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On the spectral Nevanlinna–Pick problem

by

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Abstract. We give several characterizations of the symmetrized n-disc G_n which generalize to the case $n \geq 3$ the characterizations of the symmetrized bidisc that were used in order to solve the two-point spectral Nevanlinna–Pick problem in $\mathcal{M}_2(\mathbb{C})$. Using these characterizations of the symmetrized n-disc, which give necessary and sufficient conditions for an element to belong to G_n, we obtain necessary conditions of interpolation for the general spectral Nevanlinna–Pick problem. They also allow us to give a method to construct analytic functions from the open unit disc of $\mathbb{C}$ into G_n and to obtain some of the complex geodesics on G_n.

1. INTRODUCTION

The *spectral Nevanlinna–Pick* problem is the following.

We are given distinct points $\lambda_1, \ldots, \lambda_m$ in the open unit disc $\mathbb{D}$ of the complex plane and $n \times n$ complex matrices $W_1, \ldots, W_m \in \mathcal{M}_n(\mathbb{C})$, and we would like to find necessary and sufficient conditions for the existence of an analytic $n \times n$ matrix-valued function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \quad (j = 1, \ldots, m) \tag{1.1}$$

and

$$r(F(\lambda)) \leq 1 \quad (\lambda \in \mathbb{D}). \tag{1.2}$$

(*Here $r(W)$ denotes the spectral radius of the square matrix W.*)

It is a variant of the well known *classical Nevanlinna–Pick* problem in $\mathcal{M}_n(\mathbb{C})$, that is, the problem obtained by replacing the condition (1.2) by

$$\|F(\lambda)\| \leq 1 \quad (\lambda \in \mathbb{D}). \tag{1.3}$$

(Here $\|W\|$ denotes the operator norm on $\mathcal{M}_n(\mathbb{C})$.) The classical version has a complete solution: the existence of a function F satisfying (1.1) and (1.3) can be reduced [11, Chapter X] to the determination of the semi-positivity of

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the so-called *Nevanlinna–Pick matrix* associated to the interpolation data,

$$\left[\frac{I - W_j^* W_k}{1 - \overline{\lambda}_j \lambda_k}\right]_{j,k=1}^m. \tag{1.4}$$

Lots of different approaches were developed in order to solve this classical version. The standard one is operator-theoretic, and it uses the Sz. Nagy–Foiaş commutant lifting theorem [16]. So, the first idea for the spectral version was to try to find a spectral variant of the commutant lifting theorem. It was obtained by Bercovici, Foiaş and Tannenbaum in 1991, and using this result we obtain the following theorem.

THEOREM 1.1 ([7, Theorem 4]). *For given interpolation data $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ and $W_1, \dots, W_m \in \mathcal{M}_n(\mathbb{C})$, there exists a bounded analytic function $F : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, m$ and $\sup_{|\lambda|<1} r(F(\lambda)) < 1$ if and only if we can find invertible matrices $X_1, \dots, X_m \in \mathcal{M}_n(\mathbb{C})$ and an analytic function $G : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $G(\lambda_j) = X_j W_j X_j^{-1}$ for $j = 1, \dots, m$ and $\sup_{|\lambda|<1} \|G(\lambda)\| < 1$.*

Therefore, we can solve this slightly simpler form of the spectral Nevanlinna–Pick problem if and only if we can solve the classical Nevanlinna–Pick problem for some target matrices $\{Y_j\}_{j=1}^m$ such that Y_j is conjugate to W_j for all j. The theorem does provide in principle a method of determining whether an interpolation function exists, but is rather difficult to apply in general, because there is no control on the new matrices Y_j. It involves a non-trivial search over $n^2 m$ parameters, and we cannot obtain an explicit necessary and sufficient condition of interpolation.

Clearly, if $n = 1$ then the spectral Nevanlinna–Pick problem is just the classical scalar Nevanlinna–Pick problem, which is solved. That is why throughout this paper n will always be assumed strictly greater than 1. The simplest case of the spectral Nevanlinna–Pick problem is when we consider a two-point interpolation problem, one of the matrices being the null matrix $0 \in \mathcal{M}_n(\mathbb{C})$. By using the Vesentini theorem [6, Theorem 3.4.7] on the subharmonicity of the spectral radius, we can easily see that there exists $F : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $F(0) = 0$, $F(\lambda_0) = W_0$ and $r(F(\lambda)) \leq 1$ on $\mathbb{D}$ if and only if $r(W_0) \leq |\lambda_0|$. Apart from this result, the two-point spectral Nevanlinna–Pick problem in $\mathcal{M}_2(\mathbb{C})$ is the only case for which we can find an explicit necessary and sufficient condition in the published literature. (See [5], [8], and the references therein.) The methods used to obtain this result are totally different from the ones of Bercovici, Foiaş and Tannenbaum. The purpose of this paper is to generalize to the case $n \geq 3$ some of the ingredients that were used to solve the two-point spectral Nevanlinna–Pick problem for $n = 2$.

2. THE SYMMETRIZED n-DISC

For an $n \times n$ complex matrix W, denote by $\sigma(W)$ its spectrum. (Each eigenvalue of W is counted according to its multiplicity.) Denote by Ω_n the open spectral unit ball in $\mathcal{M}_n(\mathbb{C})$, that is,

$$\Omega_n = \{W \in \mathcal{M}_n(\mathbb{C}) : r(W) < 1\}.$$

For $W \in \mathcal{M}_n(\mathbb{C})$, the fact that W belongs to Ω_n is equivalent to the fact that its characteristic polynomial $P(z) = \det(zI - W)$ has all its roots inside $\mathbb{D}$. Therefore, if we want a necessary and sufficient condition for a matrix to belong to Ω_n then clearly this condition must be on the coefficients of the characteristic polynomial. This leads us to consider the elementary symmetric functions in n variables

$$S_n^k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} \prod_{i=1}^{k} \lambda_{j_i} \quad (k = 1, \ldots, n), \tag{2.1}$$

the symmetrization map

$$\pi_n(\lambda_1, \ldots, \lambda_n) = (S_n^1(\lambda_1, \ldots, \lambda_n), \ldots, S_n^n(\lambda_1, \ldots, \lambda_n)) \tag{2.2}$$

and the set $G_n := \pi_n(\mathbb{D}^n)$, which we shall call the *open symmetrized n-disc*. It is clear that $W \in \Omega_n$ if and only if $\pi_n(\sigma(W)) \in G_n$. Therefore, we can consider $\Pi_n : \Omega_n \to G_n$ given by

$$\Pi_n(W) = \pi_n(\sigma(W)). \tag{2.3}$$

Then (2.2) and the Viète relations imply that Π_n is a well defined analytic map. It is surjective, but it is very far from being injective: for example, it does not take into account the Jordan form of W. For $(s_1, \ldots, s_n) \in G_n$, by considering the companion matrix associated to the polynomial $P(z) = z^n - s_1 z^{n-1} + \cdots + (-1)^n s_n$ we obtain the analytic map $J_n : G_n \to \Omega_n$ given by

$$J_n(s_1, \ldots, s_n) = \begin{bmatrix} 0 & 0 & \ldots & 0 & (-1)^{n-1} s_n \\ 1 & 0 & \ldots & 0 & (-1)^{n-2} s_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -s_2 \\ 0 & 0 & \ldots & 1 & s_1 \end{bmatrix}. \tag{2.4}$$

Even though $\Pi_n \circ J_n$ is the identity on G_n, one can easily see that $J_n \circ \Pi_n$ is not the identity on Ω_n. Therefore, we expect that the two interpolation problems, the one into G_n and the one into Ω_n, are not equivalent. This is indeed the case (see [2, Example 2.2]), but in the generic case, that is, when all the target matrices are non-derogatory, the following result holds.

THEOREM 2.1 ([2, Theorem 2.1]). *Let $\lambda_1,\dots,\lambda_m \in \mathbb{D}$ be distinct points, and consider m matrices $W_1,\dots,W_m \in \Omega_n$. Suppose that W_j is non-derogatory for all j (that is, W_j is conjugate to the companion matrix associated to its characteristic polynomial). Then the following assertions are equivalent.*

(i) *There exists $f : \mathbb{D} \to \Omega_n$ analytic such that $f(\lambda_j) = W_j$ for all j.*
(ii) *There exists $g : \mathbb{D} \to G_n$ analytic such that $g(\lambda_j) = \Pi_n(W_j)$ for all j.*

We think that the interpolation problem into G_n is more tractable than the interpolation problem into Ω_n. First of all, the dimension of G_n is n while the dimension of Ω_n is n^2, and therefore we are reducing an n^2-dimensional problem to an n-dimensional one. By applying Π_n to a matrix W we are erasing its Jordan form, and we consider (in an analytical way) only the spectrum of W. Also, the set G_n is bounded (and therefore compact) while Ω_n is not: therefore, a Montel type reasoning can be applied for G_n (this is not the case for Ω_n, see [7, Example 3]). The only difference in favor of Ω_n is that it is balanced (if $\lambda \in \overline{\mathbb{D}}$ and $W \in \Omega_n$ then $\lambda W \in \Omega_n$), while G_n is not. Even though it is not a balanced set, the symmetrized n-disc has a similar property. Define $\varrho : \mathbb{C}^n \to \mathbb{R}_+$ given by

$$\varrho(s_1,\dots,s_n) = \max\{|\lambda_1|,\dots,|\lambda_n| : S_n^k(\lambda_1,\dots,\lambda_n) = s_k,\ k = 1,\dots,n\}.$$

The map ϱ is the analogue for G_n of the spectral radius on Ω_n. It is clear that G_n is the set of all points $(s_1,\dots,s_n) \in \mathbb{C}^n$ for which $\varrho(s_1,\dots,s_n) < 1$, and for its closure, the *closed symmetrized n-disc* $\Gamma_n := \overline{G}_n$, we have $\Gamma_n = \{(s_1,\dots,s_n) \in \mathbb{C}^n : \varrho(s_1,\dots,s_n) \le 1\}$. The most important properties of the map ϱ are given in the following proposition.

PROPOSITION 2.1. (i) *If $\lambda \in \mathbb{C}$ and $(s_1,\dots,s_n) \in \mathbb{C}^n$ then*

$$\varrho(\lambda s_1,\dots,\lambda^n s_n) = |\lambda|\varrho(s_1,\dots,s_n).$$

(ii) *For a domain $\Delta \subseteq \mathbb{C}$ and an analytic map $g : \Delta \to \mathbb{C}^n$, the composition $\varrho \circ g$ is subharmonic on Δ.*

Proof. Part (i) is a consequence of the definition of ϱ. Part (ii) is a consequence of the Vesentini theorem [6, Theorem 3.4.7] and the fact that $\varrho \circ g = r(J_n \circ g)$ on Δ. ∎

Part (i) implies that if $(s_1,\dots,s_n) \in G_n$ and $\lambda \in \overline{\mathbb{D}}$, then $(\lambda s_1,\dots,\lambda^n s_n) \in G_n$. Part (ii) and the fact that the subharmonic functions satisfy the maximum modulus principle give the following corollary. It will be a very useful tool for us in the remainder of this paper.

COROLLARY 2.1. *Let f be analytic on a neighborhood of $\overline{\mathbb{D}}$ with values in $\mathbb{C}^n$ such that $f(\mathbb{T}) \subseteq \Gamma_n$. Then $f(\overline{\mathbb{D}}) \subseteq \Gamma_n$. Moreover, if there exists a point $\lambda_0 \in \mathbb{D}$ such that $f(\lambda_0) \in G_n$ then $f(\mathbb{D}) \subseteq G_n$, and if there exists $\lambda_0 \in \mathbb{D}$ such that $f(\lambda_0) \in \Gamma_n \setminus G_n$ then $f(\mathbb{D}) \subseteq \Gamma_n \setminus G_n$.*

3. CHARACTERIZATIONS OF G_n

For a given matrix $W \in \mathcal{M}_n(\mathbb{C})$, we have already seen that W belongs to Ω_n if and only if $(s_1, \dots, s_n) := \Pi_n(W)$ belongs to G_n, that is, if and only if the polynomial

$$P(z) = z^n - s_1 z^{n-1} + \dots + (-1)^n s_n \tag{3.1}$$

has all its roots inside $\mathbb{D}$. To verify whether all the roots of a polynomial are inside $\mathbb{D}$, the well known Schur theorem [15] is a standard tool. Denote by $P^{\#}(z)$ the reverse polynomial $z^n\overline{P(1/\overline{z})}$, that is, $P^{\#}(z) = (-1)^n \overline{s}_n z^n + \dots + (-\overline{s}_1)z + 1$, and by S the shift operator $S(x_1, \dots, x_n) = (x_2, x_3, \dots, x_n, 0)$ on $\mathbb{C}^n$. Then the Schur test asserts that P has all its roots inside $\mathbb{D}$ if and only if $\|P(S)(P^{\#}(S))^{-1}\| < 1$, that is, if and only if

$$\left\| \begin{bmatrix} (-1)^n s_n & (-1)^{n-1} s_{n-1} & \dots & -s_1 \\ 0 & (-1)^n s_n & \dots & s_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^n s_n \end{bmatrix} \cdot \begin{bmatrix} 1 & -\overline{s}_1 & \dots & (-1)^{n-1}\overline{s}_{n-1} \\ 0 & 1 & \dots & (-1)^{n-2}\overline{s}_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}^{-1} \right\| < 1. \tag{3.2}$$

Unfortunately, we cannot obtain even necessary conditions of interpolation into Ω_n by using Schur's result because of the conjugation operation that appears in (3.2), which does not respect analyticity. Characterizations of G_n (and therefore of Ω_n) which respect analyticity will be given in the rest of this section.

3.1. Characterization of G_n given by $\mathbb{D}$. For $(s_1, \dots, s_n) \in \mathbb{C}^n$, consider the polynomial $P(z)$ given by (3.1). If we put

$$Q(z) = n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1} s_{n-1} z^{n-2} + \dots + (-s_1), \tag{3.3}$$

$$R(z) = n - (n-1)s_1 z + \dots + (-1)^{n-1} s_{n-1} z^{n-1}, \tag{3.4}$$

then we can easily verify that

$$Q(z) = \frac{d}{dz}(z^n P(1/z)), \qquad R(z) = z^{n-1} P'(1/z),$$

for all z in $\mathbb{C} \setminus \{0\}$. Using Q and R, we define $f = Q/R$, that is,

$$f(z) = \frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1} s_{n-1} z^{n-2} + \dots + (-s_1)}{n - (n-1)s_1 z + \dots + (-1)^{n-1} s_{n-1} z^{n-1}}. \tag{3.5}$$

This rational function will play a fundamental role in this paper. Its importance is reflected in the next theorem.

THEOREM 3.1. *Let* $(s_1,\dots,s_n)\in\mathbb{C}^n$, *and let* f *be given by* (3.5). *The following assertions are equivalent.*

(i) *The element* $(s_1,\dots,s_n)$ *belongs to* G_n.

(ii) *We have*

$$\sup_{|z|\le 1}|f(z)|<1. \tag{3.6}$$

Proof. (i)⇒(ii): If all the zeros of the polynomial P given by (3.1) lie inside $\mathbb{D}$, then by the Lucas theorem [13, Theorem 6.1] the zeros of its derivative P' also lie inside $\mathbb{D}$. Therefore, $P'(w)\ne 0$ for all $w\in\mathbb{C}\setminus\mathbb{D}$, which gives $P'(1/z)\ne 0$ for all z in a neighborhood of $\overline{\mathbb{D}}$. Therefore, $R(z)\ne 0$ on the same neighborhood, and this implies that f is well defined and analytic on a neighborhood of $\overline{\mathbb{D}}$. To prove (3.6) it suffices to show that $|f(\xi)|<1$ for all ξ in $\mathbb{T}$. Consider therefore such a ξ. By the definition of Q, we have $Q(\xi)=n\xi^{n-1}P(1/\xi)-\xi^{n-2}P'(1/\xi)$, and therefore

$$\begin{aligned}|f(\xi)|&=\left|\frac{n\xi^{n-1}P(1/\xi)-\xi^{n-2}P'(1/\xi)}{\xi^{n-1}P'(1/\xi)}\right|\\&=\left|n\,\frac{P(1/\xi)}{P'(1/\xi)}-\frac{1}{\xi}\right|=\left|n\,\frac{P(\overline{\xi})}{\overline{\xi}P'(\overline{\xi})}-1\right|.\end{aligned}$$

We must prove that $\zeta\mapsto nP(\zeta)/(\zeta P'(\zeta))$ sends $\mathbb{T}$ into $\{w\in\mathbb{C}:|w-1|<1\}$. If we denote by $z_1,\dots,z_n\in\mathbb{D}$ the zeros of P, then for all ζ in $\mathbb{T}$ we have

$$\frac{\zeta P'(\zeta)}{nP(\zeta)}=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1-z_j/\zeta}. \tag{3.7}$$

Therefore $\zeta P'(\zeta)/(nP(\zeta))$ belongs to the convex hull of $\{1/(1-z_j/\zeta):j=1,\dots,n\}$. Since $z_j/\zeta\in\mathbb{D}$ for all $j\in\{1,\dots,n\}$, and since the map $w\mapsto 1/(1-w)$ is a conformal transformation of $\mathbb{D}$ onto $\{t\in\mathbb{C}:\mathrm{Re}(t)>1/2\}$, we obtain $\mathrm{Re}(1/(1-z_j/\zeta))>1/2$ for $j=1,\dots,n$. Now (3.7) implies that $\mathrm{Re}(\zeta P'(\zeta)/(nP(\zeta)))>1/2$. Using then the inverse of the above conformal transformation, we obtain $1-nP(\zeta)/(\zeta P'(\zeta))\in\mathbb{D}$.

