|^2 \, dt \right\} \leq E_x \left\{ |Y_\tau^x|^2 + 2 \int_0^\tau F(X_t) Y_t^x \, dt \right\},
\]

and so

\[
|u(x)|^2 + \|\nabla u\|_{0;x}^2 \leq E_x |\varphi (X_\tau)|^2 + \|F\|_{0;x}^2 + \|u\|_{0;x}^2.
\]

By (1.6),

\[
|u(x)| \leq E_x |\varphi (X_\tau)| + E_x \int_0^\tau |F(X_t)| \, dt.
\]
Obviously $\|u\|_{0,x}^2 \leq \|u\|_{0,2}^2 E_x \tau$, so from (2.5), (2.6) we see that
\[
(2.7) \quad \|\nabla u\|_{0,x}^2 \leq (1 + 2E_x \tau)\|\varphi\|_{0,2}^2 + \|F\|_{0,x}^2 \\
+ 2E_x \tau \sup_{x \in D} \left(E_x \int_0^\tau |F(X_t)| \, dt \right)^2.
\]

3. Linear equations. It is well known that in case $d > 1$ for any $\varphi \in W_2^1$ and $F \in \mathbb{L}_2$ the problem
\[
(3.1) \quad Au = -F, \quad u|_{\partial D} = \varphi
\]
has a unique weak solution in $W_2^1$, that is, there exists a unique $u \in W_2^1$ such that $u - \varphi \in W_2^1$ and $(a \nabla u, \nabla \eta)_2 - 2(\nabla u, \eta b)_2 = 2(F, \eta)_2$ for all $\eta \in W_2^1$ (see, e.g., [5, Theorem 8.3]). If, in addition, $\varphi$ satisfies (i) and $F \in \mathbb{L}_p$ with $p > d/2$ then $u$ is continuous on $\overline{D}$ ([5, Theorem 8.31]). In the case where $d = 1$ the problem (3.1) has an even more regular weak solution for $F \in \mathbb{L}_p$ with $p = 1$ (the substitution $v = au'/2$ reduces $Au = -F$ to the ordinary linear differential equation $v' + 2ba^{-1}v = -F$, so one can easily write down an explicit formula for the solution). Our purpose is to show that the condition $F \in \mathbb{L}_p$ can be replaced by
\[
(3.2) \quad \sup_{x \in D} \|F\|_{0,x} < \infty
\]
and, what is more important, that under (3.2) the pair $(u(X^\tau), (\sigma \nabla u)(X^\tau))$ is a solution of the BSDE $(\varphi, F)$. To prove this, we will need some auxiliary results.

Proposition 3.1. If $\varphi \in C(\partial D)$ and $F$ satisfies (3.2) then $u : \overline{D} \to \mathbb{R}$ defined by
\[
\begin{align*}
u(x) = E_x \left\{ \varphi(X_\tau) + \int_0^\tau F(X_t) \, dt \right\}, \quad x \in \overline{D},
\end{align*}
\]
is continuous.

Proof. Since $D$ is regular, $P_x(\tau = \tau) = 1$ for $x \in \overline{D}$. Hence, by standard arguments, \overline{D} \ni x \mapsto E_x \varphi(X_\tau)$ is continuous (see, e.g., the proof of [3, Theorem 1.23]). Therefore we only need to show that if $\{x_n\} \subset \overline{D}$, $x_n \to x$ then $E_x E\int_0^\tau F(X_t) \, dt = E_x \int_0^\tau F(X_t) \, dt = 0$, and for this purpose it suffices to show that $E_x \tau \to 0$, because by Schwarz’s inequality $|E_x \int_0^\tau F(X_t) \, dt| \leq \|F\|_{0,x}^2 (E_x \tau)^{1/2}$. By [3, Proposition 1.19], for each $t > 0$ the function $x \mapsto P_x(\tau > t)$ is upper semicontinuous in $\mathbb{R}^d$, so $\limsup_{n \to \infty} P_{x_n}(\tau > t) \leq P_x(\tau > t) = 0$. On the other hand, since $P_x(\tau > t) \leq e^{-\alpha t/2} \sup_{x \in D} E_x e^{\alpha t/2}$ for $t \geq 0$, it follows from Remark 2.1 that the functions $t \mapsto P_x(\tau > t)$ are bounded by an integrable function
uniformly in \( x \in \overline{D} \). Therefore applying Fatou’s lemma gives

\[
\limsup_{n \to \infty} E_{x_n} \tau = \limsup_{n \to \infty} \int_0^\infty P_{x_n}(\tau > t) \, dt \leq \limsup_{n \to \infty} \int_0^\infty P_{x_n}(\tau > t) \, dt = 0,
\]

and the proof is complete. □

**Lemma 3.2.** Let \( x \in D \). If \( u_n \to u \) in \( L_2 \) and \( \{u_n\} \) is bounded in \( L_0(x) \) then \( u \in L_0(x) \) and \( \|u\|_{0;x} \leq \limsup_{n \to \infty} \|u_n\|_{0;x} \). If, in addition, \( \{u_n\} \) is a Cauchy sequence in \( L_0(x) \) then \( u_n \to u \) in \( L_0(x) \).

**Proof.** Let \( \{V_m\} \) be a sequence of open sets such that \( x \notin \bigcap_{m=1}^{\infty} V_m = D \) for \( m \in \mathbb{N} \). It is known that \( \text{G}_D(x, \cdot) \) is continuous on \( D \setminus \{x\} \) (see [14]). Therefore, \( \text{G}_D(x, \cdot) \) is bounded on \( V_m \), and consequently\( \|1_{V_m}(u_n - u)\|_{0;x} = \int_{V_m} |u_n(y) - u(y)|^2 \text{G}_D(x, y) \, dy \to 0 \), that is, \( 1_{V_m}u_n \to 1_{V_m}u \) in \( L_0(x) \). On the other hand, \( \{u_n\} \) is bounded in \( L_0(x) \) and therefore weakly relatively compact. Suppose \( u_n(k) \to v \) weakly in \( L_0(x) \) for some subsequence \( \{n(k)\} \). Then \( 1_{V_m}u_n(k) \to 1_{V_m}v \) weakly in \( L_0(x) \) as well. It follows that \( \|1_{V_m}u\|_{0;x} = \|1_{V_m}v\|_{0;x} \). Since \( 1_{V_m}u^2 \uparrow 1_{D\setminus\{x\}}u^2 \) pointwise, using the monotone convergence theorem yields \( \|u\|_{0;x} = \|v\|_{0;x} \). Let \( (\cdot, \cdot)_{0;x} \) denote the scalar product in \( L_0(x) \). Since \( (u_n(k), v)_{0;x} \to (v, v)_{0;x} \), it follows that \( (u_n(k), v)_{0;x} \to (v, v)_{0;x} \), hence

\[
\|v\|_{0;x}^2 = \limsup_{k \to \infty} |(u_n(k), v)_{0;x}| \leq \limsup_{k \to \infty} \|u_n\|_{0;x} \cdot \|v\|_{0;x}
\]

\[
\leq \limsup_{n \to \infty} \|u_n\|_{0;x} \cdot \|v\|_{0;x},
\]

which proves the first assertion. The second one follows from the fact that if \( u_n \to v \) in \( L_0(x) \) then \( \|1_{V_m}(u - v)\|_{0;x} = 0 \) for \( m \in \mathbb{N} \). □

For \( n \in \mathbb{N} \) let \( a_n : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) and \( b_n : \mathbb{R}^d \to \mathbb{R}^d \) be measurable functions satisfying (1.2) and let \( \varphi_n \in W_2^1 \) and \( F_n \in \mathbb{L}_2 \). Set

\[
A^n = 2 \sum_{i,j=1}^{d} D_j(a^{ij}_n(x)D_i) + \sum_{i=1}^{d} b^i_n(x)D_i
\]

and let \( u_n \in W_2^1 \) be a weak solution of the problem

\[
A^n u_n = -F_n, \quad u_n|\partial D = \varphi_n.
\]

In the rest of this section we will assume that \( a^{ij}_n \to a^{ij} \) and \( b^n_i \to b^i \) a.e. for \( i, j = 1, \ldots, d \).

