STUDIA MATHEMATICA 170 (1) (2005)

Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces

by

Teresa Bermúdez (La Laguna), Antonio Bonilla (La Laguna),
José A. Conejero (València) and Alfredo Peris (València)

Abstract. Our aim in this paper is to prove that every separable infinite-dimensional complex Banach space admits a topologically mixing holomorphic uniformly continuous semigroup and to characterize the mixing property for semigroups of operators. A concrete characterization of being topologically mixing for the translation semigroup on weighted spaces of functions is also given. Moreover, we prove that there exists a commutative algebra of operators containing both a chaotic operator and an operator which is not a multiple of the identity and no multiple of which is chaotic. This gives a negative answer to a question of deLaubenfels and Emamirad.

1. Introduction. A bounded linear operator T on a separable complex Banach space X is said to be *hypercyclic* if there exists an $x \in X$ such that $\{T^n x\}_{n\in\mathbb{N}}$ is dense in X. In that case we say that x is a *hypercyclic vector* for T. A point x is called *periodic* for T if there exists some $n > 1$ such that $T^n x = x$. The operator T is *chaotic* if it is hypercyclic and the set of periodic points is dense in X.

This definition of chaos is consistent with the definition given by Devaney [17, p. 50] for an arbitrary continuous mapping f on a metric space, which requires that f be transitive, the set of periodic points of f be dense, and f have sensitive dependence on the initial conditions. The first two conditions imply the third [5]. For more details about hypercyclicity see the surveys of Grosse-Erdmann [23, 24] and the survey of Bonet, Martínez-Giménez and Peris [10].

2000 *Mathematics Subject Classification*: Primary 47A16; Secondary 47D03.

The first author is supported in part by Consejería de Educación del Gobierno de Canarias PI 2002/023 (Spain) and by MCYT and FEDER, BFM2003-07139.

The second author is supported by MCYT and FEDER BFM2002-02098.

The third and fourth authors are supported by FEDER and MCYT, Proyecto no. BFM2001-2670 and AVCIT Grupo 03/050.

In 1969, Rolewicz [30] gave the first example of a hypercyclic operator on a Banach space. He showed that if B is the backward shift on $l^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. He also wondered if for every separable infinite-dimensional Banach space there exists a hypercyclic operator. This question was independently answered in the affirmative by Ansari [1] and Bernal [7], and by Bonet and Peris [11] for Fréchet spaces. On the other hand, Bonet, Martínez-Giménez and Peris [9] have recently proved that the dual of a reflexive separable hereditarily indecomposable complex Banach space admits no chaotic bounded linear operator.

It is well known that T is hypercyclic if and only if for any pair of non-void open sets U, V there exists some positive integer n such that

$$(*) \qquad T^n U \cap V \neq \emptyset.$$

An operator is said to be *topologically mixing* if $(*)$ holds for every n large enough.

Costakis and Sambarino gave a sufficient condition for a linear operator to be topologically mixing and characterized those weighted backward shift operators that are topologically mixing [14].

In our context all semigroups are semigroups of operators in $L(X)$, the set of bounded linear operators from X to X.

Our semigroups have as an index set a concrete sector in the complex plane. For $\alpha \in [0, \pi/2]$ we define the *sector* $\Delta(\alpha)$ by

$$\Delta(\alpha) = \{re^{i\theta} : r \geq 0,\ \theta \in [-\alpha, \alpha]\}.$$

We also consider $\Delta(\pi) = \mathbb{C}$, the whole complex plane.

A one-parameter family $\{T(t)\}_{t \in \Delta(\alpha)}$ of bounded linear operators is a *one-parameter semigroup* of operators in $L(X)$ if it satisfies the following conditions:

(1) $T(0) = I$ (here I stands for the identity operator on X).
(2) $T(t)T(s) = T(t+s)$ for all $t, s \in \Delta(\alpha)$ (semigroup law).