(ii)⇒(i): We first prove that (3.6) implies that $R(z)\ne 0$ for all z in $\overline{\mathbb{D}}$. Suppose, for contradiction, that there exists z_0 in $\overline{\mathbb{D}}$ such that $R(z_0)=0$. Then $z_0\ne 0$ and (3.6) implies that $Q(z_0)=0$. The equality $R(z_0)=0$ gives $P'(1/z_0)=0$, and $Q(z_0)=0$ gives $nz_0^{n-1}P(1/z_0)-z_0^{n-2}P'(1/z_0)=0$. Thus $P(1/z_0)$ is also 0, and therefore $1/z_0$ is a zero of order at least 2 for the polynomial P. If we write $P(z)=(z-1/z_0)^m g(z)$ on $\mathbb{C}$, where $m\ge 2$ and g is non-zero on a neighborhood of $1/z_0$, then for z in a neighborhood of z_0

we have

$$\begin{aligned} f(z) &= \frac{nz^{n-1}P(1/z) - z^{n-2}P'(1/z)}{z^{n-1}P'(1/z)} \\ &= \frac{nz^{n-1}(1/z - 1/z_0)^m g(1/z)}{mz^{n-1}(1/z-1/z_0)^{m-1}g(1/z) + z^{n-1}(1/z-1/z_0)^m g'(1/z)} \\ &\quad - \frac{z^{n-2}(m(1/z-1/z_0)^{m-1}g(1/z) + (1/z-1/z_0)^m g'(1/z))}{mz^{n-1}(1/z-1/z_0)^{m-1}g(1/z) + z^{n-1}(1/z-1/z_0)^m g'(1/z)}, \end{aligned}$$

and therefore

$$|f(z_0)| = \left|\frac{-mz_0^{n-2}g(1/z_0)}{mz_0^{n-1}g(1/z_0)}\right| = |1/z_0| \geq 1,$$

which contradicts our hypothesis on f.

Therefore R is non-zero on $\overline{\mathbb{D}}$ and now (3.6) implies that $|Q| < |R|$ on $\mathbb{T}$. This yields $|z^{n-1}Q(1/z)| < |z^n R(1/z)|$ for all $z \in \mathbb{T}$, and therefore

$$\begin{aligned} &|n(-1)^n s_n + (n-1)(-1)^{n-1}s_{n-1}z + \cdots + (-s_1)z^{n-1}| \\ &\qquad < |nz^n - (n-1)s_1 z^{n-1} + \cdots + (-1)^{n-1}s_{n-1}z| \end{aligned}$$

on $\mathbb{T}$. Using now Rouché's theorem we find that the polynomials

$$\begin{aligned} &(n(-1)^n s_n + (n-1)(-1)^{n-1}s_{n-1}z + \cdots + (-s_1)z^{n-1}) \\ &\qquad + (nz^n - (n-1)s_1 z^{n-1} + \cdots + (-1)^{n-1}s_{n-1}z) \end{aligned}$$

and

$$nz^n - (n-1)s_1 z^{n-1} + \cdots + (-1)^{n-1}s_{n-1}z$$

have the same number of roots inside $\mathbb{D}$. The second polynomial is in fact $z^n R(1/z)$, and we have just proved that it has all its roots inside $\mathbb{D}$. Therefore the first polynomial, which is in fact nP, has n roots inside $\mathbb{D}$, and therefore $(s_1, \ldots, s_n) \in G_n$. ∎

For the particular case $n = 2$, we deduce that $(s, p) \in \mathbb{C}^2$ belongs to G_2 if and only if

$$\sup_{|z|\leq 1}\left|\frac{2zp - s}{2 - zs}\right| < 1. \tag{3.8}$$

We recover the characterization of G_2 given by Agler and Young [4, Theorem 1.1]. In fact, they proved that $(s,p) \in G_2$ if and only if $|s| < 2$ and (3.8) is satisfied. But one can easily see that (3.8) implies that $|s| < 2$. Indeed, if (3.8) is true and $|s| \geq 2$, then $2(2/s)p - s = 0$, that is, $p = s^2/4$, and then the rational function from (3.8) equals $-s/2$ on $\overline{\mathbb{D}}$. Since $|s| \geq 2$, this contradicts (3.8).

Let us remark that in the proof of Theorem 3.1 we have shown that the inequality (3.6) implies that the denominator of f does not have zeros on $\overline{\mathbb{D}}$.

Therefore, the denominator and the numerator of f do not have common zeros on $\overline{\mathbb{D}}$. But they can have common zeros on $\mathbb{C} \setminus \overline{\mathbb{D}}$, and therefore the degree of the rational function f can be strictly less than $n-1$. In fact, the degree of f can even be zero: if $\alpha \in \mathbb{D}$, then for $(\mathrm{C}_n^1\alpha, \mathrm{C}_n^2\alpha^2, \ldots, \mathrm{C}_n^n\alpha^n) \in G_n$ the associated function f is a constant, $f = -\alpha$ on $\mathbb{D}$. (Throughout this paper, C_n^k denotes the binomial coefficient $\mathrm{C}_n^k = n!/(k!(n-k)!)$.)

Using Theorem 3.1, we obtain the following necessary condition for the interpolation problem into G_n.

Corollary 3.1. *Consider m distinct points $\lambda_1, \ldots, \lambda_m$ in $\mathbb{D}$ and m elements $(s_{1,1}, \ldots, s_{n,1}), \ldots, (s_{1,m}, \ldots, s_{n,m})$ in G_n. If there exists an analytic function $g : \mathbb{D} \to G_n$ such that $g(\lambda_j) = (s_{1,j}, \ldots, s_{n,j})$ for $j = 1, \ldots, m$, then for all $z \in \overline{\mathbb{D}}$ the matrix $\left[\dfrac{1-\overline{z}_j z_k}{1-\overline{\lambda}_j\lambda_k}\right]_{j,k=1}^m$ is positive semi-definite, where*

$$z_j = \frac{n(-1)^n s_{n,j}z^{n-1} + (n-1)(-1)^{n-1}s_{n-1,j}z^{n-2} + \cdots + (-s_{1,j})}{n-(n-1)s_{1,j}z + \cdots + (-1)^{n-1}s_{n-1,j}z^{n-1}}$$

for $j = 1, \ldots, m$.

Proof. This is an immediate consequence of Theorem 3.1 and (1.4). ∎

Using Theorem 3.1 we also obtain a similar characterization of Γ_n. Here is the version of this result in the case of the closed symmetrized n-disc.

Theorem 3.2. *Let $(s_1, \ldots, s_n) \in \mathbb{C}^n$ and let f be given by (3.5). The following assertions are equivalent.*

(i) *The element $(s_1, \ldots, s_n)$ belongs to Γ_n.*
(ii) *We have*

$$\sup_{|z|\le 1} |f(z)| \le 1. \tag{3.9}$$

Proof. (i)⇒(ii): Fix $z \in \mathbb{D}$. Since the polynomial P given by (3.1) has all its roots inside $\overline{\mathbb{D}}$, the Lucas theorem shows that P' also has all its roots inside $\overline{\mathbb{D}}$. For the polynomial R given by (3.3), this implies that $R(z) \neq 0$.

By applying Theorem 3.1 to $(rs_1, \ldots, r^n s_n) \in G_n$ we obtain

$$\left|\frac{n(-1)^n r^n s_n z^{n-1} + (n-1)(-1)^{n-1}r^{n-1}s_{n-1}z^{n-2} + \cdots + (-rs_1)}{n-(n-1)rs_1 z + \cdots + (-1)^{n-1}r^{n-1}s_{n-1}z^{n-1}}\right| < 1$$

for all $r \in (0,1)$. By letting $r \to 1$ and using the fact that $R(z) \neq 0$ we deduce that $|f(z)| \le 1$. Therefore $|f(z)| \le 1$ for all z in $\mathbb{D}$ and since f is a rational function this implies that f is continuous on a neighborhood of $\overline{\mathbb{D}}$. Therefore, by continuity, $|f(z)| \le 1$ for all $z \in \overline{\mathbb{D}}$.

(ii)⇒(i): Consider a sequence $(r_k)_{k\in\mathbb{N}} \subseteq (0,1)$ such that $r_k \nearrow 1$. For each $k \in \mathbb{N}$ we have

$$\sup_{|z|\le 1}\left|\frac{n(-1)^n r_k^n s_n z^{n-1}+(n-1)(-1)^{n-1}r_k^{n-1}s_{n-1}z^{n-2}+\cdots+(-r_k s_1)}{n-(n-1)r_k s_1 z+\cdots+(-1)^{n-1}r_k^{n-1}s_{n-1}z^{n-1}}\right| = r_k \sup_{|w|\le r_k}|f(w)|\le r_k<1,$$

and now Theorem 3.1 implies that $(r_k s_1,\dots,r_k^n s_n)\in G_n$. By letting $k\to\infty$ we conclude that $(s_1,\dots,s_n)\in \Gamma_n$. ∎

For $(s_1,\dots,s_n)$ in $\Gamma_n\setminus G_n$, the denominator of f given by (3.5) has no zeros on $\mathbb{D}$, but it can have zeros on $\mathbb{T}$. In this situation, the numerator of f has at least the same zeros on $\mathbb{T}$, with at least the same multiplicity, so that, in the expression of f, the factors of the denominator giving zeros on $\mathbb{T}$ will simplify. For example, if we consider 1, 1 and 1/2 in $\overline{\mathbb{D}}$, which give the element $(5/2,2,1/2)\in\Gamma_3\setminus G_3$, we have

$$f(z)=\frac{-3z^2+8z-5}{6-10z+4z^2}=-\frac{(z-1)(3z-5)}{(z-1)(4z-6)}=-\frac{3z-5}{4z-6}.$$

One can easily verify that $\sup_{|z|\le 1}|f(z)|=1$.

In fact, using Theorems 3.1 and 3.2 we obtain the following characterization of the boundary $\Gamma_n\setminus G_n$ of Γ_n.

COROLLARY 3.2. *Let* $(s_1,\dots,s_n)\in\mathbb{C}^n$ *and let* f *be given by* (3.5). *The following assertions are equivalent.*

(i) *The element* $(s_1,\dots,s_n)$ *belongs to* $\Gamma_n\setminus G_n$.

(ii) *We have*

$$\sup_{|z|\le 1}|f(z)|=1. \tag{3.10}$$

Inside the boundary of Γ_n there is a set that will play an important role for the interpolation problem into G_n. It is the *distinguished boundary* of Γ_n, that is, the set

$$\mathrm{db}(\Gamma_n):=\pi_n(\mathbb{T}^n), \tag{3.11}$$

where π_n is given by (2.2).

For the distinguished boundary of Γ_n, we have the following characterization.

THEOREM 3.3. *For* $(s_1,\dots,s_n)\in\mathbb{C}^n$, *the following assertions are equivalent.*

(i) *The element* $(s_1,\dots,s_n)$ *belongs to* $\mathrm{db}(\Gamma_n)$.

(ii) *The complex numbers* s_j *satisfy the following relations:*

$$|s_n|=1, \tag{3.12}$$

$$s_j=\bar{s}_{n-j}s_n \qquad (1\le j\le n/2), \tag{3.13}$$

$$((n-1)s_1/n,(n-2)s_2/n,\dots,s_{n-1}/n)\in\Gamma_{n-1}. \tag{3.14}$$

Proof. (i)⇒(ii): If $(s_1, \dots, s_n) \in \mathrm{db}(\varGamma_n)$ then

$$s_1 = \sum_{j=1}^{n} \lambda_j, \ \dots, \ s_n = \prod_{j=1}^{n} \lambda_j,$$

where $|\lambda_j| = 1$ for $j = 1, \dots, n$. Then clearly $|s_n| = 1$, and

$$\begin{aligned} \bar{s}_{n-j} &= \sum \overline{\lambda}_{i_1} \cdots \overline{\lambda}_{i_{n-j}} = \sum \frac{1}{\lambda_{i_1} \cdots \lambda_{i_{n-j}}} \\ &= \frac{1}{\lambda_1 \cdots \lambda_n} \sum \lambda_{i_1} \cdots \lambda_{i_j} = \frac{s_j}{s_n} \end{aligned}$$

for $j = 1, \dots, n-1$. Also, the zeros of the polynomial $P(z) = z^n - s_1 z^{n-1} + \dots + (-1)^n s_n$ are all inside $\overline{\mathbb{D}}$ and therefore, by using once more the Lucas theorem, we see that the zeros of P' are all inside $\overline{\mathbb{D}}$. Therefore, (3.14) is true.

(ii)⇒(i): Let f be given by (3.5). The relation (3.14) implies that the denominator of f has no zeros on $\mathbb{D}$. Since $|s_n| = 1$ and $s_j = \bar{s}_{n-j} s_n$ for $j = 1, \dots, n-1$ we infer that, for z in $\mathbb{T}$ such that $n - (n-1)s_1 z + \dots + (-1)^{n-1} s_{n-1} z^{n-1} \neq 0$, we have $|f(z)| = 1$. Therefore, the rational function f has no poles on $\overline{\mathbb{D}}$. Using now the maximum modulus principle we have $|f| \le 1$ on $\overline{\mathbb{D}}$, and Theorem 3.2 implies that $(s_1, \dots, s_n) \in \varGamma_n$. Since $|s_n| = 1$ we conclude that $(s_1, \dots, s_n) \in \mathrm{db}(\varGamma_n)$. ∎

3.2. Characterizations of G_n given by G_{n-1}. The proof of Theorem 3.1 relies on the Lucas theorem. For $(s_1, \dots, s_n) \in G_n$, the fact that the denominator of the function f given by (3.5) has no zeros on $\overline{\mathbb{D}}$ was one of the motivations for choosing f of this form. But the Lucas theorem can be seen as a particular case of a more general result, which appears in the theory of apolar polynomials [13, Chapter IV].

THEOREM 3.4 ([13, Corollary (16,1a)]). *Given two complex polynomials*

$$A(z) = \sum_{k=0}^{n} \mathrm{C}_n^k a_k z^k \quad \textit{and} \quad B(z) = \sum_{k=0}^{n} \mathrm{C}_n^k b_k z^k,$$

consider

$$C(z) = \sum_{k=0}^{n} \mathrm{C}_n^k a_k b_k z^k.$$

If A has all its zeros inside the open disc of radius $r > 0$ and all the zeros of B lie inside the closed disc of radius $R > 0$, then all the zeros of C lie inside the open disc of radius rR.

As a corollary, we obtain the following result.