The following proposition is undoubtedly known. We provide a proof, because we could not find an appropriate reference.

**Proposition 3.3.** If \( \varphi_n \to \varphi \) in \( W_2^1 \) and \( F_n \to F \) in \( \mathbb{L}_2 \) then \( u_n \to u \) in \( W_2^1 \), where \( u \) is a unique weak solution of the problem (3.1).
Proof. First we show that there is $C = C(\lambda, \Lambda)$ such that
\begin{equation}
\|u\|_2 \leq C \|F\|_2
\end{equation}
for some $C = C(\lambda, \Lambda)$. We can certainly assume that $F$ is bounded, for if \{f_k\} is a sequence of bounded functions such that $f_k \to F$ in $L^2$ and $u_k$ is a weak solution of (3.1) with $F$ replaced by $f_k$, $k \in \mathbb{N}$, then $\|u\|_2 = \lim_{k \to \infty} \|u_k\|_2$ by [5, Corollary 8.7]. Let $\hat{A} = \sum_{i,j=1}^d D_j(\frac{1}{2}a^{ij}(x)D_i - b^j(x))$ be the formal adjoint of $A$. By [5, Theorem 8.6] there exists a unique weak solution $v \in \dot{W}_2^1$ of the problem $\hat{A}v = -1$, $v|_{\partial D} = 0$. Since, by [5, Theorem 8.15], $u, v$ are bounded, it follows that $uv \in \dot{W}_2^1$, and hence $-2^{-1}(a\nabla u, \nabla(uv))_2 + (\nabla u, uv) = -(F, uv)_2$. Therefore
\begin{equation}
(a\nabla u, v\nabla u)_2 + (a\nabla u, u\nabla v)_2 - 2(\nabla u, uv) = 2(F, uv)_2.
\end{equation}
On the other hand, $-2^{-1}(a\nabla v, \nabla u^2)_2 + (vb, \nabla u^2)_2 = -(u^2, 1)_2$, and consequently
\begin{equation}
\|u\|_2^2 = (a\nabla u, u\nabla v)_2 - 2(\nabla u, uv) = 2(F, uv)_2.
\end{equation}
By the above,
\begin{equation}
\|u\|_2^2 + (a\nabla u, v\nabla u)_2 = 2(F, uv)_2,
\end{equation}
and so $\|u\|_2 \leq \|v\|_\infty \|F\|_2$, which proves (3.5), because from [5, Theorem 8.16] it follows that $v$ is bounded by a constant depending only on $\lambda, \Lambda$.

Now, write $w_n = u_n - \varphi_n - u + \varphi$. Since $w_n \in \dot{W}_2^1$,
\begin{align*}
\lambda \|\nabla w_n\|_2^2 &\leq (a_n\nabla w_n, \nabla w_n)_2
\end{align*}
\begin{align*}
&= (a_n\nabla u_n, \nabla w_n)_2 - (a_n\nabla \varphi_n, \nabla w_n)_2 \\
&+ ((a - a_n)\nabla(u - \varphi), \nabla w_n)_2 - (a\nabla u, \nabla w_n)_2 + (a\nabla \varphi, \nabla w_n)_2 \\
&= 2(F_n - F, w_n)_2 + ((a - a_n)\nabla(u - \varphi), \nabla w_n)_2 + (a\nabla \varphi - \varphi_n, \nabla w_n)_2 \\
&+ 2(\nabla w_n, w_nb_n)_2 + 2(\nabla (\varphi_n - \varphi), w_nb)_2 + 2(\nabla u, w_n(b_n - b))_2.
\end{align*}
From this and (3.5) the proposition follows. ■

Let $X^n = \{(X, P^n_x); x \in \mathbb{R}^d\}$ be a Markov process corresponding to the operator $A^n$ and let $M^n$ denote the martingale additive functional of $X^n$ of the decomposition of $X$ of the form (1.3).

**Lemma 3.4.** $\mathcal{L}[(X, M^n) | P^n_x] \Rightarrow \mathcal{L}[(X, M) | P_x]$ in $C([0, \infty); \mathbb{R}^{2d})$ for each $x \in D$.

Proof. Since $P^n_x \Rightarrow P_x$ in $C([0, \infty); \mathbb{R}^d)$ (see, e.g., [16]), it suffices to repeat, with some obvious changes, the proof of [11, Lemma 3.2]. ■

In the proof of the following lemma we combine ideas from the proofs of [3, Theorem A.1] and [17, Lemma 11.1.2].
Lemma 3.5. Let \( x \in D \). There exists a sequence of regular domains \( \{D_m\} \) such that \( \{x\} \subset D_m \subset \overline{D}_m \subset D \) for \( m \in \mathbb{N} \), \( \bigcup_{m=1}^{\infty} D_m = D \) and \( \tau_m = \inf\{t \geq 0 : |X_t| \notin D_m\} \) is continuous at \( \omega \) for \( P_x \)-almost all \( \omega \in \Omega \).

Proof. Set \( \varepsilon_0 = \text{dist}(x, \partial D) \). For \( \varepsilon \in (0, \varepsilon_0) \) let \( D_\varepsilon = \{y \in D : \text{dist}(y, \partial D) > \varepsilon\} \) and let \( E_\varepsilon \) denote the connected component of \( D_\varepsilon \) containing \( x \). In the proof of [3, Theorem A.1] it is shown that the sets \( E_\varepsilon \) are regular and \( \bigcup_{0 < \varepsilon < \varepsilon_0} E_\varepsilon = D \), so it is clear that the lemma will be proved once we show that for at most countably many \( \varepsilon \)'s from \((0, \varepsilon_0)\), \( \sigma_\varepsilon = \inf\{t \geq 0 : |X_t| \notin E_\varepsilon\} \) is continuous for \( P_x \)-almost all \( \omega \) in \( \Omega \). Let \( \sigma_\varepsilon^+ = \inf\{t \geq 0 : |X_t| \notin E_\varepsilon\} \). Since \( \sigma_\varepsilon^+ = \lim_{\delta \downarrow \varepsilon} \sigma_\delta \) and \( \varepsilon \mapsto E_x \exp(-\sigma_\varepsilon) \) is a non-negative non-increasing function,

\[
E_x \exp(-\sigma_\varepsilon^+) = \lim_{\delta \downarrow \varepsilon} E_x \exp(-\sigma_\delta) = E_x \exp(-\sigma_\varepsilon)
\]

for all but a countable number of \( \varepsilon \)'s. Since \( \sigma_\varepsilon^+ \geq \sigma_\varepsilon \) this shows that \( \sigma_\varepsilon^+ = \sigma_\varepsilon \) for all but a countable number of \( \varepsilon \)'s, which is the desired conclusion because \( \sigma_\varepsilon \) is lower semicontinuous and \( \sigma_\varepsilon^+ \) is upper semicontinuous.