The *infinitesimal generator* A of a semigroup $\{T(t)\}_{t \in \Delta(\alpha)}$ is defined by

$$Ax := \lim_{h \to 0} \frac{T(h)x - x}{h}$$

for all $x \in X$ for which this limit exists.

We say that a semigroup is *strongly continuous* provided that $\lim_{t \to s} T(t) = T(s)$ for all $s \in \Delta(\alpha)$ pointwise on X. The semigroup is *uniformly continuous* if this limit holds uniformly on the unit ball of X.

If $\{T(t)\}_{\in \Delta(\alpha)}$ is a strongly continuous semigroup, then the infinitesimal generator is closed and densely defined. In the uniformly continuous case, the infinitesimal generator is everywhere defined and bounded.

For $\alpha \neq 0$ the semigroup $\{T(t)\}_{t\in\Delta(\alpha)}$ is a *holomorphic semigroup of angle* α if the mapping $t \mapsto T(t)$ is analytic on $\Delta(\alpha)$. For more details about holomorphic semigroups see [31, Chapter IX, Section 10], [28, p. 60] and [18, Chapter II, Section 4].

The most common index set, $\Delta(0) = [0,\infty)$, is widely applied in the literature due to its relation to ordinary and partial differential equations. For further information about semigroups of operators on Banach spaces we refer the reader to the book of Pazy [28], or to the recent book of Engel and Nagel [18].

A semigroup $\{T(t)\}_{t\in\Delta(\alpha)}$ is *hypercyclic* if there exists $x \in X$ such that $\{T(t)x\}_{t\in\Delta(\alpha)}$ is dense in X. Furthermore, it is *chaotic* if it is hypercyclic and the set $\{x \in X : \exists t \in \Delta(\alpha)\backslash\{0\},\ T(t)x = x\}$ is dense in X. We say that a semigroup $\{T(t)\}_{t\in\Delta(\alpha)}$ is *topologically mixing* if for every pair of non-void sets U, V there exists $t_0 \in \Delta(\alpha)$ such that

$$T(t)U \cap V \neq \emptyset \quad \text{for all } t \text{ such that } |t| \geq |t_0|.$$

In particular, all operators (except $T(0) = I$) of a topologically mixing semigroup are topologically mixing as single operators.

The theory of hypercyclic semigroups of operators has attracted the attention of many authors [6, 29, 16, 13]. In [6] it is proved that every separable infinite-dimensional complex Banach space admits a hypercyclic uniformly continuous semigroup and that there are some separable Banach spaces admitting no chaotic semigroup of operators.

Our aim in this paper is to prove that every separable infinite-dimensional complex Banach space admits a topologically mixing holomorphic uniformly continuous semigroup. We also give some examples of topologically mixing semigroups. More precisely, we characterize the topologically mixing translation semigroups on $L^p_\varrho(I)$ and $C_{0,\varrho}(I)$, where $I = \Delta(0) = [0,\infty)$ or $I = \mathbb{R}$ and ϱ is an admissible weight function on I.

Moreover, we prove that there exists a commutative algebra of operators that contains a chaotic operator and an operator T which is not a multiple of the identity such that no multiple of T is chaotic, answering in the negative a question of deLaubenfels and Emamirad [15].

2. Existence of topologically mixing holomorphic semigroups. We consider the weighted ℓ^1-space with a weight sequence $\beta = (\beta_i)_{i=1}^\infty$ of positive numbers, defined by

$$\ell^1(\beta) := \Big\{(x_i)_{i=1}^\infty : x_i \in \mathbb{C},\ \sum_{i=1}^\infty \beta_i|x_i| < \infty\Big\}$$

equipped with the norm

$$\|(x_i)\| := \sum_{i=1}^{\infty} \beta_i |x_i|.$$

If $\sup_{i\in\mathbb{N}} \beta_i/\beta_{i+1} \le M$ for some constant M, then the backward shift B defined by

$$B(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

is a bounded linear operator on $\ell^1(\beta)$.

In the case where $\beta_i = 1$ for all $i \ge 1$ we refer to this space as ℓ^1.