COROLLARY 3.3. *If* $(s_1, \dots, s_n) \in G_n$ *and* $(S_1, \dots, S_n) \in \Gamma_n$, *then*

$$((s_1 S_1)/\mathrm{C}_n^1, \dots, (s_n S_n)/\mathrm{C}_n^n) \in G_n. \tag{3.15}$$

If $(s_1, \dots, s_n) \in G_n$ *and* $(S_1, \dots, S_{n-1}) \in \Gamma_{n-1}$, *then*

$$((s_1 S_1)/\mathrm{C}_n^1, \dots, (s_{n-1} S_{n-1})/\mathrm{C}_n^{n-1}) \in G_{n-1}. \tag{3.16}$$

Proof. We obtain (3.15) by applying Theorem 3.4 to the polynomials $z^n - s_1 z^{n-1} + \dots + (-1)^n s_n$ and $z^n + S_1 z^{n-1} + \dots + S_n$, with $r = R = 1$. Then (3.16) is a particular case of (3.15), since the facts that $(S_1, \dots, S_{n-1}, 0) \in \Gamma_n$ and $(S_1, \dots, S_{n-1}) \in \Gamma_{n-1}$ are equivalent. ∎

The relation (3.15) says that, for a fixed $(S_1, \dots, S_n) \in \Gamma_n$, if we consider the diagonal matrix $D = \mathrm{diag}(S_1/\mathrm{C}_n^1, \dots, S_n/\mathrm{C}_n^n) \in \mathcal{M}_n(\mathbb{C})$, then $D(G_n) \subseteq G_n$. We therefore obtain a family of (non-trivial) linear operators from G_n into G_n. In fact, one can easily see that $(s_1, \dots, s_n) \in G_n$ if and only if $D(s_1, \dots, s_n) \in G_n$ for all D in the above family. If we consider in (3.16) the element $(\mathrm{C}_{n-1}^1, \dots, \mathrm{C}_{n-1}^{n-1}) \in \Gamma_{n-1}$, then for $(s_1, \dots, s_n)$ in G_n we have $((n-1)s_1/n, \dots, s_{n-1}/n) \in G_{n-1}$. That is, if $P(z) = z^n - s_1 z^{n-1} + \dots + (-1)^n s_n$ has all its roots inside $\mathbb{D}$, then so does $(1/n)P'(z) = z^{n-1} - (n-1)s_1 z^{n-2}/n + \dots + (-1)^{n-1} s_{n-1}/n$. This is the result that was repeatedly used in the proof of Theorem 3.1.

Having now in mind (3.16), for a fixed element $(s_1, \dots, s_n) \in \mathbb{C}^n$ the natural generalization of the function f given by (3.5) is

$$g(S_1, \dots, S_{n-1}) = \frac{S_{n-1}s_n - S_{n-2}s_{n-1}/\mathrm{C}_n^1 + S_{n-3}s_{n-2}/\mathrm{C}_n^2 + \dots + (-1)^{n-1}s_1/\mathrm{C}_n^{n-1}}{1 - S_1 s_1/\mathrm{C}_n^1 + S_2 s_2/\mathrm{C}_n^2 + \dots + (-1)^{n-1} S_{n-1}s_{n-1}/\mathrm{C}_n^{n-1}}. \tag{3.17}$$

It is defined on the subset of $\mathbb{C}^{n-1}$ where the denominator is not zero. Observe that if $(s_1, \dots, s_n) \in G_n$, then g is well defined on a neighborhood of Γ_{n-1}. Moreover, the following theorem holds.

THEOREM 3.5. *Let* $(s_1, \dots, s_n) \in \mathbb{C}^n$ *and let* g *be given by* (3.17). *The following assertions are equivalent.*

(i) *The element* $(s_1, \dots, s_n)$ *belongs to* G_n.
(ii) *The function* g *is well defined on a neighborhood of* $\Gamma_{n-1} \subseteq \mathbb{C}^{n-1}$, *and*

$$\sup_{(S_1, \dots, S_{n-1}) \in \Gamma_{n-1}} |g(S_1, \dots, S_{n-1})| < 1. \tag{3.18}$$

In fact, if $(s_1, \dots, s_n) \in G_n$, *then*

$$\sup_{(S_1, \dots, S_{n-1}) \in \Gamma_{n-1}} |g(S_1, \dots, S_{n-1})| = \sup_{|z| \le 1} |f(z)|, \tag{3.19}$$

where f *is the rational function given by* (3.5).

Proof. Observe that $f(z) = (-1)^n g(\mathrm{C}_{n-1}^1 z, \mathrm{C}_{n-1}^2 z^2, \ldots, \mathrm{C}_{n-1}^{n-1} z^{n-1})$ for all $z \in \overline{\mathbb{D}}$. Using Theorem 3.1, we obtain the implication "(ii)⇒(i)". The same identity also implies the inequality "$\geq$" in (3.19). If we now show the inequality "$\leq$" in (3.19), the theorem will be proved. For this, it is sufficient to prove that if

$$\sup_{|z|\leq 1}\left|\frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1}s_{n-1}z^{n-2} + \cdots + (-s_1)}{n-(n-1)s_1 z + \cdots + (-1)^{n-1}s_{n-1}z^{n-1}}\right| < r,$$

then

$$\sup_{(S_1,\ldots,S_{n-1})\in\Gamma_{n-1}}\left|\frac{S_{n-1}s_n - S_{n-2}s_{n-1}/\mathrm{C}_n^1 + S_{n-3}s_{n-2}/\mathrm{C}_n^2 + \cdots + (-1)^{n-1}s_1/\mathrm{C}_n^{n-1}}{1 - S_1 s_1/\mathrm{C}_n^1 + S_2 s_2/\mathrm{C}_n^2 + \cdots + (-1)^{n-1}S_{n-1}s_{n-1}/\mathrm{C}_n^{n-1}}\right| \leq r$$

If the first inequality holds, then

$$\frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1}s_{n-1}z^{n-2} + \cdots + (-s_1)}{n-(n-1)s_1 z + \cdots + (-1)^{n-1}s_{n-1}z^{n-1}} \neq rw$$

for all $|z| \leq 1$ and $|w| \geq 1$. This gives

$$\begin{aligned}&(nrw + s_1) - z((n-1)rws_1 + 2s_2) + \cdots \\ &+ (-1)^{n-2}z^{n-2}(2rws_{n-2} + (n-1)s_{n-1}) + (-1)^{n-1}z^{n-1}(rws_{n-1} + ns_n) \neq 0\end{aligned}$$

for all $|z| \leq 1$ and $|w| \geq 1$. Set

$$\widetilde{S}_1 = \frac{(n-1)rws_1 + 2s_2}{nrw + s_1}, \ \ldots$$

$$\widetilde{S}_{n-2} = \frac{2rws_{n-2} + (n-1)s_{n-1}}{nrw + s_1}, \qquad \widetilde{S}_{n-1} = \frac{rws_{n-1} + ns_n}{nrw + s_1}.$$

Then

$$1 - \widetilde{S}_1 z + \cdots + (-1)^{n-2}z^{n-2}\widetilde{S}_{n-2} + (-1)^{n-1}z^{n-1}\widetilde{S}_{n-1} \neq 0$$

for all $|z| \leq 1$. This gives

$$z^{n-1} - \widetilde{S}_1 z^{n-2} + \cdots + (-1)^{n-2}z\widetilde{S}_{n-2} + (-1)^{n-1}\widetilde{S}_{n-1} \neq 0$$

for all $|z| \geq 1$, and therefore $(\widetilde{S}_1, \ldots, \widetilde{S}_{n-1}) \in G_{n-1}$. Using now (3.15), for $(S_1, \ldots, S_{n-1}) \in \Gamma_{n-1}$ we have $(S_1\widetilde{S}_1/\mathrm{C}_{n-1}^1, \ldots, S_{n-1}\widetilde{S}_{n-1}/\mathrm{C}_{n-1}^{n-1}) \in G_{n-1}$. This implies

$$1 - S_1\widetilde{S}_1/\mathrm{C}_{n-1}^1 + \cdots + (-1)^{n-2}S_{n-2}\widetilde{S}_{n-2}/\mathrm{C}_{n-1}^{n-2} + (-1)^{n-1}S_{n-1}\widetilde{S}_{n-1}/\mathrm{C}_{n-1}^{n-1} \neq 0.$$

By reversing the above calculations, we obtain

$$\frac{(-1)^n S_{n-1}s_n + (-1)^{n-1}S_{n-2}s_{n-1}/\mathrm{C}_n^1 + (-1)^{n-2}S_{n-3}s_{n-2}/\mathrm{C}_n^2 + \cdots + (-s_1/\mathrm{C}_n^{n-1})}{1 - S_1 s_1/\mathrm{C}_n^1 + S_2 s_2/\mathrm{C}_n^2 + \cdots + (-1)^{n-1}S_{n-1}s_{n-1}/\mathrm{C}_n^{n-1}} \neq rw$$

for all $(S_1, \ldots, S_{n-1}) \in \Gamma_{n-1}$ and $|w| \geq 1$. Therefore,

$$\sup_{(S_1,\ldots,S_{n-1})\in\Gamma_{n-1}}\left|\frac{S_{n-1}s_n - S_{n-2}s_{n-1}/\mathrm{C}_n^1 + S_{n-3}s_{n-2}/\mathrm{C}_n^2 + \cdots + (-1)^{n-1}s_1/\mathrm{C}_n^{n-1}}{1 - S_1 s_1/\mathrm{C}_n^1 + S_2 s_2/\mathrm{C}_n^2 + \cdots + (-1)^{n-1}S_{n-1}s_{n-1}/\mathrm{C}_n^{n-1}}\right| \leq r$$

and the theorem is proved. ∎

It is clear that to test whether $(s_1, \ldots, s_n) \in G_n$, it is much easier to use (3.6). A result similar to Corollary 3.1 can also be obtained: using the function g given by (3.17), we can easily deduce necessary conditions for the interpolation problem into G_n. In the general case, we do not know whether we obtain more necessary conditions besides the ones given by Corollary 3.1. For the interpolation problem with two interpolation points, one of them being $(0, \ldots, 0) \in G_n$, the equality (3.19) says that we obtain the same necessary conditions.

The most important fact about the theorem we have just proved is that it gives the following new characterization of G_n in terms of G_{n-1}.

COROLLARY 3.4. *For $(s_1, \ldots, s_n)$ in $\mathbb{C}^n$, the following assertions are equivalent.*

(i) *The element $(s_1, \ldots, s_n)$ belongs to G_n.*
(ii) *For all z in $\overline{\mathbb{D}}$ we have $(\widetilde{s}_1(z), \ldots, \widetilde{s}_{n-1}(z)) \in G_{n-1}$, where*

$$\widetilde{s}_j(z) = \mathrm{C}_{n-1}^j \, \frac{s_j/\mathrm{C}_n^j - z s_{j+1}/\mathrm{C}_n^{j+1}}{1 - z s_1/\mathrm{C}_n^1} \tag{3.20}$$

for $j = 1, \ldots, n-1$.

Proof. The elements of Γ_{n-1} are of the form

$$(t_1 + z, t_2 + z t_1, \ldots, t_{n-2} + z t_{n-3}, z t_{n-2}),$$

where $z \in \overline{\mathbb{D}}$ and $(t_1, \ldots, t_{n-2}) \in \Gamma_{n-2}$. Theorem 3.5 shows that $(s_1, \ldots, s_n) \in G_n$ if and only if the function g given by (3.17) is analytic on a neighborhood of Γ_{n-1} and

$$\sup_{z \in \overline{\mathbb{D}}} \ \sup_{(t_1, \ldots, t_{n-2}) \in \Gamma_{n-2}} |g(t_1 + z, t_2 + z t_1, \ldots, t_{n-2} + z t_{n-3}, z t_{n-2})| < 1.$$

Therefore, $(s_1, \ldots, s_n) \in G_n$ if and only if for all z in $\overline{\mathbb{D}}$ we have

$$1 - t_1 \widetilde{s}_1(z)/\mathrm{C}_{n-1}^1 + t_2 \widetilde{s}_2(z)/\mathrm{C}_{n-1}^2 + \cdots + (-1)^{n-2} t_{n-2} \widetilde{s}_{n-2}(z)/\mathrm{C}_{n-1}^{n-2} \neq 0$$

on a neighborhood of Γ_{n-2} and

$$\sup_{(t_1, \ldots, t_{n-2}) \in \Gamma_{n-2}} \left| \frac{t_{n-2} \widetilde{s}_{n-1}(z) - t_{n-3} \widetilde{s}_{n-2}(z)/\mathrm{C}_{n-1}^1 + \cdots + (-1)^{n-2} \widetilde{s}_1(z)/\mathrm{C}_{n-1}^{n-2}}{1 - t_1 \widetilde{s}_1(z)/\mathrm{C}_{n-1}^1 + t_2 \widetilde{s}_2(z)/\mathrm{C}_{n-1}^2 + \cdots + (-1)^{n-2} t_{n-2} \widetilde{s}_{n-2}(z)/\mathrm{C}_{n-1}^{n-2}} \right| < 1.$$

By Theorem 3.5 once more, this is equivalent to $(\widetilde{s}_1(z), \ldots, \widetilde{s}_{n-1}(z)) \in G_{n-1}$ for all z in $\overline{\mathbb{D}}$. ∎

3.3. Parametrization of G_n. The last characterization of G_n we give is totally different from the preceding ones. We shall obtain, in fact, a parametrization of G_n. It cannot be used in order to obtain necessary interpolation conditions, but, as we shall see in Section 5, it gives the extremal rational function of degree 1 from $\mathbb{D}$ into G_n.

THEOREM 3.6. *For $(s_1, \ldots, s_n)$ in $\mathbb{C}^n$, the following assertions are equivalent.*

(i) *The element $(s_1, \ldots, s_n)$ belongs to G_n.*
(ii) *We have $|s_n| < 1$, and there exists $(S_1, \ldots, S_{n-1})$ in G_{n-1} such that*

$$s_j = S_j + \overline{S}_{n-j} s_n \qquad \text{for } j = 1, \ldots, n-1.$$

Proof. (i)⇒(ii): If $(s_1, \ldots, s_n) \in G_n$ then $s_n \in \mathbb{D}$ and therefore $|s_n| < 1$. We calculate the S_j such that $s_j = S_j + \overline{S}_{n-j} s_n$ for all j, and we obtain $S_j = (s_j - \overline{s}_{n-j} s_n)/(1 - |s_n|^2)$ for $j = 1, \ldots, n-1$. We must now prove that $(S_1, \ldots, S_{n-1}) \in G_{n-1}$. Consider $h : \mathbb{C} \to \mathbb{C}^n$ given by

$$h(\lambda) = (S_1 + \lambda \overline{S}_{n-1}, \ldots, S_{n-1} + \lambda \overline{S}_1, \lambda). \tag{3.21}$$

If we write $h(\lambda) = (t_1(\lambda), \ldots, t_n(\lambda))$ on $\mathbb{C}$, then $t_j = \overline{t}_{n-j} t_n$ on $\mathbb{T}$ for all $j \in \{1, \ldots, n-1\}$. Consider

$$\begin{aligned} F(z, \lambda) &= \frac{n(-1)^n t_n(\lambda) z^{n-1} + (n-1)(-1)^{n-1} t_{n-1}(\lambda) z^{n-2} + \cdots + (-t_1(\lambda))}{n - (n-1) t_1(\lambda) z + \cdots + (-1)^{n-1} t_{n-1}(\lambda) z^{n-1}} \\ &= \frac{\lambda A(z) + B(z)}{\lambda C(z) + D(z)}, \end{aligned}$$

where

$$\begin{aligned} A(z) &= n(-1)^n z^{n-1} + (n-1)(-1)^{n-1} \overline{S}_1 z^{n-2} + \cdots + (-\overline{S}_{n-1}), \\ B(z) &= (n-1)(-1)^{n-1} S_{n-1} z^{n-2} + (n-2)(-1)^{n-2} S_{n-2} z^{n-3} + \cdots + (-S_1), \\ C(z) &= -(n-1) \overline{S}_{n-1} z + (n-2) \overline{S}_{n-2} z^2 + \cdots + (-1)^{n-1} \overline{S}_1 z^{n-1}, \\ D(z) &= n - (n-1) S_1 z + (n-2) S_2 z^2 + \cdots + (-1)^{n-1} S_{n-1} z^{n-1}. \end{aligned}$$

Fix $z \in \mathbb{T}$. Using the fact that $|t_n| = 1$ and $t_j = \overline{t}_{n-j} t_n$ on $\mathbb{T}$ we obtain $|F(z, \lambda)| = 1$ for all λ in $\mathbb{T}$ for which $\lambda C(z) + D(z) \neq 0$. Also, Theorem 3.1 and the fact that $h(s_n) = (s_1, \ldots, s_n) \in G_n$ give $|F(z, s_n)| < 1$. Therefore, the map $\lambda \mapsto F(z, \lambda)$ is a Möbius transformation on $\mathbb{D}$. This gives $\lambda C(z) + D(z) \neq 0$ for all $|\lambda| < 1$ and $|z| = 1$, and

$$|F(z, \lambda)| < 1 \qquad (|\lambda| < 1,\ |z| = 1). \tag{3.22}$$

If $(S_1, \ldots, S_{n-1}) \notin G_{n-1}$, then $h(0) \notin G_n$. Since $h(s_n) \in G_n$, by continuity we can find λ_0 in the line segment between 0 and s_n such that $h(\lambda_0) \in \Gamma_n \setminus G_n$. Using Corollary 3.2, we find $z_0 \in \mathbb{T}$ such that $|F(z_0, \lambda_0)| = 1$, and this contradicts (3.22).