Lemma 3.6. For given \( x \in D \) let \( \{D_m\} \) be the sequence of domains of Lemma 3.5 and let \( \{\tau_m\} \) be the sequence of the corresponding stopping times. Assume \( F_n, F : D \rightarrow \mathbb{R} \) and \( G_n = (G_1^n, \ldots, G_d^n) \), \( G = (G_1, \ldots, G_d) : \mathbb{R}^d \rightarrow \mathbb{R} \) are measurable functions such that \( F_n \rightarrow F \) and \( G_i^n \rightarrow G_i \), \( i = 1, \ldots, d \), in \( \mathbb{L}_2 \). Then for any \( m, N \in \mathbb{N} \) and \( \delta > 0 \),

\[
\mathcal{L}\left[\int_{\delta}^{(\wedge N \wedge \tau_m) \vee \delta} F_n(X_t) \, dt, \int_{\delta}^{(\wedge N \wedge \tau_m) \vee \delta} \langle G_n(X_t), dM_t^n \rangle \right] \quad | P^n_x |
\]

\[
\Rightarrow \mathcal{L}\left[\int_{\delta}^{(\wedge N \wedge \tau_m) \vee \delta} F(X_t) \, dt, \int_{\delta}^{(\wedge N \wedge \tau_m) \vee \delta} \langle G(X_t), dM_t \rangle \right] \quad | P_x |
\]

in \( C([0, \infty); \mathbb{R}^{d+2}) \) as \( n \rightarrow \infty \).

Proof. By Lemmas 3.4 and 3.5,

\[
\mathcal{L}\left[\int (N \wedge \tau_m) \vee \delta, \quad X_{(\wedge N \wedge \tau_m) \vee \delta}, \quad |M^n| \quad | P^n_x | \right] \quad \Rightarrow \mathcal{L}\left[\int (N \wedge \tau_m) \vee \delta, \quad X_{(\wedge N \wedge \tau_m) \vee \delta}, \quad |M| \quad | P_x | \right]
\]

in \( \mathbb{R} \times C([0, \infty); \mathbb{R}^{2d}) \) as \( n \rightarrow \infty \). On the other hand, by (2.1), for any \( h \in \mathbb{L}_1 \),

\[
E^n_x \int_{\delta}^{(N \wedge \tau_m) \vee \delta} |h(X_t)| \, dt \quad \leq \quad C(\lambda, A) \int_{\delta}^{(N \wedge \tau_m) \vee \delta} t^{-d/2} \, dt \int_{D_m} |h(y)| \, dy \quad \leq \quad C(\lambda, A, d, N, \delta) \|h\|_1,
\]
and similar estimates hold if we replace $E^n_x$ by $E_x$. Therefore the lemma follows by the same method as in the proof of [12, Theorem 1].

We are now ready to prove our main result on the Dirichlet problem for linear equations.

**Theorem 3.7.** If $\varphi$ satisfies (i) and $F \in \mathbb{L}_2$ satisfies (3.2) then there exists a unique weak solution $u \in W^1_2 \cap C(\overline{D})$ of the problem (3.1). Moreover, for each $x \in D$ the pair (1.8) is a solution, in $\mathcal{H}^{1+d}_{0:x}$, of the BSDE $(\varphi, F)$ associated with $(X, P_x)$.

**Proof.** We first prove the theorem under the additional assumption that $F$ is bounded. Let $j(x) = c \exp(-1/(1 - |x|^2))$ if $|x| < 1$ and $j(x) = 0$ if $|x| \geq 1$, where $c$ is chosen so that $\int_{\mathbb{R}^d} j(x) \, dx = 1$, and let $j_n(x) = n^d j(nx)$. Set

\begin{equation}
(3.7) \quad a_{ij}^n = a_{ij} \ast j_n, \quad b^i_n = b^i \ast j_n, \quad \varphi_n = \varphi \ast j_n, \quad F_n = F \ast j_n
\end{equation}

(we first extend $\varphi$ to a continuous function on $\mathbb{R}^d$ and $F$ to be zero outside $D$) and let $u_n \in C^2(D) \cap C(\overline{D})$ be a (unique) solution to the problem (3.4). Then for any $n, m, N \in \mathbb{N}$,

\begin{equation}
(3.8) \quad u_n(X_{t \wedge N \wedge \tau_m}) = u_n(X_{N \wedge \tau_m}) + \int_{t \wedge N \wedge \tau_m}^{N \wedge \tau_m} F_n(X_s) \, ds
\end{equation}

\[ - \int_{t \wedge N \wedge \tau_m}^{N \wedge \tau_m} \langle \nabla u_n(X_s), dM^n_s \rangle, \quad t \geq 0, \quad P^n_x\text{-a.s.} \]

By Proposition 3.3, $u_n \to u$ in $W^1_2$, where $u \in W^1_2$ is a unique solution of the problem (3.1). On the other hand, by (2.6), $\{u_n\}$ is bounded on $D$ and hence, by De Giorgi's continuity theorem (see [5, Theorem 8.24]), equicontinuous in each compact subset of $D$. Therefore there is a bounded continuous (on $D$) version of $u$ (still denoted by $u$) such that $u_n \to u$ uniformly on compact subsets of $D$. As a consequence, by Lemma 3.6 and the continuous mapping theorem (see [2, Theorem 5.5]),

\[
\mathcal{L} \left[ (u_n(X_{(\cdot \wedge N \wedge \tau_m) \vee \delta}), \int_\delta^{(\cdot \wedge N \wedge \tau_m) \vee \delta} F_n(X_t) \, dt, \int_\delta^{(\cdot \wedge N \wedge \tau_m) \vee \delta} \langle \nabla u_n(X_t), dM^n_t \rangle \right] \bigg| P^n_x \bigg]
\]

\[
\Rightarrow \mathcal{L} \left[ (u(X_{(\cdot \wedge N \wedge \tau_m) \vee \delta}), \int_\delta^{(\cdot \wedge N \wedge \tau_m) \vee \delta} F(X_t) \, dt, \int_\delta^{(\cdot \wedge N \wedge \tau_m) \vee \delta} \langle \nabla u(X_t), dM_t \rangle \right] \bigg| P_x \bigg]
\]

in $C([0, \infty); \mathbb{R}^3)$. By the above and (3.8), using once again the continuous mapping theorem, we get
\[ u(X_{(t \wedge N \wedge \tau_m) \setminus \delta}) = u(X_{(N \wedge \tau_m) \setminus \delta}) + \int_{(t \wedge N \wedge \tau_m) \setminus \delta}^{(N \wedge \tau_m) \setminus \delta} F(X_s) \, ds \]
\[ - \int_{(t \wedge N \wedge \tau_m) \setminus \delta}^{(N \wedge \tau_m) \setminus \delta} \langle \nabla u(X_s), dM_s \rangle, \quad t \geq 0, \ P_x\text{-a.s.} \]