Following some ideas of [16, Theorem 5.2], where it was proved that $\{e^{tB}\}_{t\ge 0}$ is hypercyclic in $\ell^1(\beta)$, we prove that $\{e^{tB^2}\}_{t\in\Delta(\pi/2)}$ is also a topologically mixing semigroup there.

LEMMA 2.1. *Let $\beta = (\beta_i)_{i=1}^{\infty}$ be a sequence of positive numbers and B the backward shift operator. If $\sup_{i\in\mathbb{N}} \beta_i/\beta_{i+1} \le M$ for some constant M, then the uniformly continuous semigroup $\{e^{tB^2}\}_{t\in\Delta(\alpha)}$ is topologically mixing in $\ell^1(\beta)$ for every $\alpha \in [0, \pi/2] \cup \{\pi\}$.*

Proof. Fix $\alpha \in [0, \pi/2] \cup \{\pi\}$. Consider

$$\varphi := \{(x_i)_{i=1}^{\infty} \subset \mathbb{C} : \exists m\ \forall i > m,\ x_i = 0\},$$

which is dense in $\ell^1(\beta)$.

Given any pair $y = (y_i)_{i=1}^{\infty}$ and $z = (z_i)_{i=1}^{\infty}$ of elements in φ, and $\varepsilon > 0$, our purpose is to construct vectors $v(t) = (v_i(t))_{i=1}^{\infty} \in \ell^1(\beta)$, $t \in \Delta(\alpha)$, such that, for $T(t) := e^{tB^2}$, there exists $t_0 \in \Delta(\alpha)$ with

$$\|v(t) - y\| < \varepsilon \text{ and } \|T(t)v(t) - z\| < \varepsilon, \text{ for all } t \in \Delta(\alpha) \text{ with } |t| > |t_0|.$$

To do this we first select and fix $k \in \mathbb{N}$ with

$$y_i = z_i = 0 \quad \text{for } i \ge 2k.$$

We now define $v(t)$ as follows:

$$v_i(t) = \begin{cases} y_i & \text{for } i = 1, \dots, 2k, \\ 0 & \text{for } i = 2k+1, \dots, 4k \text{ and for } i > 6k. \end{cases}$$

We will define $v_i(t)$ for $i = 4k+1, \dots, 6k$ in such a way that

$$(T(t)v(t))_i = \sum_{j=0}^{3k} \frac{t^j}{j!} v_{2j+i}(t) \quad \text{for } i = 1, \dots, 2k.$$

In fact, we choose $v_i(t)$ for $i = 4k+1, \dots, 6k$ as the solutions of the following system:

$$A \begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix} + D \begin{pmatrix} v_{4k+1}(t) \\ \vdots \\ v_{6k}(t) \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix}, \tag{2.1}$$

where

$$A := \begin{pmatrix} 1 & 0 & \mu_1 & 0 & \mu_2 & 0 & \cdots & 0 & \mu_{k-1} & 0 \\ 0 & 1 & 0 & \mu_1 & 0 & \mu_2 & \cdots & \mu_{k-2} & 0 & \mu_{k-1} \\ \cdot & 0 & 1 & 0 & \mu_1 & 0 & \cdots & 0 & \mu_{k-2} & 0 \\ \cdot & \cdot & 0 & 1 & 0 & \mu_1 & \cdots & \cdot & 0 & \mu_{k-2} \\ \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdots & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

with $\mu_i = t^i/i!$ for $i = 1, \dots, k-1$ and

$$D := \begin{pmatrix} \lambda_0 & 0 & \lambda_1 & 0 & \cdots & \lambda_{k-1} & 0 \\ 0 & \lambda_0 & 0 & \lambda_1 & \cdots & 0 & \lambda_{k-1} \\ \lambda_{-1} & 0 & \lambda_0 & 0 & \cdots & \cdot & 0 \\ 0 & \lambda_{-1} & 0 & \lambda_0 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_{-k+1} & 0 & \cdot & \cdot & \cdots & \lambda_0 & 0 \\ 0 & \lambda_{-k+1} & \cdot & \cdot & \cdots & 0 & \lambda_0 \end{pmatrix},$$

where $\lambda_i = t^{2k+i}/(2k+i)!$ for $i = -k+1, \dots, k-1$.