(ii)⇒(i): Consider h given by (3.21). We want to prove that $h(\lambda) \in G_n$ for all $\lambda \in \mathbb{D}$. By Theorem 3.1, $(S_1, \ldots, S_{n-1}, 0) \in G_n$ implies

$$\sup_{|z| \le 1} \left| \frac{(-1)^{n-1}(n-1) S_{n-1} z^{n-2} + (-1)^{n-2}(n-2) S_{n-2} z^{n-3} + \cdots + (-S_1)}{n - (n-1) S_1 z + (n-2) S_2 z^2 + \cdots + (-1)^{n-1} S_{n-1} z^{n-1}} \right| < 1$$

and this inequality gives

$$\sup_{|z|\le 1} \frac{|C(z)|}{|D(z)|} < 1.$$

Fix now $\lambda_0 \in \mathbb{T}$. Then the map $z \mapsto F(z,\lambda_0)$ is well defined and analytic on a neighborhood of $\overline{\mathbb{D}}$. We have already seen that $|F(z,\lambda_0)| = 1$ for all $z \in \mathbb{T}$ and, using the maximum modulus principle, we obtain $|F(z,\lambda_0)| \le 1$ for all $z \in \overline{\mathbb{D}}$. Theorem 3.1 implies now that $h(\lambda_0) \in \Gamma_n$. Therefore $h(\mathbb{T}) \subseteq \Gamma_n$, and since $h(0) \in G_n$, Corollary 2.1 shows that $h(\mathbb{D}) \subseteq G_n$. In particular $h(s_n) \in G_n$, which gives $(s_1,\dots,s_n) \in G_n$. ∎

The above theorem contains a generalization to the case $n \ge 3$ of the parametrization formula for the elements of G_2 given by Agler and Young [4, Theorem 1.1]. The idea to seek for such a parametrization of G_n was given by the expression of the function f from (3.5). The proof of the above theorem shows why the condition on the S_j and the relations between the S_j and the s_j are quite natural. Even though we have stated and proved Theorem 3.6 having in mind the function f, for example, for the implication "(ii)⇒(i)" a simpler proof can be given. It relies on the fact that the Blaschke products send $\mathbb{C}\setminus\overline{\mathbb{D}}$ into $\mathbb{C}\setminus\overline{\mathbb{D}}$. Indeed, let $(S_1,\dots,S_{n-1}) \in G_{n-1}$. Then for all z in $\mathbb{C}\setminus\overline{\mathbb{D}}$ and λ in $\mathbb{D}$ we have

$$z\,\frac{z^{n-1} - S_1 z^{n-2} + \dots + (-1)^{n-1} S_{n-1}}{1 - \overline{S}_1 z + \dots + (-1)^{n-1}\overline{S}_{n-1} z^{n-1}} \ne (-1)^{n-1}\lambda.$$

Therefore, for all λ in $\mathbb{D}$ and z in $\mathbb{C}\setminus\overline{\mathbb{D}}$ we have

$$\begin{aligned} z^n - (S_1 + \lambda\overline{S}_{n-1})z^{n-1} + (S_2 + \lambda\overline{S}_{n-2})z^{n-2} + \cdots \\ + (-1)^{n-1}(S_{n-1} + \lambda\overline{S}_1)z + (-1)^n\lambda \ne 0, \end{aligned}$$

which is equivalent to

$$(S_1 + \lambda\overline{S}_{n-1}, S_2 + \lambda\overline{S}_{n-2}, \dots, S_{n-1} + \lambda\overline{S}_1, \lambda) \in G_n \quad (\lambda \in \mathbb{D}).$$

For the closed symmetrized n-disc, an analogous result holds.

Theorem 3.7. *For $(s_1,\dots,s_n)$ in $\mathbb{C}^n$, the following assertions are equivalent.*

(i) *The element $(s_1,\dots,s_n)$ belongs to Γ_n.*
(ii) *We have $|s_n| \le 1$, and there exists $(S_1,\dots,S_{n-1})$ in Γ_{n-1} such that*

$$s_j = S_j + \overline{S}_{n-j} s_n \quad \textit{for } j = 1,\dots,n-1.$$

Proof. (i)⇒(ii): If $(s_1,\dots,s_n) \in \Gamma_n$, then by considering a sequence $(r_k)_{k\in\mathbb{N}} \subseteq [0,1)$ with $r_k \to 1$ we obtain $(r_k s_1,\dots,r_k^n s_n) \in G_n$ for all k. Theorem 3.6 now yields a sequence $((S_{k,1},\dots,S_{k,n-1}))_{k\in\mathbb{N}} \subseteq G_{n-1}$ such that $r_k^j s_j = S_{k,j} + \overline{S}_{k,n-j} r_k^n s_n$ for all j and k. The symmetrized n-disc is bounded and so the sequences $(S_{k,j})_{k\in\mathbb{N}}$ are also bounded. Therefore, we may suppose that $S_{k,j} \to S_j$ for $j = 1, \dots, n-1$. Then $(S_1, \dots, S_{n-1}) \in \Gamma_{n-1}$, and $s_j = S_j + \overline{S}_{n-j} s_n$ for $j = 1, \dots, n-1$.

(ii)⇒(i): By applying Theorem 3.6 for $rs_n \in \mathbb{D}$ and $(rS_1, \dots, r^{n-1}S_{n-1}) \in G_{n-1}$ we obtain

$$(rS_1 + (r^{n-1}\overline{S}_{n-1})(rs_n), \dots, r^{n-1}S_{n-1} + (r\overline{S}_1)(rs_n), rs_n) \in G_n$$

for all $r \in [0, 1)$. By letting $r \to 1$, we see that $(s_1, \dots, s_n) \in \Gamma_n$. ■

4. THE RATIONAL Γ_n-INNER FUNCTIONS

By its definition (3.11), the distinguished boundary of Γ_n is the counterpart of $\mathbb{T} \subseteq \overline{\mathbb{D}}$ for the set Γ_n. In accordance with this definition, an analytic function $f : \mathbb{D} \to G_n$ will be called Γ_n*-inner* if $f(\xi) \in \mathrm{db}(\Gamma_n)$ for almost all ξ in $\mathbb{T}$. If we write $f = (s_1, \dots, s_n)$, then the fact that f is Γ_n-inner implies that s_n is a scalar inner function on $\mathbb{D}$. Also, if f is a rational Γ_n-inner function, then s_n is necessarily a finite Blaschke product.

For the interpolation problem from $\mathbb{D}$ into $\mathbb{D}$ we know [12, Theorem 1.2.2] that if there exists $f : \mathbb{D} \to \mathbb{D}$ analytic such that $f(\lambda_j) = z_j$ for $j = 1, \dots, m$ then there exists a Blaschke product of degree at most m which solves the same interpolation problem. The Blaschke products are the rational analytic functions on $\mathbb{D}$ which are unimodular on $\mathbb{T}$. For the classical Nevanlinna–Pick problem in $\mathcal{M}_n(\mathbb{C})$, their role is played by the rational analytic functions $F : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ that are *inner*, that is, $F(\xi)$ is unitary for all $\xi \in \mathbb{T}$. In this case, an analogous result holds.

THEOREM 4.1 ([1, p. 181]). *Given distinct points $\lambda_1, \dots, \lambda_m$ in $\mathbb{D}$ and matrices $W_1, \dots, W_m$ in $\mathcal{M}_n(\mathbb{C})$, if there exists $F : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ analytic such that $\|F(\lambda)\| \le 1$ for all $\lambda \in \mathbb{D}$ and $F(\lambda_j) = W_j$ for all j, then there exists $G : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ analytic, with entries rational functions with poles in the exterior of $\overline{\mathbb{D}}$, such that $G(\xi)$ is unitary for all $\xi \in \mathbb{T}$ and $G(\lambda_j) = W_j$ for $j = 1, \dots, m$.*

The version of this result for the symmetrized n-disc is the following.

THEOREM 4.2. *Given distinct points $\lambda_1, \dots, \lambda_m$ in $\mathbb{D}$ and elements $(s_{1,1}, \dots, s_{n,1}), \dots, (s_{1,m}, \dots, s_{n,m})$ in G_n, if there exists an analytic function $f : \mathbb{D} \to G_n$ such that $f(\lambda_j) = (s_{1,j}, \dots, s_{n,j})$ for all j, then there exists a rational Γ_n-inner function $g : \mathbb{D} \to G_n$ such that $g(\lambda_j) = (s_{1,j}, \dots, s_{n,j})$ for all j.*

Proof. If such an f exists, then $F = J_n \circ f$ is a bounded analytic function from $\mathbb{D}$ into Ω_n. Then for $r \in (0, 1)$ we can apply Theorem 1.1 to rF to obtain an analytic function $G_r : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $G_r(\lambda_j) \sim rF(\lambda_j)$ for all j and $\sup_{|\lambda|<1} \|G_r(\lambda)\| < 1$. We thus obtain a uniformly bounded family of analytic functions $(G_r)_{r\in(0,1)}$, and using the Montel theorem we find a sequence $(r_n)_{n\in\mathbb{N}} \subseteq (0,1)$ and an analytic function $G : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $r_n \to 1$ and $G_{r_n} \to G$ locally uniformly on $\mathbb{D}$. Then $\|G(\lambda)\| \le 1$ on $\mathbb{D}$. Also, the continuity properties of the eigenvalues give $\sigma(G(\lambda_j)) = \sigma(F(\lambda_j))$ for $j = 1, \dots, m$. (Recall that the eigenvalues are counted with their multiplicities.) Now Theorem 4.1 gives a rational inner function $\widetilde{G} : \mathbb{D} \to \mathcal{M}_n(\mathbb{C})$ such that $\widetilde{G}(\lambda_j) = G(\lambda_j)$ for all j. Define $g = \Pi_n \circ \widetilde{G}$. Then g is a rational analytic function on $\mathbb{D}$ and since the spectrum of a unitary matrix is always a subset of $\mathbb{T}$ we conclude that $g : \mathbb{D} \to G_n$ is Γ_n-inner. We also have

$$g(\lambda_j) = \Pi_n(\widetilde{G}(\lambda_j)) = \Pi_n(G(\lambda_j)) = \Pi_n(F(\lambda_j)) = f(\lambda_j) = (s_{1,j}, \dots, s_{n,j})$$

for $j = 1, \dots, m$. ∎

The important fact about the rational Γ_n-inner functions is that their form can be calculated explicitly. Indeed, let $f = (s_1, \dots, s_n) : \mathbb{D} \to G_n$ be such a function. As we have already seen, s_n is necessarily a finite Blaschke product, so let

$$s_n(\lambda) = \xi \prod_{k=1}^{m} \frac{\lambda - \alpha_k}{1 - \overline{\alpha}_k \lambda}, \tag{4.1}$$

where $\xi \in \mathbb{T}$ and $\alpha_k \in \mathbb{D}$ for all k. Using (3.13) and the fact that $f(\mathbb{T}) \subseteq \mathrm{db}(\Gamma_n)$ we deduce that $s_j = \overline{s}_{n-j} s_n$ on $\mathbb{T}$ for $j = 1, \dots, n-1$. Consider now the Hardy space H^2 (on $\mathbb{T}$). For a fixed $j \in \{1, \dots, n-1\}$, the function s_j belongs to H^∞, and so to H^2. Therefore, $\overline{s}_{n-j} s_n \in H^2$. Then $s_{n-j}\overline{s}_n \in \overline{H}^2$, where $\overline{H}^2 = \{f \in L^2 : \overline{f} \in H^2\}$. If we define $H^2_- = \{f \in L^2 : \widehat{f}(k) = 0, \forall k \ge 0\}$, then $\overline{H}^2 = \lambda H^2_-$, where λ is the identity function from $\mathbb{D}$ into $\mathbb{D}$. Since $|s_n| = 1$ on $\mathbb{T}$ and $s_{n-j}\overline{s}_n \in \overline{H}^2$ we obtain $s_{n-j} \in s_n \overline{H}^2$. Therefore, $s_{n-j} \in (\lambda s_n) H^2_-$. Since s_{n-j} also belongs to H^2, we deduce that $s_{n-j} \in H^2 \cap (\lambda s_n) H^2_-$. For the inner function (in fact, finite Blaschke product) λs_n, the set $H^2 \cap (\lambda s_n) H^2_-$ is its model space. We have [14, p. 228]

$$H^2 \cap (\lambda s_n) H^2_- = H^2 \ominus (\lambda s_n) H^2.$$

The fact that λs_n is a finite Blaschke product implies that $H^2 \ominus (\lambda s_n) H^2$ is of finite dimension (in fact, the dimension is exactly the degree of λs_n, that is, $m+1$). Its elements are of the form Q/P, where Q is a polynomial of degree at most m and $P(\lambda) = \prod_{j=1}^m (1 - \overline{\alpha}_j \lambda)$. Since $s_k \in H^2 \ominus (\lambda s_n) H^2$ for $k = 1, \dots, n-1$, we can find a polynomial P_k of degree at most m such that $s_k = P_k/P$. We also have $s_k = \overline{s}_{n-k} s_n$ on $\mathbb{T}$, and therefore

$$\frac{s_{n-k} s_k}{s_n} = |s_{n-k}|^2 \ge 0 \quad \text{on } \mathbb{T},$$

for all $k = 1, \dots, n-1$. We have

$$\frac{s_{n-k}s_k}{s_n} = \frac{P_{n-k}P_k}{\xi(\prod_{j=1}^m (1-\overline{\alpha}_j\lambda))(\prod_{j=1}^m(\lambda-\alpha_j))},$$

and therefore $s_{n-k}s_k/s_n$ is a rational function which is positive on $\mathbb{T}$. The form of those rational functions can be calculated explicitly [10, p. 137]: we can find $r \geq 0$, t and q in $\mathbb{N} \cup \{0\}$ with $t+q=m$, $\beta_1, \dots, \beta_q \in \mathbb{D}$ and $\xi_1, \dots, \xi_t \in \mathbb{T}$ such that

$$\frac{s_{n-k}s_k}{s_n} = r\,\frac{(\prod_{j=1}^t(\lambda+\xi_j)^2)(\prod_{j=1}^q(1-\overline{\beta}_j\lambda)(\lambda-\beta_j))}{(\prod_{j=1}^t \xi_j)(\prod_{j=1}^m(1-\overline{\alpha}_j\lambda)(\lambda-\alpha_j))}.$$

Since $s_k = \overline{s}_{n-k}s_n$ on $\mathbb{T}$, this implies, for $k \neq n-k$, that we can find a real number R and $\eta, \delta \in \mathbb{T}$ with $\eta\delta\prod_{j=1}^t \xi_j = \xi$ such that

$$s_k = R\eta\,\frac{(\prod_{j=1}^t(\lambda+\xi_j))(\prod_{j=1}^q f_j)}{\prod_{j=1}^m(1-\overline{\alpha}_j\lambda)},$$

$$s_{n-k} = R\delta\,\frac{(\prod_{j=1}^t(\lambda+\xi_j))(\prod_{j=1}^q g_j)}{\prod_{j=1}^m(1-\overline{\alpha}_j\lambda)}$$

where, for $j = 1, \dots, q$, either $f_j = 1-\overline{\beta}_j\lambda$ and $g_j = \lambda-\beta_j$, or $f_j = \lambda-\beta_j$ and $g_j = 1-\overline{\beta}_j\lambda$. If $n-k=k$, then s_k must be of the form

$$s_k = R\eta\,\frac{(\prod_{j=1}^t(\lambda+\xi_j))(\prod_{j=1}^s(1-\overline{\beta}_j\lambda)(\lambda-\beta_j))}{\prod_{j=1}^m(1-\overline{\alpha}_j\lambda)},$$

where $t+2s=m$, R is a real number and $\eta^2\prod_{j=1}^t \xi_j = \xi$. Using now the above calculations and Theorem 4.2, we obtain the following result.