Letting \( \delta \downarrow 0 \) and then \( N \uparrow \infty \) we see that

\[ u(X_{t \wedge \tau_m}) = u(X_{\tau_m}) + \int_{t \wedge \tau_m}^{\tau_m} F(X_s) \, ds - \int_{t \wedge \tau_m}^{\tau_m} \langle \nabla u(X_s), dM_s \rangle \ P_x\text{-a.s.} \]

for \( t \geq 0 \). Since \( \tau_m \uparrow \tau \ P_x\text{-a.s.} \) and \( F \) is bounded, \( \int_{t \wedge \tau_m}^{\tau_m} F(X_s) \, ds \to \int_t^{\tau} F(X_s) \, ds \ P_x\text{-a.s.} \). Furthermore, by (2.5) and boundedness of \( \{u_n\} \), \( \{\nabla u_n\} \) is bounded in \( L_0(x) \) for each \( x \in D \). From this and Lemma 3.2 we conclude that \( \nabla u \in L_0(x) \) for \( x \in D \), hence that \( \int_{t \wedge \tau_m}^{\tau} \langle \nabla u(X_s), dM_s \rangle \to \int_t^{\tau} \langle \nabla u(X_s), dM_s \rangle \ P_x\text{-a.s.} \). Finally, by Proposition 3.1, \( u \) is continuous in \( \overline{D} \) and \( u = \varphi \) on \( \partial D \). Hence \( u(X_{t \wedge \tau_m}) \to u(X_{t \wedge \tau}) \) and \( u(X_{\tau_m}) \to \varphi(X_\tau) \ P_x\text{-a.s.} \). Thus, for each \( x \in D \),

\[ u(X_{t \wedge \tau}) = \varphi(X_\tau) + \int_{t \wedge \tau}^{\tau} F(X_s) \, ds - \int_{t \wedge \tau}^{\tau} \langle \nabla u(X_s), dM_s \rangle \ P_x\text{-a.s.} \]

for each \( t \geq 0 \). Since the processes on the left and right-hand side of the above equality are continuous, this proves the proposition for bounded \( F \).

To prove the general case, we set \( F_n = (\sigma_n \nabla u_n)(X_\tau) \). By what has already been proved, the pair \( (u_n(X_\tau), (\sigma \nabla u_n)(X_\tau)) \), where \( u_n \in W^1_2 \cap C(\overline{D}) \) is a solution of the problem \( A u_n = -F_n, u_n|_{\partial D} = \varphi \), solves the BSDE \( (\varphi, F_n) \). In particular,

\[ u_n(X_{t \wedge \tau_m}) = \varphi(X_\tau) + \int_{t \wedge \tau_m}^{\tau} F_n(X_s) \, ds \]
\[ - \int_{t \wedge \tau_m}^{\tau} \langle \nabla u_n(X_s), dM_s \rangle, \quad t \geq 0, \ P_x\text{-a.s.} \]

Clearly \( |F_n - F| \to 0 \) and \( |F_n - F| \leq F \). Therefore, since \( F \) satisfies (3.2), the Lebesgue dominated convergence theorem shows that

\[ \lim_{n \to \infty} \|F_n - F\|_{0;x}^2 = \lim_{n \to \infty} \int_D |F_n - F|^2(y)G_D(x,y) \, dy = 0 \]

for \( x \in D \). Similarly, since \( F \in L_2 \), we have \( F_n \to F \) in \( L_2 \), and consequently, by Proposition 3.3, \( u_n \to u \) in \( W^1_2 \), where \( u \) is a unique solution of (3.1). Moreover, for each \( x \in D \),

\[ \nabla u_n \to \nabla u \text{ in } L_0(x). \]
Indeed, from (2.7) it follows that for any \( n, m \in \mathbb{N} \),
\[
\|\nabla (u_n - u_m)\|_{0;x}^2 \leq \|F_n - F_m\|_{0;x}^2 + 2(E_x \tau) \sup_{x \in D} \left( \int_0^\tau |F_n - F_m|(X_t) \, dt \right)^2.
\]
By (3.10), \( \lim_{n,m \to \infty} \|F_n - F_m\|_{0;x} = 0 \). At the same time, since \( |F_n - F_m| \leq |F| \) and \( |F_n - F_m| = 0 \) if \( |F| \leq N \) and \( n, m \geq N \), for any \( n, m \geq N \) we have
\[
(3.12) \quad \sup_{x \in D} E_x \int_0^\tau |F_n - F_m|(X_t) \, dt \leq \sup_{x \in D} E_x \int_0^\tau 1_{\{|F| > N\}} |F|(X_t) \, dt
\]
\[
\leq \sup_{x \in D} \left( E_x \int_0^\tau 1_{\{|F| > N\}}(X_t) \, dt \cdot E_x \int_0^\tau |F(X_t)|^2 \, dt \right)^{1/2}
\]
\[
\leq N^{-1} \sup_{x \in D} \|F\|_{0;x}^2.
\]
Therefore \( \{\nabla u_n\} \) is a Cauchy sequence in \( L_0(x) \). On the other hand, for any \( V_m \) defined in the proof of Lemma 3.2 we have
\[
\limsup_{n \to \infty} \|\mathbf{1}_{V_m} \nabla (u_n - u)\|_{0;x}^2
\]
\[
\leq \sup_{y \in V_m} G_D(x, y) \cdot \limsup_{n \to \infty} \int_{V_m} |\nabla (u_n - u)|^2(y) \, dy = 0,
\]
so (3.11) follows. We now observe that by De Giorgi’s continuity theorem (see [5, Theorem 8.24]), for any compact subset \( K \) of \( D \) and any \( x, y \in K \) and \( N \in \mathbb{N} \) we have
\[
\sup_{n \leq N} |u_n(x) - u_n(y)| \leq C_1 \sup_{n \leq N} (\|u_n\|_2 + \lambda^{-1} \|F_n\|_d) |x - y|^\alpha
\]
where \( C_1 = C_1(\lambda, A, d, \text{dist}(K, \partial D)) \) and \( \alpha = \alpha(\lambda, A, d) > 0 \). Moreover,
\[
\sup_{n > N} |u_n(x) - u_n(y)|
\]
\[
\leq |u_N(x) - u_N(y)| + \sup_{n > N} (|u_n(x) - u_N(x)| + |u_n(y) - u_N(y)|)
\]
\[
\leq |u_N(x) - u_N(y)| + 2 \sup_{n > N} \sup_{x \in K} E_x \int_0^\tau |F_n - F_N|(X_t) \, dt.
\]
Analysis similar to that in the proof of (3.12) shows that the second term on the right-hand side of the last inequality is bounded by \( 2N^{-1} \sup_{x \in K} \|F\|_{0;x}^2 \). Since \( \|u_n\|_2 \leq C_2(\|\varphi\|_{W^1_2} + \|F\|_2) \) for some \( C_2 \) not depending on \( n \) (see [5, Corollary 8.7] or the proof of Proposition 3.3), putting together the above
estimates yields
\[
\sup_{n \in \mathbb{N}} |u_n(x) - u_n(y)| \leq 2C_1(C_2(\|\varphi\|_{W^1_2} + \|F\|_2) + \lambda^{-1}N|D|^d)|x - y|^\alpha + 2N^{-1} \sup_{x \in K} \|F\|_{0;x}^2.
\]
This proves that \( \{u_n\} \) is equicontinuous in \( K \). By (2.6) and (3.2) it is also bounded on any compact subset of \( D \), and hence, by the Ascoli–Arzelà theorem, converges to \( u \) uniformly on any such subset. Using this and (3.10), (3.11), and letting \( n \to \infty \) in (3.9) gives
\[
u(X_{t \wedge \tau_m}) = \varphi(X_\tau) + \int_{t \wedge \tau_m}^{\tau} F(X_s) \, ds - \int_{t \wedge \tau_m}^{\tau} \langle \nabla u(X_t), dM_s \rangle, \quad t \geq 0,
\]
P\(_x\)-a.s. Hence, letting \( m \to \infty \) we obtain (3.9), because \( \tau_m \uparrow \tau \) P\(_x\)-a.s. and \( u \) is continuous on \( \bar{D} \). This proves the theorem. □

4. Semilinear PDEs and BSDEs. In this section we prove our main theorem on connections between weak solutions of the problem (1.5) and solutions of BSDEs of the form (1.6). We begin with results on uniqueness, existence and regularity of solutions of (1.5).

Before proving the first theorem we observe that if \( \sup_x E_x e^{\alpha \tau} < \infty \) then \( U_\alpha \mathbb{1} \) is a bounded weak solution of the problem \( (\alpha + A)U_\alpha \mathbb{1} = -1 \), \( U_\alpha \mathbb{1}|_{\partial D} = 0 \). This follows from the fact that for bounded \( f \) the function \( G_D f \) is a weak solution of the problem \( AG_D f = -f \), \( G_D f|_{\partial D} = 0 \) and \( G_D(\alpha U_\alpha \mathbb{1}) = U_\alpha \mathbb{1} - G_D 1 \), because using the Markov property and integration by parts gives

\[
E_x \left( E_{X_t} \left( e^{\alpha s} \, ds \right) \right) dt = E_x \left( (e^{\alpha \tau} - 1) \mid F_t \right) \alpha e^{-\alpha t} dt = E_x (e^{\alpha \tau} - 1).
\]

**Theorem 4.1.** Assume that \( \varphi \in W^1_2 \) and \( f \) satisfies (ii)-(v). If either \( \mu + L^2 \leq 0 \), or \( b = 0 \) and \( \sup_{x \in D} E_x e^{\alpha \tau} < \infty \) for some \( \alpha \geq 2|\mu| + L^2 \), then the problem (1.5) cannot have more than one weak solution in \( W^1_2 \).

**Proof.** Suppose that \( u_1, u_2 \in W^1_2 \) are weak solutions of (1.5). Then \( u = u_1 - u_2 \in W^1_2 \) and \( u \) solves the problem (3.1) with \( F = F_{u_1} - F_{u_2}, \varphi = 0 \). Let \( \{F_n\} \) be a sequence of bounded measurable functions such that \( F_n \to F \) in \( L^2 \), and for \( n \in \mathbb{N} \) let \( w_n \in W^1_2 \) be a weak solution of the problem \( Aw_n = -F_n, w_n|_{\partial D} = 0 \). Assume that \( b = 0 \). As in the proof of (3.6) one can show that

\[
\|w_n\|_2^2 + \alpha(w_n, w_n U_\alpha \mathbb{1})_2 + (a \nabla w_n, U_\alpha 1 \nabla w_n)_2 = 2(F_n, w_n U_\alpha \mathbb{1})_2.
\]
By Proposition 3.3, \( w_n \to u \) in \( W^1_2 \), so letting \( n \to \infty \) we obtain
\[
(4.1) \quad \|u\|_2^2 + \alpha(u, u\alpha_1) + (a\nabla u, U\alpha_1\nabla u) = 2(F, u\alpha_1). 
\]
On the other hand, by (iii), (iv),
\[
2(F, u\alpha_1) \leq 2\mu(u, u\alpha_1) + 2L(|u| \cdot |\sigma\nabla u|, U\alpha_1)
\leq 2\mu(u, u\alpha_1) + (a\nabla u, U\alpha_1\nabla u) + L^2(u, u\alpha_1).
\]
Hence \( \|u\|_2^2 + \alpha(u, u\alpha_1) \leq (2\mu + L^2)(u, u\alpha_1) \), and the theorem follows.

Consider now the case \( b \neq 0 \). Let \( \hat{A} \) be the formal adjoint of \( A \) and let \( \hat{G}_D1(x) = \int_D G(y, x) dy, \ x \in D \). Since \( \hat{G}_D1 \in W^1_2 \) is a weak solution of the problem \( \hat{A}\hat{G}_D1 = -1, \ \hat{G}_D1|_{\partial D} = 0 \), analysis similar to that in the proof of (4.1) shows that
\[
(4.2) \quad \|u\|_2^2 + (a\nabla u, \hat{G}_D1\nabla u) = 2(F, u\hat{G}_D1).
\]
Hence \( \|u\|_2^2 \leq (2\mu + L^2)(u, u\hat{G}_D1) \), and the proof is complete. ■

**Theorem 4.2.** Assume (i)-(v) and (vi') or (vi''). Then there exists a (unique) solution \( u \in W^1_2 \cap C(\bar{D}) \) of the problem (1.5). Moreover, \( u \in \mathcal{W}_{\alpha, 0}(x) \) for each \( x \in D \) and \( \sup_{x \in D} \|F_u\|_{\alpha, 0, x} < \infty \) with \( \alpha = 0 \) in the first case and any \( \alpha \) satisfying condition (vi'') in the second one.

**Proof.** By Remark 2.1 we may and will assume that \( \alpha > 0 \). Throughout the proof \( \gamma = \alpha \) if \( b = 0 \) and \( 2\mu + L^2 > 0 \), and \( \gamma = \alpha_0/q' \) if \( 2\mu + L^2 \leq 0 \), where \( \alpha_0 \) is the constant from Remark 2.1 and \( q' \) is the Hölder conjugate of some \( q \in (1, p/(2 \vee d)) \). Moreover, we write \( \alpha_1 = \delta + 2\mu + (1 + \varepsilon)L^2 \), where \( \delta, \varepsilon > 0 \) are chosen so that \( \alpha_1 < \gamma, \delta + \varepsilon L^2 < \alpha_0/2 \) and \( \varepsilon L^2 \|G_1\|_{\infty} < 1 \).

The proof will be divided into 5 steps.