THEOREM 4.3. *Let $\lambda_1, \dots, \lambda_N$ be distinct points in $\mathbb{D}$ and let $(s_{1,1}, \dots, s_{n,1}), \dots, (s_{1,N}, \dots, s_{n,N})$ in G_n be such that we can find an analytic function f from $\mathbb{D}$ into G_n with $f(\lambda_j) = (s_{1,j}, \dots, s_{n,j})$ for $j = 1, \dots, N$.*

(i) *Suppose that $n = 2k+1$, where $k \in \mathbb{N}$. Then we can find $m \in \mathbb{N}$, $\xi \in \mathbb{T}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{D}$ such that, for every $j \in \{1, \dots, k\}$, there exist $t_j, q_j \in \mathbb{N} \cup \{0\}$, $r_j \in \mathbb{R}$, $\eta_j, \delta_j \in \mathbb{T}$, $\xi_{j,1}, \dots, \xi_{j,t_j} \in \mathbb{T}$ and $\beta_{j,1}, \dots, \beta_{j,q_j} \in \mathbb{D}$ such that*

$$t_j + q_j = m, \qquad \eta_j\delta_j\Big(\prod_{i=1}^{t_j}\xi_{j,i}\Big) = \xi \qquad (1 \leq j \leq k),$$

and if s_n is given by (4.1) *and*

$$s_j = r_j\eta_j\,\frac{(\prod_{i=1}^{t_j}(\lambda+\xi_{j,i}))(\prod_{i=1}^{q_j} f_{j,i})}{\prod_{i=1}^m(1-\overline{\alpha}_i\lambda)} \qquad (1 \leq j \leq k), \tag{4.2}$$

$$(4.3)\qquad s_{n-j}=r_j\delta_j\,\frac{(\prod_{i=1}^{t_j}(\lambda+\xi_{j,i}))(\prod_{i=1}^{q_j}g_{j,i})}{\prod_{i=1}^{m}(1-\overline{\alpha}_i\lambda)}\qquad(1\le j\le k),$$

where, for $i=1,\dots,q_j$, either $f_{j,i}=1-\overline{\beta}_{j,i}\lambda$ and $g_{j,i}=\lambda-\beta_{j,i}$, or $f_{j,i}=\lambda-\beta_{j,i}$ and $g_{j,i}=1-\overline{\beta}_{j,i}\lambda$, then $g=(s_1,\dots,s_n)$ solves the same interpolation problem as f.

(ii) *If $n=2k$, where $k\in\mathbb{N}$, then almost the same statement as in* (i) *holds, the only difference being that s_k must be of the form*

$$(4.4)\qquad s_k=r_k\eta_k\frac{(\prod_{i=1}^{t_k}(\lambda+\xi_{k,i}))(\prod_{i=1}^{q_k}(1-\overline{\beta}_{k,i}\lambda)(\lambda-\beta_{k,i}))}{\prod_{i=1}^{m}(1-\overline{\alpha}_i\lambda)},$$

where $t_k,q_k\in\mathbb{N}\cup\{0\}$ and $t_k+2q_k=m$, $r_k\in\mathbb{R}$, $\beta_{k,1},\dots,\beta_{k,q_k}\in\mathbb{D}$, and $\eta_k,\xi_{k,1},\dots,\xi_{k,t_k}\in\mathbb{T}$ are such that $\eta_k^2\prod_{i=1}^{t_k}\xi_{k,i}=\xi$.

Now consider a rational analytic function $g=(s_1,\dots,s_n)$ from $\mathbb{D}$ into $\mathbb{C}^n$ of the form given by the above theorem. By our construction, $|s_n|=1$ on $\mathbb{T}$ and $s_j=\overline{s}_{n-j}s_n$ for $j=1,\dots,n-1$. By Corollary 2.1 and Theorem 3.2, in order to have $g(\overline{\mathbb{D}})\subseteq\Gamma_n$ we must have

$$(4.5)\qquad \sup_{|z|\le1}\left|\frac{n(-1)^ns_n(\xi)z^{n-1}+(n-1)(-1)^{n-1}s_{n-1}(\xi)z^{n-2}+\dots+(-s_1(\xi))}{n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}}\right|\le1$$

for all $\xi\in\mathbb{T}$. For z in $\mathbb{T}$, if $n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}\ne0$ then

$$\begin{aligned}&\left|\frac{n(-1)^ns_n(\xi)z^{n-1}+(n-1)(-1)^{n-1}s_{n-1}(\xi)z^{n-2}+\dots+(-s_1(\xi))}{n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}}\right|\\
&=\left|\frac{n(-1)^n\,\overline{s_n(\xi)}\,\overline{z}^{n-1}+(n-1)(-1)^{n-1}\,\overline{s_{n-1}(\xi)}\,\overline{z}^{n-2}+\dots+(-\overline{s_1(\xi)})}{n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}}\right|\\
&=\left|\frac{(-1)^n\,\overline{s_n(\xi)}\,\overline{z}^{n-1}(n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1})}{n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}}\right|=1,\end{aligned}$$

and therefore, in order to have (4.5) it is necessary and sufficient to have

$$n-(n-1)s_1(\xi)z+\dots+(-1)^{n-1}s_{n-1}(\xi)z^{n-1}\ne0$$

for all $z\in\mathbb{D}$ and $\xi\in\mathbb{T}$. This yields

$$(4.6)\qquad ((n-1)s_1(\xi)/n,(n-2)s_2(\xi)/n,\dots,s_{n-1}(\xi)/n)\in\Gamma_{n-1}\qquad(\xi\in\mathbb{T}).$$

If (4.6) is satisfied, then g sends $\mathbb{D}$ into Γ_n. If, for example, we also have $g(0)\in G_n$, then by Corollary 2.1, g sends $\mathbb{D}$ into G_n.

Remark. Let $f=(s_1,\dots,s_n):\mathbb{D}\to G_n$ be a rational Γ_n-inner function of degree $k\ge1$ (observe that the degree of f is the degree of s_n as a finite Blaschke product). Then, for all $1\le j\le n-1$,

$$(4.7)\qquad s_j(\lambda)=\overline{s_{n-j}(1/\overline{\lambda})}s_n(\lambda)\qquad(\lambda\in\overline{\mathbb{D}}).$$

Also, if we write $s_n = Q/P$, where $P(\lambda) = \prod_{i=1}^k (1-\overline{\alpha}_i\lambda)$ and $Q(\lambda) = \zeta\prod_{i=1}^k(\lambda-\alpha_i)$, with $\zeta \in \mathbb{T}$ and $\alpha_1,\dots,\alpha_k \in \mathbb{D}$, then there exist polynomials $P_1,\dots,P_{n-1}$, all of degree less than or equal to k, such that

$$s_j(\lambda) = \frac{P_j(\lambda)}{P(\lambda)} = \frac{\overline{P_{n-j}(1/\overline{\lambda})}}{\overline{Q(1/\overline{\lambda})}} \quad (\lambda \in \overline{\mathbb{D}}) \tag{4.8}$$

for $1 \le j \le n-1$. Indeed, if we define $S_j(\lambda) = \overline{s_{n-j}(1/\overline{\lambda})}s_n(\lambda)$, then S_j is a well defined analytic function on a neighborhood of $\overline{\mathbb{D}}$. Since $f(\mathbb{T}) \subseteq \mathrm{db}(\Gamma_n)$ we also have $S_j(\xi) = \overline{s_{n-j}(\xi)}s_n(\xi) = s_j(\xi)$ for all $\xi \in \mathbb{T}$. Therefore $S_j = s_j$, and (4.7) holds. Then, for $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned} s_j(\lambda) &= \zeta\,\frac{\overline{P_{n-j}(1/\overline{\lambda})}}{\overline{P(1/\overline{\lambda})}}\,\frac{Q(\lambda)}{P(\lambda)} = \zeta\,\frac{\lambda^k\overline{P_{n-j}(1/\overline{\lambda})}}{\prod_{i=1}^k(\lambda-\alpha_i)}\,\frac{\prod_{i=1}^k(\lambda-\alpha_i)}{\prod_{i=1}^k(1-\overline{\alpha}_i\lambda)} \\ &= \frac{\overline{P_{n-j}(1/\overline{\lambda})}}{\overline{\zeta\prod_{i=1}^k(1/\overline{\lambda}-\alpha_i)}} = \frac{\overline{P_{n-j}(1/\overline{\lambda})}}{\overline{Q(1/\overline{\lambda})}}. \end{aligned}$$

For the interpolation problem into G_n, Theorem 4.3 gives a method to construct "enough" interpolation functions. The main problem is now that the condition (4.6) is very difficult to verify. This will reduce the applicability of the above theorem only to some very particular cases, which will appear in the remainder of this paper.

5. COMPLEX GEODESICS ON G_n

A two-point interpolation problem from $\mathbb{D}$ into a domain $\mathcal{D} \subseteq \mathbb{C}^n$ is closely related to the theory of Carathéodory and Kobayashi pseudo-distances on $\mathcal{D}$. By definition (see [9, Chapter 4]), the *Carathéodory pseudo-distance* between $p, q \in \mathcal{D}$ is given by $C_{\mathcal{D}}(p,q) = \sup d(F(p),F(q))$, where the supremum is over all analytic functions $F : \mathcal{D} \to \mathbb{D}$, and d denotes the *hyperbolic distance* on $\mathbb{D}$, that is,

$$d(\lambda_1,\lambda_2) = \tanh^{-1}\left|\frac{\lambda_1-\lambda_2}{1-\overline{\lambda}_1\lambda_2}\right| \quad (\lambda_1,\lambda_2 \in \mathbb{D}).$$

An analytic function $\varphi : \mathbb{D} \to \mathcal{D}$ is called a *complex geodesic* on $\mathcal{D}$ if $C_{\mathcal{D}}(\varphi(\lambda_1),\varphi(\lambda_2)) = d(\lambda_1,\lambda_2)$ for all $\lambda_1,\lambda_2 \in \mathbb{D}$. Observe that if $\varphi : \mathbb{D} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathbb{D}$ are analytic functions such that $F \circ \varphi$ is an automorphism of $\mathbb{D}$, then φ is a complex geodesic on $\mathcal{D}$. For $p, q \in \mathcal{D}$, we define $\delta_{\mathcal{D}}(p,q) = \inf d(\lambda_1,\lambda_2)$, where the infimum is over all $\lambda_1,\lambda_2 \in \mathbb{D}$ for which there exists $\varphi : \mathbb{D} \to \mathcal{D}$ analytic such that $\varphi(\lambda_1) = p$ and $\varphi(\lambda_2) = q$. The map $\delta_{\mathcal{D}}$ is usually called the *Lempert function*, and, for general domains $\mathcal{D}$, it does not satisfy the triangle inequality, and therefore it is not a pseudo-distance. By definition, the *Kobayashi pseudo-distance* $K_{\mathcal{D}}$ on $\mathcal{D}$

is the largest pseudo-distance on $\mathcal{D}$ smaller than $\delta_{\mathcal{D}}$. Then (see [9, Chapter 4]) $C_{\mathcal{D}} \le K_{\mathcal{D}} \le \delta_{\mathcal{D}}$ on $\mathcal{D}$, and if φ is a complex geodesic on $\mathcal{D}$, then $C_{\mathcal{D}} = K_{\mathcal{D}} = \delta_{\mathcal{D}}$ on $\varphi(\mathbb{D})$. Also observe that if $\mathcal{D} = G_n$ or Ω_n, where $n \ge 2$, then for $\lambda_1, \lambda_2 \in \mathbb{D}$ and $p, q \in \mathcal{D}$, there exists an analytic function $f : \mathbb{D} \to \mathcal{D}$ such that $f(\lambda_1) = p$ and $f(\lambda_2) = q$ if and only if $\delta_{\mathcal{D}}(p, q) \le d(\lambda_1, \lambda_2)$, and therefore a necessary and sufficient condition for the two-point interpolation problem from $\mathbb{D}$ into G_n and Ω_n is given by δ_{G_n} and δ_{Ω_n}, respectively.

5.1. Complex geodesics of degree 1 and 2 on G_n. By using Theorem 4.3, we can find an explicit formula for some of the complex geodesics on G_n.

5.1.1. *Complex geodesics of degree* 1 *on* G_n. The simplest case in the statement of Theorem 4.3 is when $m = 1$. Consider $s_n(\lambda) = \lambda$ for $\lambda \in \mathbb{D}$. Then for a fixed $j \in \{1, \dots, n-1\}$, for the function s_j given by (4.2), (4.3) or (4.4) the following cases can occur:

(i) $j \ne n/2$ and $s_j(\lambda) = r\eta(\lambda + \xi)$, $s_{n-j}(\lambda) = r\delta(\lambda + \xi)$, where $r \in \mathbb{R}$ and $\eta, \delta, \xi \in \mathbb{T}$, with $\eta\delta\xi = 1$. By putting $a = r\overline{\eta}$ and $b = r\eta\xi$ we obtain $s_j(\lambda) = \overline{a}\lambda + b$ and $s_{n-j}(\lambda) = \overline{b}\lambda + a$.

(ii) $j \ne n/2$ and $s_j(\lambda) = r\eta(\lambda - \alpha)$, $s_{n-j}(\lambda) = r\delta(1 - \overline{\alpha}\lambda)$, where $r \in \mathbb{R}$, $\eta, \delta \in \mathbb{T}$ with $\eta\delta = 1$, and $\alpha \in \mathbb{D}$. Put $a = r\overline{\eta}$ and $b = -r\eta\alpha$. Once more, we have $s_j(\lambda) = \overline{a}\lambda + b$ and $s_{n-j}(\lambda) = \overline{b}\lambda + a$.

(iii) $j \ne n/2$ and $s_j(\lambda) = r\delta(1 - \overline{\alpha}\lambda)$, $s_{n-j}(\lambda) = r\eta(\lambda - \alpha)$, where $r \in \mathbb{R}$, $\eta, \delta \in \mathbb{T}$ with $\eta\delta = 1$, and $\alpha \in \mathbb{D}$. Put $a = -r\eta\alpha$ and $b = r\overline{\eta}$. Then again $s_j(\lambda) = \overline{a}\lambda + b$ and $s_{n-j}(\lambda) = \overline{b}\lambda + a$.

(iv) $j = n/2$ and $s_j(\lambda) = r\eta(\lambda + \xi)$, where $r \in \mathbb{R}$ and $\eta, \xi \in \mathbb{T}$ with $\eta^2\xi = 1$. By putting $a = r\eta$ we obtain $r\eta\xi = r\overline{\eta} = \overline{a}$ and $s_j(\lambda) = \overline{a}\lambda + a$ on $\mathbb{D}$.

Therefore, we obtain a function of the form

$$f(\lambda) = (\overline{S}_{n-1}\lambda + S_1, \dots, \overline{S}_1\lambda + S_{n-1}, \lambda) \quad (\lambda \in \mathbb{D}),$$

and by (4.6) a necessary and sufficient condition for $f(\mathbb{D}) \subseteq G_n$ is that $((n-1)(\overline{S}_{n-1}\xi + S_1)/n, \dots, (\overline{S}_1\xi + S_{n-1})/n) \in \Gamma_{n-1}$ for all $\xi \in \mathbb{T}$. Even for this very particular case, the last condition is not trivial to check. Instead of using this condition, we shall use the proof of Theorem 3.6 to see that $f(\mathbb{D}) \subseteq G_n$ if and only if $(S_1, \dots, S_{n-1}) \in G_{n-1}$. If this is the case, and $F : G_n \to \mathbb{D}$ is given by $F(s_1, \dots, s_n) = s_n$, then $F \circ f$ is the identity on $\mathbb{D}$. Therefore, f is a complex geodesic on G_n. We have proved the following theorem.

THEOREM 5.1. *For* $b(\lambda) = \xi(\lambda - \alpha)/(1 - \overline{\alpha}\lambda)$, *where* $\xi \in \mathbb{T}$ *and* $\alpha \in \mathbb{D}$, *and* $(S_1, \dots, S_{n-1}) \in G_{n-1}$, *the map*

$$\lambda \mapsto (\overline{S}_{n-1}b(\lambda) + S_1, \dots, \overline{S}_1 b(\lambda) + S_{n-1}, b(\lambda)) \tag{5.1}$$

is a complex geodesic on G_n.