**Step 1.** We first prove that for any square-integrable \( w : D \to \mathbb{R}^d \) such that \( \sup_{x \in D} \|w\|_{\mathcal{W}_\gamma}(x) < \infty \) the problem
\[
(4.3) \quad Av = -\tilde{F}_v, \quad v|_{\partial D} = \varphi,
\]
where \( \tilde{F}_v(x) = \tilde{f}(x, v(x)), \ \tilde{f}(x, y) = f(x, y, w(x)), \ x \in D, \ y \in \mathbb{R}, \) is uniquely solvable in \( W^1_2 \). For this purpose, for \( n \in \mathbb{N} \) define \( a_n, b_n \) by (3.7), and \( h_n = (h_n^1, \ldots, h_n^d) : D \to \mathbb{R}^d \) and \( f_n : D \times \mathbb{R} \to \mathbb{R} \) by
\[
h_n^i(x) = (-n) \vee \psi^i(x) \wedge n, \quad i = 1, \ldots, d, \quad f_n(x, y) = f(x, y, h_n(x)).
\]
Let \( \{(X, P^n_x); x \in \mathbb{R}^d\} \) be a diffusion corresponding to \( A^n \) defined by (3.3). By [4, Theorem 3.4] (see also [8, Theorem 3.1]), for each \( x \in D \) there is a unique solution \( (Y^{x, n}, Z^{x, n}) \) of the BSDE (\( \varphi, \tilde{f} \)) associated with \( (X, P^n_x) \), and moreover, by [4, Lemma 6.2], \( Y^{x, n} = v_n(X^\tau) P^n_x \)-a.s., where \( v_n(x) = Y^{x, n}_0 \) for \( x \in D \). We will show that \( v_n \) is a unique solution of the problem
\[
(4.4) \quad A^n v_n = -\tilde{F}_n, \quad v_n|_{\partial D} = \varphi
\]
in $W^1_2$, where $\tilde{F}_n : D \to \mathbb{R}$, $\tilde{F}_n(x) = f_n(x, v_n(x))$. Since $\alpha_1 < \gamma$, by (2.4) applied to $v_n$ we have
\begin{equation}
|v_n(x)|^2 \leq \|\varphi\|^2 \mathbb{E} x^{2 \gamma} + 2 \delta^{-1}(\|g\|^2_{\gamma, x} + L^2\|w\|^2_{\gamma, x}).
\end{equation}
If $2\mu + L^2 > 0$ then $\|g\|^2_{\gamma, x} \leq U_\gamma(x)\|g\|_\infty$, and if $2\mu + L^2 \leq 0$ then by Hölder's inequality,
\[\|g\|^2_{\gamma, x} \leq (U_{\gamma q'}(x))^{1/q'} \left( \int_D (G_D(x, y))^{(p/2q')'} \, dy \right)^1/(p/2q)' \|g\|^2_p,\]
where $(p/2q)'$ is the Hölder conjugate to $p/2q$. By [13, Corollary A.1.2] and [5, Theorem 8.16] (for $d = 1$ see remarks at the beginning of Section 3), $\sup_{x \in D} \|G_D(x, \cdot)\|_r < \infty$ for any $r > 1$ whose Hölder conjugate is greater than $(2 \vee d)/2$. Hence, in both cases, $\sup_{x \in D} \|g\|^2_{\gamma, x} < \infty$. From this and (4.5) we see that $v_n \in L_2$, hence $\tilde{F}_n \in L_2$. Set now $\varphi_k = \varphi \ast j_k$ and $\tilde{F}_{n,k} = ((-k) \vee \tilde{F}_n \wedge k) \ast j_k$ for $k \in \mathbb{N}$, where $j_k$ is defined as in the proof of Theorem 3.7. Since $D$ is regular, $\varphi_k \in C^\beta(D) = C^{0,\beta}(\overline{D})$ and $\tilde{F}_{n,k}$ are bounded and belong to $C^\beta(D)$, it follows from [5, Theorem 6.11] that there is a unique solution $v_{n,k} \in C^{2,\beta}(D) \cap C(\overline{D})$ of the problem $A^n v_{n,k} = -\tilde{F}_{n,k}$, $v_{n,k}|_{\partial D} = \varphi_k$. By Itô’s formula, $(Y_t^{n,k}, Z_t^{n,k}) = (v_{n,k}(X_t^n), (\sigma_n \nabla v_{n,k}(X_t^n)))$, $t \geq 0$, is a solution of the BSDE $(\varphi_k, \tilde{F}_{n,k})$ associated with $(X, P^n_x)$. Therefore
\begin{equation}
Y_t^{n,k} = \varphi_k(X_T) + \int_0^T \tilde{F}_{n,k}(X_s) \, ds \quad \text{for} \quad t \leq T,
\end{equation}
Obviously $\varphi_k \to \varphi$ uniformly on $\overline{D}$ and $\tilde{F}_{n,k} \to \tilde{F}_n$ in $L_2$ as $k \to \infty$. Hence, by Proposition 3.3, $v_{n,k} \to \tilde{v}_n$ in $W^1_2$), where $\tilde{v}_n \in W^1_2$ is a unique solution of the problem $A^n \tilde{v}_n = -\tilde{F}_n$, $\tilde{v}_n|_{\partial D} = \varphi$. On the other hand, by (2.6) we have
\[|v_{n,k}(x)| \leq \|\varphi\|_\infty + \|g\|_{0, x}^{1/2} \ast j_k + K\|v_n\|_\infty + L n,
\]
since $\|\varphi\|_\infty \leq \|\varphi\|_\infty$ and $|\tilde{F}_n| \leq |\tilde{F}_n| \ast j_k$. Consequently, $\{v_{n,k}\}_{k \in \mathbb{N}}$ is bounded in $D$. Using the above properties of $\{\varphi_k\}_{k \in \mathbb{N}}$, $\{\tilde{F}_{n,k}\}_{k \in \mathbb{N}}$, $\{v_{n,k}\}_{k \in \mathbb{N}}$ and the fact that $\tilde{F}_n \in L_0(x)$ for $x \in D$, as in the proof of (3.9) we deduce from (4.6) that there is a continuous version $\tilde{v}_n$ of $\tilde{v}_n$ such that
\[\tilde{v}_n(X_t) = \varphi(X_T) + \int_0^T \tilde{F}_n(X_s) \, ds \quad \text{for} \quad t \leq T,
\]
$P^n_x$-a.s. At the same time,
\[ Y_t^{x,n} = \varphi(X_T) + \int_{t}^{\tau} \tilde{F}_n(X_s) \, ds - \int_{t}^{\tau} \langle Z_s^{x,n}, \sigma^{-1}_n(X_s) \, dM_s^n \rangle, \quad t \geq 0, \]

\( P^n_x \)-a.s. Putting

\[ \xi_t = Y_t^{x,n} - \nu_n(X_t), \quad N_t = \int_{0}^{t} \langle Z_s^{x,n}, (\sigma_n \nabla \nu_n)(X_s), \sigma^{-1}_n(X_s) \, dM_s^n \rangle \]

we have \( \xi_t = N_{t \wedge \tau} - N_{\tau}, \ t \geq 0, \) and hence

\[ \xi_t = \xi_{t \wedge \tau} = E_{\tau}^n(\xi_{t \wedge \tau} \mid {\mathcal{F}}_{t \wedge \tau}) = E_{\tau}^n(-N_{\tau} \mid {\mathcal{F}}_{t \wedge \tau}) + N_{t \wedge \tau} = 0. \]

Therefore \( \nu_n(X_\tau) = Y_\tau^{x,n} = Y_0^{x,n} = \nu_n(x). \)

From what has already been proved it follows that \( \nu_n \) is a unique, in \( W^1_2 \), weak solution of the problem (4.4), and moreover, \( (\nu_n(X_\tau), (\sigma_n \nabla \nu_n)(X_\tau)) \)

is a solution of the BSDE \((\varphi, \tilde{f})\) associated with \((X, P^n_x)\). By the definition of a weak solution, \( \nu_n - \varphi \in W^1_2 \) and

\[ (a_n \nabla (\nu_n - \varphi), \nabla \eta)_2 = 2(\tilde{F}_n, \eta)_2 - (a_n \nabla \varphi, \nabla \eta)_2 + 2(\nabla (\nu_n - \varphi), \eta b_n)_2 + 2(\nabla \varphi, \eta b_n)_2 \]

for any \( \eta \in W^1_2 \). Putting \( \eta = \nu_n - \varphi \) we obtain

\[ (a_n \nabla (\nu_n - \varphi), \nabla (\nu_n - \varphi))_2 \leq 2(\tilde{F}_n - \tilde{F}_\varphi, \nu_n - \varphi)_2 + 2(\tilde{F}_\varphi, \nu_n - \varphi)_2 + A \|
abla \varphi\|_2 \cdot \|
abla (\nu_n - \varphi)\|_2 \]

\[ + 2A \|
abla (\nu_n - \varphi)\|_2 \cdot \|\nu_n - \varphi\|_2 + 2A \|
abla \varphi\|_2 \cdot \|\nu_n - \varphi\|_2 \]

\leq 2 \mu \|\nu_n - \varphi\|_2^2 + 2(\|g\|_2 + K \|\varphi\|_2 + L \|w\|_2) \cdot \|\nu_n - \varphi\|_2 \]

\[ + \lambda^{-1}A^2 \|\nabla \varphi\|_2^2 + 4 \lambda^{-1}A^2 \|\nu_n - \varphi\|_2^2 + 2 \lambda \|
abla (\nu_n - \varphi)\|_2^2 \]

\[ + 2A \|
abla \varphi\|_2 \cdot \|\nu_n - \varphi\|_2 , \]

and hence \( \|
abla (\nu_n - \varphi)\|_2 \leq C(\|g\|_2 + \|\varphi\|_{W^1_2} + \|w\|_2 + \|\nu_n - \varphi\|_2) \) for some \( C = C(\lambda, A, \mu, K, L) \). From this and (4.5) we see that \( \{\nu_n - \varphi\}_{n \in \mathbb{N}} \) is bounded in \( W^1_2 \), and therefore, by Rellich’s theorem (see, e.g., [5, Theorem 7.22]), precompact in \( L_2 \). Suppose that for some subsequence (still denoted by \( n \)) \( \nu_n \to v \) in \( L_2 \). Then \( \tilde{F}_{\nu_n} \to \tilde{F}_v \) in \( L_2 \) by [6, Theorem 2.1]. From this and Proposition 3.3 we can conclude that \( v \) is a weak solution of (4.3).

**Step 2.** There exists a version \( u \) of \( v \) such that \( u \) is continuous on \( \overline{D} \), \( \text{sup}_{x \in D} \|u\|_{W^1_2(x)} < \infty \) and for each \( x \in D \) the pair (1.8) is a solution, in \( H^1_{\gamma;D} \), of the BSDE \((\varphi, \tilde{f})\) associated with \((X, P_x)\). Indeed, by Lemma 3.2 and (4.5), \( \text{sup}_{x \in D} \|v\|_{0;x} < \infty \). Hence \( \text{sup}_{x \in D} \|\tilde{F}_v\|_{0;x} < \infty \), and so, by Theorem 3.7, there is a unique solution \( u \in W^1_2 \cap C(\overline{D}) \) of (3.1) with \( F \) replaced by \( \tilde{F}_v \). Since the problem (3.1) is uniquely solvable in \( W^1_2 \), it follows that
$u = v$ a.e. Moreover, again by Theorem 3.7, the pair (1.8) solves the BSDE $(\varphi, \tilde{f})$ in $\mathcal{H}_0^{1+d}$. Since (2.2) is satisfied and $\sup_{x \in D} \|\tilde{f}(\cdot, 0)\|_{\gamma;x} < \infty$, it follows from Proposition 2.3 that $u$ is in fact a solution of class $\mathcal{H}_{\gamma;x}$ for every $x$ and its norms in $\mathcal{H}_{\gamma;x}^{1+d}$ are bounded uniformly in $x \in D$.

**Step 3.** There is a weak solution $u \in W_2^1$ of (1.5). To prove this, denote by $V$ the space $W_2^1$ equipped with the norm

$$
\|u\|_V^2 = \begin{cases}
\|u\|_2^2 + (\alpha - \alpha_1)(u, u U_a)_2 + (a \nabla u, U_a \nabla u)_2 & \text{if } b = 0,
\|u\|_2^2 + (a \nabla u, \widehat{G}_D \nabla u)_2 & \text{if } b \neq 0,
\end{cases}
$$

and let $V_\gamma = \{u \in V : \sup_{x \in D} \|u\|_{V_\gamma(x)} < \infty\}$. Define the mapping $\Phi : V_\gamma \to V$ by letting $\Phi(\tilde{w})$ be the solution of (4.3) with $w = \sigma \nabla \tilde{w}$. By what has already been proved, $\Phi$ is well defined and in fact $\Phi : V_\gamma \to V_\gamma$. Set $\Phi(w_i) = v_i$, $i = 1, 2$, $v = v_1 - v_2$, $w = w_1 - w_2$ and $F = f(\cdot, v_1, \sigma \nabla w_1) - f(\cdot, v_2, \sigma \nabla w_2)$. If $b = 0$ then as in the proof of Theorem 4.1 we show that (4.1) holds with $v$ in place of $u$. On the other hand,

$$
2(F, v \alpha U_a)_2 = 2(f(\cdot, v_1, \sigma \nabla w_1) - f(\cdot, v_2, \sigma \nabla w_1), v \alpha U_a)_2
\leq 2\mu(v, v \alpha U_a)_2 + 2L(\|\sigma \nabla w_1\|, \|v \alpha U_a\|)_2
\leq (2\mu + (1 + \varepsilon)L^2)(v, v \alpha U_a)_2 + (1 + \varepsilon)^{-1}(a \nabla w, U_a \nabla w)_2,
$$

the last inequality being a consequence of an elementary inequality $2ab \leq \eta a^2 + \eta^{-1}b^2$ with $\eta = (1 + \varepsilon)L$. Hence

$$
\|v\|_2^2 + \alpha(v, v \alpha U_a)_2 + (a \nabla v, \alpha \nabla v)_2
\leq (2\mu + (1 + \varepsilon)L^2)(v, v \alpha U_a)_2 + (1 + \varepsilon)^{-1}(a \nabla w, U_a \nabla w)_2.
$$

If $b \neq 0$ then (4.2) holds with $v$ in place of $u$ and, since $2\mu \leq -L^2$, we have

$$
(F, v \widehat{G}_D)_2 \leq \varepsilon L^2 \|\widehat{G}_D\|_\infty \|v\|_2^2 + (1 + \varepsilon)^{-1}(a \nabla w, \widehat{G}_D \nabla w)_2.
$$

Thus, in both cases $\Phi$ is contractive. Set $u_0 = 0$, $u_n = \Phi(u_{n-1})$, $n \in \mathbb{N}$. Then $\{u_n\}$ is a Cauchy sequence in $V$ and hence bounded in $L_2$. In fact, it is bounded in $W_2^1$. Indeed, we have

$$
\lambda \|\nabla (u_n - \varphi)\|_2^2 \leq (a \nabla (u_n - \varphi), \nabla (u_n - \varphi))_2
= -(a \nabla \varphi, \nabla (u_n - \varphi))_2 + 2(\nabla u_n, (u_n - \varphi)b)_2
+ 2(f(\cdot, u_n, \sigma \nabla u_{n-1}), u_n - \varphi)_2
\leq 4^{-1}\lambda \|\nabla (u_n - \varphi)\|_2^2 + \lambda^{-1}A^2\|\nabla \varphi\|_2^2
+ 2^{-1}\lambda \|\nabla (u_n - \varphi)\|_2^2 + 2^{-1}\lambda \|\nabla \varphi\|_2^2 + 4\lambda^{-1}A^2\|u_n - \varphi\|_2^2
+ 2(g + K\|u_n\| + L\|\sigma \nabla u_{n-1}\|, |u_n - \varphi|)_2.
$$
Hence $\|\nabla u_n\|^2 \leq C_1 + C_2 \|\nabla u_{n-1}\|^2$ for some $C_1, C_2$ depending only on $\lambda, A, K, \Lambda, \|g\|_2$ and $\sup_{n \geq 0} \|u_n\|_2$, which implies boundedness of $\{u_n\}$ in $W^1_2$. Therefore $\{u_n\}$ is weakly relatively compact in $W^1_2$. Since we know that $\{u_n\}$ is Cauchy in $V$ as well, it converges in $V$ to some $\tilde{u}$.