Are those the only complex geodesics of degree 1 on G_n? For the general case $n \geq 2$, we do not have an answer. In fact, we would like to have an answer to the following more general question.

QUESTION. *If $f = (s_1, \ldots, s_n) : \mathbb{D} \to G_n$ is a complex geodesic on G_n, must s_n be a Blaschke product of degree at most n?*

If the answer to the last question is yes (for $n = 2$, this is indeed the case [8]), then every complex geodesic of degree 1 on G_n is of the form given by Theorem 5.1.

5.1.2. *Complex geodesics of degree* 2 *on G_n passing through the origin.* If we consider $m = 2$ in the statement of Theorem 4.3, then (4.1)–(4.4) give the form of the rational analytic functions $f = (s_1, \ldots, s_n) : \mathbb{D} \to \mathbb{C}^n$ of degree 2 such that $|s_n| = 1$ on $\mathbb{T}$ and $s_j = \overline{s}_{n-j} s_n$ on $\mathbb{T}$ for $j = 1, \ldots, n-1$. In order to have $f(\mathbb{D}) \subseteq G_n$, the function f must satisfy (4.6) and $f(0) \in G_n$. Even though the degree of f is small, the relations in (4.6) are very difficult to check in the general case. To simplify the calculations, suppose that f also satisfies $f(0) = (0, \ldots, 0)$. Let therefore $\xi \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Consider also $\vartheta_1, \ldots, \vartheta_{n-1} \in \mathbb{C}$ such that $\vartheta_j = \overline{\vartheta}_{n-j} \xi$ for $j = 1, \ldots, n-1$. Then put

$$f(\lambda) = \left(\frac{\vartheta_1 \lambda}{1 - \overline{\alpha}\lambda}, \ldots, \frac{\vartheta_{n-1}\lambda}{1 - \overline{\alpha}\lambda}, \xi\lambda \, \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda} \right) \quad (\lambda \in \mathbb{D}). \tag{5.2}$$

If we write $f = (s_1, \ldots, s_n)$ on $\mathbb{D}$, then in order to have

$$n - (n-1)s_1(\lambda)z + \cdots + (-1)^{n-1} s_{n-1}(\lambda) z^{n-1} \neq 0$$

for all $z \in \mathbb{D}$ and $\lambda \in \mathbb{T}$ we must impose the condition

$$\lambda(n\overline{\alpha} + (n-1)\vartheta_1 z - (n-2)\vartheta_2 z^2 + \cdots + (-1)^n \vartheta_{n-1} z^{n-1}) \neq n \qquad (|z| < 1,\ |\lambda| = 1).$$

This is equivalent to

$$\sup_{|z| \leq 1} |n\overline{\alpha} + (n-1)\vartheta_1 z - (n-2)\vartheta_2 z^2 + \cdots + (-1)^n \vartheta_{n-1} z^{n-1}| \leq n. \tag{5.3}$$

Therefore, if the last supremum is $\leq n$, then $f(\mathbb{D}) \subseteq G_n$. If the supremum is exactly n and it is attained at some $z_0 \in \mathbb{T}$, then we see that $h : \mathbb{D} \to \mathbb{D}$ defined by

$$h(\lambda) = \frac{n(-1)^n s_n(\lambda) z_0^{n-1} + (n-1)(-1)^{n-1} s_{n-1}(\lambda) z_0^{n-2} + \cdots + (-s_1(\lambda))}{n - (n-1)s_1(\lambda) z_0 + \cdots + (-1)^{n-1} s_{n-1}(\lambda) z_0^{n-1}}$$

for $\lambda \in \mathbb{D}$ is a rational function of degree at most 1 such that $|h(\lambda)| = 1$ for all $\lambda \in \mathbb{T}$. Since $h(0) = 0$ we obtain $h(\lambda) = \eta\lambda$ for some $\eta \in \mathbb{T}$, and so $f : \mathbb{D} \to G_n$ is a complex geodesic on G_n. Let us also remark that if the answer to the final question from the above subsection is yes, then every complex geodesic of degree 2 on G_n passing through the origin is of the form (5.2).

For the case $n = 2$, if (s_0, p_0) is a fixed element in $G_2 \setminus \{(0,0)\}$ then there is a complex geodesic on G_2 of the form (5.1) or (5.2) passing through $(0,0)$ and (s_0, p_0). This allows us to obtain a Schwarz lemma for the symmetrized bidisc ([3, Theorem 1.1], [8, Theorem 5]). In fact, if $n = 2$ then Theorem 4.3 can be used ([8, Theorem 7]) in order to explicitly calculate the set of all complex geodesics on G_2, and to solve completely the two-point interpolation problem into G_2.

For $n \geq 3$, observe that if equality occurs in (5.3) then we can find $z_0, \xi \in \mathbb{T}$ such that $n - (n-1)s_1(\xi)z_0 + \cdots + (-1)^{n-1}s_{n-1}(\xi)z_0^{n-1} = 0$. Then the polynomial $Q(z) = nz^{n-1} - (n-1)s_1(\xi)z^{n-2} + \cdots + (-1)^{n-1}s_{n-1}(\xi)$ has a zero at $\bar{z}_0 \in \mathbb{T}$. Since $(s_1(\xi), \ldots, s_n(\xi)) \in \Gamma_n$, the polynomial $P(z) = z^n - s_1(\xi)z^{n-1} + \cdots + (-1)^n s_n(\xi)$ has all its roots inside $\overline{\mathbb{D}}$. Since $Q = P'$, the Lucas theorem implies that P has a zero of order at least 2 at $\bar{z}_0$. Therefore, the complex geodesic f on G_n given by (5.2) is a rational Γ_n-inner function on $\mathbb{D}$ of degree 2 for which we can find $\xi \in \mathbb{T}$ such that $f(\xi) = \pi_n(\bar{z}_0, \bar{z}_0, w_1, \ldots, w_{n-2}) \in \mathrm{db}(\Gamma_n)$. As we shall see in the next subsection, this situation can be generalized to the case of rational Γ_n-inner functions on $\mathbb{D}$ of degree $k \leq n$: by studying the relation between the interpolation problem into G_n and the interpolation problem into G_{n-1} we shall give a method to construct complex geodesics of arbitrary order $k \leq n$ on G_n.

5.2. Relations between the case n and $n-1$. Complex geodesics of order $k \leq n$. Let us start by proving a generalization of the implication "(i)⇒(ii)" of Corollary 3.4. The idea of the proof of Lemma 5.1 comes from the proof of Theorem 3.5.

LEMMA 5.1. *Let $(s_1, \ldots, s_n) \in G_n$ and define $R = \sup_{|z|\leq 1} |f(z)|$, where f is given by* (3.5), *so that $0 \leq R < 1$. Set*

$$\widetilde{s}_j(w) = \mathrm{C}_{n-1}^j \frac{s_j/\mathrm{C}_n^j - (w/R)(s_{j+1}/\mathrm{C}_n^{j+1})}{1 - (w/R)(s_1/\mathrm{C}_n^1)} \quad (w \in \mathbb{D}) \tag{5.4}$$

for $j = 1, \ldots, n-1$. Then $(\widetilde{s}_1(w), \ldots, \widetilde{s}_{n-1}(w)) \in G_{n-1}$ for all $w \in \mathbb{D}$. Moreover, if $(s_1, \ldots, s_n) \notin \{\pi_n(\alpha, \ldots, \alpha) : \alpha \in \mathbb{D}\}$, then there exists $w_0 \in \mathbb{T}$ such that $(\widetilde{s}_1(w_0), \ldots, \widetilde{s}_{n-1}(w_0)) \in \Gamma_{n-1} \setminus G_{n-1}$.

Proof. We know that

$$\frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1}s_{n-1}z^{n-2} + \cdots + (-s_1)}{n - (n-1)s_1 z + \cdots + (-1)^{n-1}s_{n-1}z^{n-1}} \neq -\frac{R}{w}$$

for $|z| \leq 1$ and $0 < |w| < 1$, and that there exist $|z_0| = 1$ and $|w_0| = 1$ such that equality occurs. If the above rational function in z is not constant, then $|s_1| < nR$ and

$$1-\frac{(n-1)s_1-2(w/R)s_2}{n-(w/R)s_1}\,z+\frac{(n-2)s_2-3(w/R)s_3}{n-(w/R)s_1}\,z^2+\cdots$$
$$+(-1)^{n-1}\frac{s_{n-1}-n(w/R)s_n}{n-(w/R)s_1}\,z^{n-1}\neq 0$$

for all $|z|\le 1$ and $|w|<1$. Therefore

$$\lambda^{n-1}-\widetilde{s}_1(w)\lambda^{n-1}+\cdots+(-1)^{n-1}\widetilde{s}_{n-1}(w)\neq 0$$

for all $w\in\mathbb{D}$ and $\lambda\in\mathbb{C}\setminus\mathbb{D}$, which means that $(\widetilde{s}_1(w),\dots,\widetilde{s}_{n-1}(w))\in G_{n-1}$ for all $w\in\mathbb{D}$. Since also $\overline{z}_0^{n-1}-\widetilde{s}_1(w_0)\overline{z}_0^{n-2}+\cdots+(-1)^{n-1}\widetilde{s}_{n-1}(w_0)=0$, this implies that $(\widetilde{s}_1(w_0),\dots,\widetilde{s}_{n-1}(w_0))\in\Gamma_{n-1}\setminus G_{n-1}$. If the above function in z is constant, then we can find $\alpha\in\mathbb{D}$ such that $(s_1,\dots,s_n)=(\mathrm{C}_n^1\alpha,\dots,\mathrm{C}_n^n\alpha^n)$, and then $\widetilde{s}_j(w)=\mathrm{C}_{n-1}^j\alpha^j$ for all $w\in\mathbb{D}$. ∎

By using (3.20) or the above lemma, we obtain a relation between the Γ_n-inner and Γ_{n-1}-inner functions.

Lemma 5.2. *For an analytic function $f=(s_1,\dots,s_n):\mathbb{D}\to G_n$ and for $z\in\overline{\mathbb{D}}$, the map $f_z=(\widetilde{s}_1,\dots,\widetilde{s}_{n-1})$ sends $\mathbb{D}$ into G_{n-1}, where*

$$\widetilde{s}_j(\lambda)=\mathrm{C}_{n-1}^j\,\frac{s_j(\lambda)/\mathrm{C}_n^j-zs_{j+1}(\lambda)/\mathrm{C}_n^{j+1}}{1-zs_1(\lambda)/\mathrm{C}_n^1}\qquad(\lambda\in\mathbb{D})\tag{5.5}$$

for $j=1,\dots,n-1$. Moreover, if f is Γ_n-inner and $z\in\mathbb{T}$, then f_z is Γ_{n-1}-inner, and if f is rational and Γ_n-inner and $z\in\mathbb{T}$, then f_z is rational and Γ_{n-1}-inner.

Proof. The fact that for all $z\in\overline{\mathbb{D}}$ the function f_z is analytic from $\mathbb{D}$ into G_{n-1} is a consequence of the above lemma. If $f(\xi)\in\mathrm{db}(\Gamma_n)$ for almost all ξ in $\mathbb{T}$ and if $z\in\mathbb{T}$, then

$$|s_{n-1}(\xi)/\mathrm{C}_n^{n-1}-zs_n(\xi)/\mathrm{C}_n^n|=|1-zs_1(\xi)/\mathrm{C}_n^1|$$

almost everywhere on $\mathbb{T}$, and therefore $|\widetilde{s}_{n-1}|=1$ almost everywhere on $\mathbb{T}$. Since $f_z(\xi)\in\Gamma_{n-1}$ almost everywhere on $\mathbb{T}$ this implies that $f_z(\xi)\in\mathrm{db}(\Gamma_{n-1})$ almost everywhere on $\mathbb{T}$. ∎

Lemma 5.2 gives a way to construct analytic functions from $\mathbb{D}$ into G_{n-1}, by starting with analytic functions from $\mathbb{D}$ into G_n. But how to solve the inverse problem? That is, we start with analytic functions from $\mathbb{D}$ into G_{n-1} and we want to construct analytic functions from $\mathbb{D}$ into G_n. It is clear that if $g=(s_1,,\dots,s_{n-1}):\mathbb{D}\to G_{n-1}$ and $b:\mathbb{D}\to\mathbb{D}$ are analytic functions, then by considering $f=(s_1+b,s_1b+s_2,\dots,s_{n-2}b+s_{n-1},s_{n-1}b)$ we find that g is analytic from $\mathbb{D}$ into G_n. But in the definition of g we have separated, analytically, $n-1$ values in $\mathbb{D}$ from one value in $\mathbb{D}$. Therefore, for the interpolation problem into G_n, it is clear that this method is not sufficient. A general way of reversing the construction in the above lemma (that is, $g=(\widetilde{s}_1,\dots,\widetilde{s}_{n-1})$ analytic from $\mathbb{D}$ into G_{n-1} is given, and we want to find

$z \in \overline{\mathbb{D}}$ and $f = (s_1, \dots, s_n) : \mathbb{D} \to G_n$ analytic such that $f_z = g$) seems to be difficult to obtain.

It is proved in Lemma 5.2 that if $f = (s_1, \dots, s_n) : \mathbb{D} \to G_n$ is Γ_n-inner and $z \in \mathbb{T}$, then $g = (\widetilde{s}_1, \dots, \widetilde{s}_{n-1})$ given by (5.5) is Γ_{n-1}-inner. But if f is a complex geodesic on G_n, does it follow that there exists $z \in \mathbb{T}$ such that g is a complex geodesic on G_{n-1}? We do not know the answer to this question. A partial result is given by the proof of the following theorem. It will also allow us to obtain some of the complex geodesics of order n on G_n.

THEOREM 5.2. *If $f = (s_1, \dots, s_n) : \mathbb{D} \to G_n$ is a rational Γ_n-inner function of degree less than or equal to n such that there exists $\xi \in \mathbb{T}$ with $f(\xi) = \pi_n(\eta, \dots, \eta)$ for some $\eta \in \mathbb{T}$, then f is a complex geodesic of degree n on G_n.*

Proof. Induction on n. For $n = 2$, let $f = (s, p)$ from $\mathbb{D}$ into G_2 be a rational function of degree at most 2 with poles off $\overline{\mathbb{D}}$ such that $f(\mathbb{T}) \subseteq \mathrm{db}(\Gamma_2)$ and $\|s\|_\infty = 2$. Then f cannot be of degree 1 since as we have already seen in Theorem 5.1 it must then be of the form $f(\lambda) = (\overline{\beta} b(\lambda) + \beta, b(\lambda))$, where $\beta \in \mathbb{D}$ and b is a Möbius transformation on $\mathbb{D}$, which contradicts the fact that $\|s\|_\infty = 2$. Let $f(\xi) = (2\eta, \eta^2)$ where $\xi, \eta \in \mathbb{T}$, and consider $q : G_2 \to \mathbb{D}$ given by

$$q(s,p) = \frac{2\overline{\eta}p - s}{2 - \overline{\eta}s}.$$

Then $q \circ f$ is a rational function of degree at most 1 on $\mathbb{D}$ which is also inner. The relation (3.8) and the fact that $f(\mathbb{D}) \subseteq G_2$ imply $|q \circ f| < 1$ on $\mathbb{D}$, and therefore $q \circ f$ cannot be a constant function. Hence $q \circ f$ is a Möbius transformation on $\mathbb{D}$, and therefore f is a complex geodesic on G_2.