Step 4. $\tilde{u} \in V_\gamma$. To prove this, for fixed $x \in D$ we denote by $\mathcal{V}(x)$ the space $\mathcal{W}_\gamma(x)$ equipped with a norm $\| \cdot \|_{\mathcal{V}(x)}$ equivalent to $\| \cdot \|_{W^1_2(x)}$, defined by

$$\|u\|^2_{\mathcal{V}(x)} = (\gamma - \alpha_1)\|u\|^2_{\gamma; x} + \|\sigma \nabla u\|^2_{\gamma; x}.$$ 

Define $\{u_n\}$ as in Step 3. By Step 2, $\Phi : \mathcal{V}(x) \to \mathcal{V}(x)$ and for each $n \in \mathbb{N}$ the pair $(w_{n+1}(X^\tau), (\sigma \nabla w_{n+1})(X^\tau))$, where $w_n = u_n - u_{n-1}$, is a solution, in $H^{\gamma+d}_{\gamma; x}$, of the BSDE $(0, F_{n+1} - F_n)$ with $F_n = f(\cdot, u_n, \sigma \nabla u_{n-1})$. By Itô's formula,

$$0 = e^{\gamma \tau} |w_{n+1}(X_\tau)|^2 = |w_{n+1}(X_0)|^2 + \gamma \int_0^\tau e^{\gamma s} |w_{n+1}(X_s)|^2 ds$$

$$+ \int_0^\tau e^{\gamma s} w_{n+1}(X_s) d(w_{n+1}(X_s)) + \int_0^\tau e^{\gamma s} d(w_{n+1}(X_s)).$$

Hence

$$E_x \int_0^\tau e^{\gamma t} \{\gamma |w_{n+1}(X_t)|^2 + (a \nabla w_{n+1}, \nabla w_{n+1})(X_t)\} dt$$

$$\leq 2E_x \int_0^\tau e^{\gamma t} (F_{n+1} - F_n) \cdot w_{n+1}(X_t) dt.$$ 

We have

$$2(F_{n+1} - F_n) \cdot w_{n+1} = 2(f(\cdot, u_{n+1}, \sigma \nabla u_n) - f(\cdot, u_n, \sigma \nabla u_{n-1})) \cdot w_{n+1}$$

$$= 2(f(\cdot, u_{n+1}, \sigma \nabla u_n) - f(\cdot, u_n, \sigma \nabla u_n)) \cdot w_{n+1}$$

$$+ 2(f(\cdot, u_n, \sigma \nabla u_n) - f(\cdot, u_n, \sigma \nabla u_{n-1})) \cdot w_{n+1}$$

$$\leq 2\mu |w_{n+1}|^2 + 2L |\sigma \nabla w_n| \cdot |w_{n+1}|$$

$$\leq (2\mu + (1 + \varepsilon)L^2) |w_{n+1}|^2 + (1 + \varepsilon)^{-1} (a \nabla w_n, \nabla w_n).$$

By the above,

$$E_x \int_0^\tau e^{\gamma t} \{(\gamma - 2\mu - (1 + \varepsilon)L^2) |w_{n+1}(X_t)|^2 + (a \nabla w_{n+1}, \nabla w_{n+1})(X_t)\} dt$$

$$\leq (1 + \varepsilon)^{-1} E_x \int_0^\tau e^{\gamma t} (a \nabla w_n, \nabla w_n)(X_t) dt,$$

which shows that $\|w_{n+1}\|^2_{\mathcal{V}(x)} \leq (1 + \varepsilon)^{-1} \|w_n\|^2_{\mathcal{V}(x)}$. Hence $\{u_n\}$ is a Cauchy sequence in $\mathcal{V}(x)$ and therefore converges to some $u$. In particular, $u_n \to u$.
in $\mathcal{L}_0(x)$. On the other hand, since $u_n \to \tilde{u}$ in $V$, $u_n \to \tilde{u}$ in $\mathbb{L}_2$, and hence $u = \tilde{u}$ a.e. by Lemma 3.2. Consequently, $u_n \to \tilde{u}$ in $\mathcal{V}(x)$ and
\[\|u_1 - \tilde{u}\|_{\mathcal{V}(x)} \leq q(1 - q)^{-1}\| u_1 - u_0\|_{\mathcal{V}(x)} \quad \text{with} \quad q = (1 + \varepsilon)^{-1/2}.\]
Hence $\|\tilde{u}\|_{\mathcal{V}(x)} \leq (1 - q)^{-1}\| u_1\|_{\mathcal{V}(x)}$, and so $u \in \mathcal{V}_\gamma$ since $u_1 \in \mathcal{V}_\gamma$.

Step 5. By Step 4, $\Phi(\tilde{u})$ is well defined. Therefore $\tilde{u}$ is a fixed point of $\Phi$, because $\|\tilde{u} - \Phi(\tilde{u})\|_V \leq \|\tilde{u} - u_n\|_V + \|\Phi(u_{n-1}) - \Phi(\tilde{u})\|_V$ for $n \in \mathbb{N}$. Consequently, $\tilde{u} \in W^1_2$ is a weak solution of (3.2). Since $F_{\tilde{u}} \in \mathcal{V}_\gamma$ it follows from Theorems 3.7 and 4.1 that $\tilde{u}$ has a version $u \in C(\overline{D})$, which completes the proof. $
$
From the above proof it is evident that if $a \in C^{1,\beta}(\overline{D})$ for some $\beta \in (0, 1)$ then in case $b = 0$ condition (vi’‘) may be replaced by $\sup_{x \in D} E_x e^{a_T} < \infty$ (in Step 1 we need not approximate $a$ by $\{a_n\}$). We do not know whether (vi’‘) may be replaced by the latter condition in the case of arbitrary $a$ satisfying (1.2).

**Theorem 4.3.** Assume (i)–(vi) and define $\alpha$ as in Theorem 4.2. If $u \in W^1_2 \cap C(\overline{D})$ is a solution of (1.5) then for each $x \in D$ the pair (1.8) is a unique, in the class $\mathcal{H}^{1+d}_{\alpha \vee 0; x}$, solution of the BSDE $(\varphi, f)$ associated with $(X, P_x)$.

**Proof.** This follows from Theorems 3.7 and 4.2. $
$
**References**


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