Suppose now that every rational analytic function $g : \mathbb{D} \to G_{n-1}$ of degree $\le n-1$ such that $g(\mathbb{T}) \subseteq \mathrm{db}(\Gamma_{n-1})$ and $g(\xi) = \pi_{n-1}(\eta, \dots, \eta)$ for some $\xi \in \mathbb{T}$ and $\eta \in \mathbb{T}$, is a complex geodesic of degree $n-1$ on G_{n-1}. Let f be as in the statement. Without loss of generality, we may suppose that $f(1) = (\mathrm{C}_n^1, \dots, \mathrm{C}_n^n)$. Define $g : \mathbb{D} \to G_{n-1}$ by $g(\lambda) = (\widetilde{s}_1(\lambda), \dots, \widetilde{s}_{n-1}(\lambda))$ on $\mathbb{D}$, where

$$\widetilde{s}_j(\lambda) = \mathrm{C}_{n-1}^j \frac{s_j(\lambda)/\mathrm{C}_n^j - s_{j+1}(\lambda)/\mathrm{C}_n^{j+1}}{1 - s_1(\lambda)/\mathrm{C}_n^1} \quad (1 \le j \le n-1).$$

Then g is a rational function of degree $\le n-1$, its poles are in the exterior of $\overline{\mathbb{D}}$, and $g(\mathbb{T}) \subseteq \mathrm{db}(\Gamma_{n-1})$ by Lemma 5.2. We want to prove that $\widetilde{s}_1(1) = n-1$. For z in $\overline{\mathbb{D}}$, put

$$\varphi_z(\lambda) = \frac{s_1(\lambda)/\mathrm{C}_n^1 - z s_2(\lambda)/\mathrm{C}_n^2}{1 - z s_1(\lambda)/\mathrm{C}_n^1} \quad (\lambda \in \mathbb{D}).$$

Lemma 5.2 shows that $\varphi_z(\mathbb{D}) \subseteq \mathbb{D}$, the function φ_z being rational, with poles off $\overline{\mathbb{D}}$. Observe that $\varphi_z(1) = 1$ for all $z \in \overline{\mathbb{D}} \setminus \{1\}$. We want to prove that

$\widetilde{s}_1(1) = n-1$, that is, $\varphi_1(1) = 1$. It will be shown in the final part of the proof that $s_1'(1) \neq 0$. Then

$$\varphi_1(1) = -\frac{s_1'(1)/\mathrm{C}_n^1 - s_2'(1)/\mathrm{C}_n^2}{s_1'(1)/\mathrm{C}_n^1},$$

and therefore we must prove that

$$-s_1'(1)/\mathrm{C}_n^1 + s_2'(1)/\mathrm{C}_n^2 = s_1'(1)/\mathrm{C}_n^1. \tag{5.6}$$

For each $z \in \overline{\mathbb{D}} \setminus \{1\}$ there exists a neighborhood V_z of $1 \in \mathbb{C}$ such that φ_z is well defined and analytic on V_z. We have $|\varphi_z(1)| = 1$ and $|\varphi_z| \le 1$ on $V_z \cap \overline{\mathbb{D}}$. Put $h_z(\theta) = |\varphi_z(e^{i\theta})|^2$ for θ in a neighborhood of $0 \in \mathbb{R}$. The function h_z is at least of class C^1 near $\theta = 0$, and it has a maximum at this point. This implies that $h_z'(0) = 0$. Since

$$\begin{aligned}\frac{dh_z}{d\theta}(\theta) &= -ie^{-i\theta}\,\overline{\varphi_z'(e^{i\theta})}\varphi_z(e^{i\theta}) + ie^{i\theta}\,\overline{\varphi_z(e^{i\theta})}\,\varphi_z'(e^{i\theta})\\ &= i(-\overline{e^{i\theta}\,\overline{\varphi_z(e^{i\theta})}\varphi_z'(e^{i\theta})} + e^{i\theta}\,\overline{\varphi_z(e^{i\theta})}\,\varphi_z'(e^{i\theta})),\end{aligned}$$

we obtain $\mathrm{Im}(\varphi_z'(1)) = 0$ for all $z \in \overline{\mathbb{D}} \setminus \{1\}$. Therefore

$$\mathrm{Im}\,\frac{s_1'(1)/\mathrm{C}_n^1 - z(-s_1'(1)/\mathrm{C}_n^1 + s_2'(1)/\mathrm{C}_n^2)}{1-z} = 0 \qquad (z \in \overline{\mathbb{D}} \setminus \{1\}),$$

which implies that $s_1'(1)/\mathrm{C}_n^1 = -s_1'(1)/\mathrm{C}_n^1 + s_2'(1)/\mathrm{C}_n^2$. Therefore, the equality (5.6) is true. Hence $\widetilde{s}_1(1) = n-1$, and now the fact that $(\widetilde{s}_1(1), \dots, \widetilde{s}_{n-1}(1)) \in \Gamma_{n-1}$ gives $g(1) = (\mathrm{C}_{n-1}^1, \dots, \mathrm{C}_{n-1}^{n-1})$. By the induction hypothesis, the degree of g is $n-1$ and there exists $G : G_{n-1} \to \mathbb{D}$ analytic such that $G \circ g$ is a Möbius transformation on $\mathbb{D}$. Define $F : G_n \to \mathbb{D}$ by

$$F(S_1, \dots, S_n) = G\left(\mathrm{C}_{n-1}^1\,\frac{S_1/\mathrm{C}_n^1 - S_2/\mathrm{C}_n^2}{1 - S_1/\mathrm{C}_n^1}, \dots, \mathrm{C}_{n-1}^{n-1}\,\frac{S_{n-1}/\mathrm{C}_n^{n-1} - S_n/\mathrm{C}_n^n}{1 - S_1/\mathrm{C}_n^1}\right).$$

Then $F \circ f = G \circ g$, and therefore $F \circ f$ is a Möbius transformation on $\mathbb{D}$. We conclude that f is a complex geodesic of degree n on G_n.

Let us now justify the remaining claim. Suppose, for contradiction, that $s_1'(1) = 0$. We know that

$$\lambda \mapsto \frac{s_1(\lambda)/\mathrm{C}_n^1 - s_2(\lambda)/\mathrm{C}_n^2}{1 - s_1(\lambda)/\mathrm{C}_n^1}$$

is a rational function sending $\mathbb{D}$ into $\mathbb{D}$. Since $s_1(1) = \mathrm{C}_n^1$ and $s_2(1) = \mathrm{C}_n^2$, the fact that $s_1'(1) = 0$ implies $s_1'(1)/\mathrm{C}_n^1 - s_2'(1)/\mathrm{C}_n^2 = 0$. Therefore, $s_2'(1) = 0$. Considering then

$$\lambda \mapsto \frac{s_2(\lambda)/\mathrm{C}_n^2 - s_3(\lambda)/\mathrm{C}_n^3}{1 - s_1(\lambda)/\mathrm{C}_n^1},$$

we deduce in the same way that $s_3'(1) = 0$. After $n-1$ steps we obtain

$s_n'(1) = 0$. If we write

$$s_n(\lambda) = \delta \frac{\lambda^m - S_1\lambda^{m-1} + \cdots + (-1)^{m-1}S_{m-1}\lambda + (-1)^m S_m}{1 - \overline{S}_1\lambda + \cdots + (-1)^{m-1}\overline{S}_{m-1}\lambda^{m-1} + (-1)^m\overline{S}_m\lambda^m},$$

where $\delta \in \mathbb{T}$ and $(S_1, \ldots, S_m) \in G_m$, the fact that $s_n'(1) = 0$ implies

$$|m - (m-1)S_1 + \cdots + (-1)^{m-1}S_{m-1}| \\ = |-\overline{S}_1 + \cdots + (m-1)(-1)^{m-1}\overline{S}_{m-1} + m(-1)^m\overline{S}_m|.$$

Therefore

$$\left|\frac{-S_1 + \cdots + (m-1)(-1)^{m-1}S_{m-1} + m(-1)^m S_m}{m - (m-1)S_1 + \cdots + (-1)^{m-1}S_{m-1}}\right| = 1,$$

and this contradicts (3.6), since $(S_1, \ldots, S_m) \in G_m$. ∎

We have seen (Theorems 3.6 and 5.1) that for every $(s_1, \ldots, s_n) \in G_n$ there is a complex geodesic on G_n of degree 1 passing through this point. Using Theorem 5.2, we also obtain the following fact.

Corollary 5.1. *For a given element $(s_1, \ldots, s_n) \in G_n$ there is a complex geodesic of degree n passing through it.*

Proof. Let $(s_1, \ldots, s_n) = \pi_n(z_1, \ldots, z_n)$ with $z_1, \ldots, z_n \in \mathbb{D}$, and fix $\xi \in \mathbb{T}$. Consider then the Möbius transformations $b_1, \ldots, b_n$ on $\mathbb{D}$ such that $b_j(0) = z_j$ and $b_j(1) = \xi$ for $j = 1, \ldots, n$. Now Theorem 5.2 shows that $\lambda \mapsto \pi_n(b_1(\lambda), \ldots, b_n(\lambda))$ is a complex geodesic of degree n on G_n passing through $(s_1, \ldots, s_n)$. ∎

Now we want to construct complex geodesics of order $k < n$ on G_n. An important ingredient is the following lemma.

Lemma 5.3. *Let $(s_1, \ldots, s_n) = \pi_n(1, \ldots, 1, z_1, \ldots, z_{n-k}) \in \mathrm{db}(\Gamma_n)$, where $1 \le k < n$ and at least one of the z_j is different from $1 \in \mathbb{T}$. Put*

$$\widetilde{s}_j = \mathrm{C}^j_{n-1}\frac{s_j/\mathrm{C}^j_n - s_{j+1}/\mathrm{C}^{j+1}_n}{1 - s_1/\mathrm{C}^1_n} \qquad (j = 1, \ldots, n-1).$$

Then $(\widetilde{s}_1, \ldots, \widetilde{s}_{n-1}) = \pi_{n-1}(1, \ldots, 1, w_1, \ldots, w_{n-k-1})$ for some $w_1, \ldots$ $\ldots, w_{n-k-1} \in \mathbb{T}$.

Proof. We already know (see Lemma 5.2) that $(\widetilde{s}_1, \ldots, \widetilde{s}_{n-1}) \in \mathrm{db}(\Gamma_{n-1})$. Consider the polynomial $P(z) = z^n - s_1 z^{n-1} + \cdots + (-1)^n s_n$ on $\mathbb{C}$. We know that 1 is a zero of P of order at least $k \ge 1$. In particular, $P(1) = 0$, that is, $1 - s_1 + \cdots + (-1)^n s_n = 0$. We rewrite this equality as

$$n(-1)^n s_n + (n-1)(-1)^{n-1}s_{n-1} + \cdots + (-s_1) \\ = -(n - (n-1)s_1 + \cdots + (-1)^{n-1}s_{n-1}),$$

and observe that in the proof of Lemma 5.1 it is shown that this implies that $1 - \widetilde{s}_1 + \cdots + (-1)^{n-1}\widetilde{s}_{n-1} = 0$. Thus 1 is a root of $Q(z) = z^{n-1} - \widetilde{s}_1 z^{n-2} + \cdots + (-1)^{n-1}\widetilde{s}_{n-1}$.

If $k \geq 2$, then 1 is also a root of the polynomial

$$\frac{P'(z)}{n} = z^{n-1} - \frac{n-1}{n}\, s_1 z^{n-2} + \cdots + (-1)^{n-1}\frac{1}{n}\, s_{n-1}.$$

By applying the above considerations to $P'(z)/n$ we obtain

$$1 - \mathrm{C}^1_{n-2}\, \frac{\frac{n-1}{n} s_1/\mathrm{C}^1_{n-1} - \frac{n-2}{n} s_2/\mathrm{C}^2_{n-1}}{1 - \frac{n-1}{n} s_1/\mathrm{C}^1_{n-1}} + \cdots \\ + (-1)^{n-2}\mathrm{C}^{n-2}_{n-2}\, \frac{\frac{2}{n} s_{n-2}/\mathrm{C}^{n-2}_{n-1} - \frac{1}{n} s_{n-1}/\mathrm{C}^{n-1}_{n-1}}{1 - \frac{n-1}{n} s_1/\mathrm{C}^1_{n-1}} = 0,$$

that is, $Q'(1)/(n-2) = 0$. Therefore, 1 is also a zero of Q of order at least 2. By repeating the above reasoning k times, we conclude that 1 is a zero of Q of order at least k, and this is exactly what we wanted to prove. ∎

Observe that the above reasoning can be reversed to show that, in the context of Lemma 5.3, if $(s_1, \ldots, s_n) \in \mathrm{db}(\Gamma_n) \setminus \{\pi_n(1, \ldots, 1)\}$ and $(\widetilde{s}_1, \ldots, \widetilde{s}_{n-1}) = \pi_{n-1}(1, \ldots, 1, w_1, \ldots, w_{n-k-1})$, where $1 \leq k < n$ and $w_1, \ldots, w_{n-k-1} \in \mathbb{T}$, then $(s_1, \ldots, s_n) = \pi_n(1, \ldots, 1, z_1, \ldots, z_{n-k})$ for some $z_1, \ldots, z_{n-k} \in \mathbb{T}$.

The above lemma gives a way to recognize some of the complex geodesics on G_n of degree less than or equal to n.

Theorem 5.3. *Let $n, k \in \mathbb{N}$ with $n \geq 2$ and $1 \leq k \leq n$. If $f : \mathbb{D} \to G_n$ is a rational Γ_n-inner function of degree at most k for which there exist $\zeta, \eta \in \mathbb{T}$ such that $f(\zeta) = \pi_n(\eta, \ldots, \eta, z_1, \ldots, z_{n-k})$ for some $z_1, \ldots, z_{n-k} \in \mathbb{T}$, then f is a complex geodesic of degree k on G_n.*

Proof. Induction on $n \geq 2$. The case $n = 2$ is proved in Theorem 5.1. Suppose now that $n \geq 2$ and that for all $k \in \{1, \ldots, n\}$, if f is a rational Γ_n-inner function of degree at most k for which there exist $\zeta, \eta \in \mathbb{T}$ such that $f(\zeta) = \pi_n(\eta, \ldots, \eta, z_1, \ldots, z_{n-k})$ for some $z_1, \ldots, z_{n-k} \in \mathbb{T}$, then f is a complex geodesic of degree k on G_n. Let now $g = (S_1, \ldots, S_{n+1}) : \mathbb{D} \to G_{n+1}$ be a rational Γ_{n+1}-inner function of degree at most $k \leq n+1$ for which, for example, $g(1) = \pi_{n+1}(1, \ldots, 1, w_1, \ldots, w_{n+1-k})$, for some $w_1, \ldots, w_{n+1-k} \in \mathbb{T}$. If $k = n+1$, then Theorem 5.2 shows that g is a complex geodesic of degree $n+1$ on G_{n+1}. If not, then at least one of the w_j is different from 1, and consider $h : \mathbb{D} \to G_n$ given by

$$h = \left(\mathrm{C}^1_n\, \frac{S_1/\mathrm{C}^1_{n+1} - S_2/\mathrm{C}^2_{n+1}}{1 - S_1/\mathrm{C}^1_{n+1}}, \ldots, \mathrm{C}^n_n\, \frac{S_n/\mathrm{C}^n_{n+1} - S_{n+1}/\mathrm{C}^{n+1}_{n+1}}{1 - S_1/\mathrm{C}^1_{n+1}}\right).$$

Lemma 5.2 shows that h is a rational Γ_n-inner function. Its degree is at most k, and $h(1) = \pi_n(1, \dots, 1, z_1, \dots, z_{n-k}) \in \mathrm{db}(\Gamma_n)$ by Lemma 5.3. The induction hypothesis now implies that h is a complex geodesic of degree k on G_n, and hence (see the proof of Theorem 5.2) g is a complex geodesic of degree k on G_{n+1}. ∎

Using the above theorem one can see now that for any fixed $1 \le k \le n$ there are complex geodesics of order k on G_n. Furthermore, the following corollary holds.

Corollary 5.2. *If $(s_1, \dots, s_n) \in G_n$, then for each $1 \le k \le n$ there is a complex geodesic of degree k on G_n of the form given by Theorem* 5.3 *which passes through this point.*

Proof. Induction on n. If $n = 2$, then for a fixed element in G_2 by Theorems 3.6 and 5.1 there is a complex geodesic of order 1 on G_2 passing through it, and by Corollary 5.1 there is a complex geodesic $f = (s, p)$ of order 2 passing through it such that $\|s\|_\infty = 2$. Suppose now that if $1 \le k \le n$ and $(s_1, \dots, s_n) \in G_n$, then there is a complex geodesic of degree k on G_n of the form given by Theorem 5.3 such that its image contains $(s_1, \dots, s_n)$. Let $1 \le j \le n+1$ and $(S_1, \dots, S_n, S_{n+1}) \in G_{n+1}$. Write $(S_1, \dots, S_n, S_{n+1}) = \pi_{n+1}(z_1, \dots, z_{n+1})$ for some $z_1, \dots, z_{n+1} \in \mathbb{D}$. If $j = 1$, we have already seen that there is a complex geodesic of order 1 on G_{n+1} of the form given by Theorem 5.3 which passes through $(S_1, \dots, S_n, S_{n+1})$. If $j > 1$, then by our induction hypothesis we can find a rational Γ_n-inner function $f = (s_1, \dots, s_n)$ of degree $j - 1$ such that $f(\lambda_0) = \pi_n(z_1, \dots, z_n)$ for a $\lambda_0 \in \mathbb{D}$ and such that, for some $\zeta \in \mathbb{T}$, we have $f(\zeta) = \pi_n(\eta, \dots, \eta, w_1, \dots, w_{n-j+1})$ for some $\eta, w_1, \dots, w_{n-j+1} \in \mathbb{T}$. Let b be the Möbius transformation on $\mathbb{D}$ such that $b(\lambda_0) = z_{n+1}$ and $b(\zeta) = \eta$. Put $g = (s_1 + b, s_1 b + s_2, \dots, s_n b)$ on $\mathbb{D}$. Then g is a rational Γ_{n+1}-inner function of degree j such that $g(\lambda_0) = (S_1, \dots, S_n, S_{n+1})$ and $g(\zeta) = \pi_n(\eta, \dots, \eta, w_1, \dots, w_{n-j+1}, \eta)$. ∎

5.3. Uniqueness. For a fixed element $z \in \overline{\mathbb{D}}$, denote by $f_z^{(n)} : G_n \to \mathbb{D}$ the analytic map

$$(5.7)\quad f_z^{(n)}(s_1, \dots, s_n) = \frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1} s_{n-1} z^{n-2} + \dots + (-s_1)}{n - (n-1)s_1 z + \dots + (-1)^{n-1} s_{n-1} z^{n-1}}.$$

For a complex geodesic on G_n of the form given by Theorem 5.3, that is, a rational Γ_n-inner function of degree $k \le n$ for which there exist $\zeta, z_0 \in \mathbb{T}$ such that $f(\zeta) = \pi_n(z_0, \dots, z_0, z_1, \dots, z_{n-k})$ for some $z_1, \dots, z_{n-k} \in \mathbb{T} \setminus \{z_0\}$, the proof of Theorem 5.3 shows that $f_{\bar{z}_0}^{(n)} \circ g$ is an automorphism of $\mathbb{D}$. Our next theorem asserts that those are the only complex geodesics on G_n which can be obtained by using the functions $f_z^{(n)}$ given by (5.7).

THEOREM 5.4. *If $g : \mathbb{D} \to G_n$ is an analytic function for which there exists $z_0 \in \overline{\mathbb{D}}$ such that $b := f_{z_0}^{(n)} \circ g$ is a Möbius transformation on $\mathbb{D}$ then g is a rational Γ_n-inner function of degree $k \le n$ for which we can find $\zeta \in \mathbb{T}$ such that $g(\zeta) = \pi_n(\overline{z}_0, \dots, \overline{z}_0, z_1, \dots, z_{n-k})$ for some $z_1, \dots, z_{n-k} \in \mathbb{T}$.*

We shall first prove the following weaker version of the statement.

LEMMA 5.4. *If $g : \mathbb{D} \to G_n$ is a rational Γ_n-inner function of degree $k \ge 1$ for which there is $z_0 \in \mathbb{T}$ such that $b := f_{z_0}^{(n)} \circ g$ is a Möbius transformation on $\mathbb{D}$ then $k \le n$ and we can find $\zeta \in \mathbb{T}$ such that $g(\zeta) = \pi_n(\overline{z}_0, \dots, \overline{z}_0, z_1, \dots, z_{n-k})$ for some $z_1, \dots, z_{n-k} \in \mathbb{T}$.*

Proof. Once more, induction on $n \ge 2$. For $n = 2$, the statement is a particular case of [8, Theorem 7]. Suppose the statement is true for an $n \ge 2$, and let $g = (S_1, \dots, S_{n+1}) : \mathbb{D} \to G_{n+1}$ be a rational Γ_{n+1}-inner function of degree $k \ge 1$ for which there is $z_0 \in \mathbb{T}$ such that $b := f_{z_0}^{(n+1)} \circ g$ is a Möbius transformation on $\mathbb{D}$. Without loss of generality we may suppose that $z_0 = 1$. Therefore,

$$\begin{aligned} &b(\lambda) \\ &= \frac{(n+1)(-1)^{n+1}S_{n+1}(\lambda) + n(-1)^n S_n(\lambda) + \dots + (-S_1(\lambda))}{(n+1) - nS_1(\lambda) + \dots + (-1)^n S_n(\lambda)} \quad (\lambda \in \overline{\mathbb{D}}). \end{aligned} \tag{5.8}$$

Put

$$\begin{aligned} &h(\lambda) \\ &= \left(\mathrm{C}_n^1 \frac{S_1(\lambda)/\mathrm{C}_{n+1}^1 - S_2(\lambda)/\mathrm{C}_{n+1}^2}{1 - S_1(\lambda)/\mathrm{C}_{n+1}^1}, \dots, \mathrm{C}_n^n \frac{S_n(\lambda)/\mathrm{C}_{n+1}^n - S_{n+1}(\lambda)/\mathrm{C}_{n+1}^{n+1}}{1 - S_1(\lambda)/\mathrm{C}_{n+1}^1} \right) \end{aligned} \tag{5.9}$$

for $\lambda \in \overline{\mathbb{D}}$. Write $h = (s_1, \dots, s_n) : \mathbb{D} \to G_n$, and observe that, on the s_j, (5.8) and (5.9) imply that

$$\frac{n(-1)^n s_n(\lambda) + (n-1)(-1)^{n-1} s_{n-1}(\lambda) + \dots + (-s_1(\lambda))}{n - (n-1)s_1(\lambda) + \dots + (-1)^{n-1} s_{n-1}(\lambda)} = b(\lambda) \tag{5.10}$$

for all $\lambda \in \overline{\mathbb{D}}$. The following cases can occur.

(i) $S_1 \ne \mathrm{C}_{n+1}^1$ on $\mathbb{T}$. In this case, h is a rational Γ_n-inner function of degree k. Indeed, we can write $S_{n+1} = Q/P$, $S_1 = R/P$ and $S_n = T/P$, where R and T are polynomials of degree at most k, $P(\lambda) = \prod_{i=1}^k (1 - \overline{\alpha}_i \lambda)$ and $Q(\lambda) = \xi \prod_{i=1}^k (\lambda - \alpha_i)$ with $\xi \in \mathbb{T}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{D}$. Then $s_n = (T - (n+1)Q)/((n+1)P - R)$, and we know that s_n is a Blaschke product on $\mathbb{D}$. We want to prove first that the degree of $T - (n+1)Q$ is k. If the degree is strictly less than k, then $T(\lambda) = a_k \lambda^k + \dots + a_1 \lambda + a_0$ with $a_k = (n+1)\xi$. Now since for $\lambda \in \mathbb{T}$ we have

$$|S_n(\lambda)| = \left| \frac{\overline{a}_k + \dots + \overline{a}_1 \lambda^{k-1} + \overline{a}_0 \lambda^k}{\prod_{i=1}^k (1 - \overline{\alpha}_i \lambda)} \right|,$$

it follows that

$$\sup_{|\lambda|\le 1}\left|\frac{\overline{a}_k+\cdots+\overline{a}_1\lambda^{k-1}+\overline{a}_0\lambda^k}{\prod_{i=1}^k(1-\overline{\alpha}_i\lambda)}\right|\le \|S_n\|_\infty\le n+1.$$

The analytic function inside the supremum has value $(n+1)\overline{\xi}$ at $\lambda=0$, so by the maximum modulus principle it is constant and equal to $(n+1)\overline{\xi}$ on $\mathbb{D}$. This implies that $S_n(\lambda)=(n+1)S_{n+1}(\lambda)$ on $\mathbb{D}$, and since $S_1=\overline{S}_nS_{n+1}$ on $\mathbb{T}$ we obtain $S_1\equiv n+1$, a contradiction. Therefore, the degree of $T-(n+1)Q$ is k, and it remains to prove that $T-(n+1)Q$ and $(n+1)P-R$ have no common zeros in $\mathbb{C}$. If $T(\alpha)=(n+1)Q(\alpha)$ and $R(\alpha)=(n+1)P(\alpha)$ for some $\alpha\in\mathbb{C}$, then the second equality implies that $|\alpha|>1$ (indeed, if $\alpha\in\overline{\mathbb{D}}$ then $P(\alpha)\ne 0$ and $S_1(\alpha)=n+1$, and this contradicts our supposition). Therefore, $1/\overline{\alpha}\in\mathbb{D}$. By using (4.8) we get $S_1(1/\overline{\alpha})=\overline{T(\alpha)}/\overline{Q(\alpha)}=n+1$, and once more we obtain a contradiction. Therefore, the degree of h is k, and then by (5.10) and our induction hypothesis we infer that $k\le n$ and that we can find $\zeta\in\mathbb{T}$ such that $h(\zeta)=\pi_n(1,\dots,1,z_1,\dots,z_{n-k})$ for some $z_1,\dots,z_{n-k}\in\mathbb{T}$. Now the remark following the proof of Lemma 5.3 shows that $g(\zeta)=\pi_{n+1}(1,\dots,1,w_1,\dots,w_{n-k+1})$ for some $w_1,\dots,w_{n-k+1}\in\mathbb{T}$.

(ii) There is $\zeta\in\mathbb{T}$ such that $S_1(\zeta)=\mathrm{C}^1_{n+1}$. Then $g(\zeta)=(\mathrm{C}^1_{n+1},\dots,\mathrm{C}^{n+1}_{n+1})$. We now use what is proved in Theorem 5.2. For the function $h=(s_1,\dots,s_n)$ given by (5.9), we have $h(\zeta)=(\mathrm{C}^1_n,\dots,\mathrm{C}^n_n)$, and so we can apply the same argument for h to see that if we set

$$h_1(\lambda)=\left(\mathrm{C}^1_{n-1}\,\frac{s_1(\lambda)/\mathrm{C}^1_n-s_2(\lambda)/\mathrm{C}^2_n}{1-s_1(\lambda)/\mathrm{C}^1_n},\dots,\mathrm{C}^{n-1}_{n-1}\,\frac{s_{n-1}(\lambda)/\mathrm{C}^{n-1}_n-s_n(\lambda)/\mathrm{C}^n_n}{1-s_1(\lambda)/\mathrm{C}^1_n}\right)$$

for $\lambda\in\overline{\mathbb{D}}$, then $h_1(\zeta)=(\mathrm{C}^1_{n-1},\dots,\mathrm{C}^{n-1}_{n-1})$. After n steps, we obtain $h_{n-1}(\zeta)=\mathrm{C}^1_1=1$. Now observe that

$$h_{n-1}=-\frac{(n+1)(-1)^{n+1}S_{n+1}(\lambda)+n(-1)^nS_n(\lambda)+\cdots+(-S_1(\lambda))}{(n+1)-nS_1(\lambda)+\cdots+(-1)^nS_n(\lambda)}$$

on $\overline{\mathbb{D}}$, and hence, by using (5.8), we obtain $b(\zeta)=-1$. Since b is bijective on $\mathbb{T}$, there is only one $\zeta\in\mathbb{T}$ such that $S_1(\zeta)=\mathrm{C}^1_{n+1}$. Since $S_1'(\zeta)\ne 0$ (see the proof of Theorem 5.2), the h given by (5.9) is of degree $k-1$. (By what is proved at (i), the only points where cancellations can occur, which could decrease the degree of s_n, are the points on $\mathbb{T}$ where S_1 equals $n+1$.) Since (5.10) holds for h, by applying the induction hypothesis we deduce that the degree of h is $\le n$. Therefore the degree of g is at most $n+1$, and since $g(\zeta)=(\mathrm{C}^1_{n+1},\dots,\mathrm{C}^{n+1}_{n+1})$, by Theorem 5.2 we conclude that $k=n+1$. ■

Proof of Theorem 5.4. Suppose first that $z_0\in\mathbb{D}$. Consider two distinct points λ_1 and λ_2 in $\mathbb{D}$. Then by Theorem 4.2 we can find a rational Γ_n-inner function $\widetilde{g}:\mathbb{D}\to G_n$ such that $\widetilde{g}(\lambda_k)=g(\lambda_k)$ for $k=1,2$. Now the analytic function $f^{(n)}_{z_0}\circ\widetilde{g}$ sends $\mathbb{D}$ into $\mathbb{D}$, and $(f^{(n)}_{z_0}\circ\widetilde{g})(\lambda_k)=(f^{(n)}_{z_0}\circ g)(\lambda_k)=$

$b(\lambda_k)$ for $k = 1, 2$. Therefore $f_{z_0}^{(n)} \circ \widetilde{g} = b$. If we write $\widetilde{g} = (s_1, \ldots, s_n)$, then

$$\frac{n(-1)^n s_n(\lambda) z_0^{n-1} + (n-1)(-1)^{n-1} s_{n-1}(\lambda) z_0^{n-2} + \cdots + (-s_1(\lambda))}{n - (n-1) s_1(\lambda) z_0 + \cdots + (-1)^{n-1} s_{n-1}(\lambda) z_0^{n-1}} = b(\lambda) \tag{5.11}$$

for all λ in a neighborhood of $\overline{\mathbb{D}}$. For $\zeta \in \mathbb{T}$, since $(s_1(\zeta), \ldots, s_n(\zeta)) \in \Gamma_n$ and $z_0 \in \mathbb{D}$ we see that $n - (n-1)s_1(\zeta) z_0 + \cdots + (-1)^{n-1} s_{n-1}(\zeta) z_0^{n-1} \neq 0$. Therefore,

$$\frac{|n(-1)^n s_n(\zeta) z_0^{n-1} + (n-1)(-1)^{n-1} s_{n-1}(\zeta) z_0^{n-2} + \cdots + (-s_1(\zeta))|}{|n - (n-1) s_1(\zeta) z_0 + \cdots + (-1)^{n-1} s_{n-1}(\zeta) z_0^{n-1}|} = 1.$$

Since by (3.9) the analytic function

$$z \mapsto \frac{n(-1)^n s_n(\zeta) z^n + (n-1)(-1)^{n-1} s_{n-1}(\zeta) z^{n-1} + \cdots + (-s_1(\zeta))}{n - (n-1) s_1(\zeta) z + \cdots + (-1)^{n-1} s_{n-1}(\zeta) z^{n-1}}$$

sends $\mathbb{D}$ into $\overline{\mathbb{D}}$, by the maximum modulus principle it must be constant. One can easily see that this implies that there is $\eta \in \mathbb{T}$ such that $(s_1(\zeta), \ldots, s_n(\zeta)) = \pi_n(\eta, \ldots, \eta)$. Since $\zeta \in \mathbb{T}$ was arbitrary, we infer that there is a Blaschke product B such that $\widetilde{g} = \pi_n(B, \ldots, B)$ on $\overline{\mathbb{D}}$. But $g(\lambda_1) = \widetilde{g}(\lambda_1)$ and $\lambda_1 \in \mathbb{D}$ was arbitrary; therefore, there exists $f : \mathbb{D} \to \mathbb{D}$ analytic such that $g = \pi_n(f, \ldots, f)$ on $\mathbb{D}$. Then (5.11) gives $f = -b$, and this easily implies that $f_z^{(n)} \circ g = -b$ for all $z \in \overline{\mathbb{D}}$. Therefore, without loss of generality we may suppose that $z_0 \in \mathbb{T}$.

Let $(\lambda_k)_{k \in \mathbb{N}} \subseteq \mathbb{D}$ be a dense sequence. For each $j \geq 2$, by applying Theorem 4.2 we obtain a rational Γ_n-inner function $g_j : \mathbb{D} \to G_n$ such that $g_j(\lambda_k) = g(\lambda_k)$ for $k = 1, \ldots, j$. As above, $f_{z_0}^{(n)} \circ g_j = b$ for each $j \geq 2$. Then by Lemma 5.4, for each $j \geq 2$ the last component of g_j is a Blaschke product of degree $\leq n$. Since $G_n \subseteq \mathbb{C}^n$ is bounded, by Montel's theorem we can find a subsequence $(j_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ and $h : \mathbb{D} \to \Gamma_n$ analytic such that $g_{j_i} \to h$ locally uniformly on $\mathbb{D}$. Then clearly the last component of h is either constant or a Blaschke product of degree at most n. But $g_{j_i}(\lambda_k) \to g(\lambda_k)$ for all k, and therefore $h(\lambda_k) = g(\lambda_k)$ for all k. By density we deduce that $h = g$, and therefore g is a rational Γ_n-inner function. Now we apply Lemma 5.4 to g to obtain the statement. ∎

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