Observe that $D = UWS$, where

$$U := \begin{pmatrix} t^{2k} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^{2k} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^{2k-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^{2k-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^{k+1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t^{k+1} \end{pmatrix},$$

$$W := \begin{pmatrix} \eta_0 & 0 & \eta_1 & 0 & \cdots & \eta_{k-1} & 0 \\ 0 & \eta_0 & 0 & \eta_1 & \cdots & 0 & \eta_{k-1} \\ \eta_{-1} & 0 & \eta_0 & 0 & \cdots & \cdot & 0 \\ 0 & \eta_{-1} & 0 & \eta_0 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta_{-k+1} & 0 & \cdot & \cdot & \cdots & \eta_0 & 0 \\ 0 & \eta_{-k+1} & \cdot & \cdot & \cdots & 0 & \eta_0 \end{pmatrix}$$

with $\eta_i = 1/(2k+i)!$ for $i = -k+1, \ldots, k-1$, and

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^{k-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t^{k-1} \end{pmatrix}$$

are regular $2k \times 2k$ matrices for $t \neq 0$. Thus the solution of (2.1) is given by

$$\begin{pmatrix} v_{4k+1}(t) \\ \vdots \\ v_{6k}(t) \end{pmatrix} = S^{-1}W^{-1}U^{-1}\left[\begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix} - A\begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix}\right].$$

Let $(w_i)_{i=1}^{2k}$ be defined as

$$\begin{pmatrix} w_1 \\ \vdots \\ w_{2k} \end{pmatrix} = U^{-1}\left[\begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix} - A\begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix}\right].$$

Then, if we consider t with $|t| > 1$, there exists C_1 independent of t such that

$$|w_j| \le C_1 |t|^{-k-1} \quad \text{for } j = 1, \ldots, 2k.$$

Thus there exists C_2 independent of t satisfying

$$|v_i(t)| \le C_2 |t|^{k-i/2} \quad \text{for } i = 4k+1, \ldots, 6k.$$

Since $v_i(t) = y_i$ for $i \le 4k$ and $y_i = 0$ for $i > 2k$, we have

$$\|v(t) - y\| = \sum_{i=4k+1}^{6k} \beta_i |v_i(t)| \le C_2 \sum_{i=4k+1}^{6k} \beta_i |t|^{-k}.$$

Hence, for sufficiently large $|t|$, we get

$$\|v(t) - y\| < \varepsilon.$$

Moreover

$$\begin{aligned}\|T(t)v(t) - z\| &= \sum_{i=2k+1}^{6k} \beta_i |(T(t)v(t))_i| = \sum_{i=2k+1}^{6k} \beta_i \Big| \sum_{j=0}^{2k-1} \frac{t^j}{j!} v_{2j+i}(t) \Big| \\ &\le \sum_{i=2k+1}^{6k} \beta_i \Big(\sum_{j=0}^{2k-1} \frac{|t|^j}{j!} |t|^{k-j-i/2} \Big) \le \sum_{i=2k+1}^{6k} \beta_i \Big(\sum_{j=0}^{2k-1} \frac{|t|^{-1/2}}{j!} \Big).\end{aligned}$$

Therefore we conclude that $\|T(t)v(t) - z\| < \varepsilon$ if $|t|$ is sufficiently large. ■

The following lemma is, in part, a generalization of [26, Lemma 2.1]. Its proof follows the same lines.

LEMMA 2.2. *Let X_1, X_2 be separable Banach spaces and let $\Phi : X_1 \to X_2$ be a continuous mapping with dense range.*

(1) *Let $T_i \in L(X_i)$ be an operator for $i = 1, 2$ such that $T_2\Phi = \Phi T_1$. If T_1 is topologically mixing (resp. hypercyclic), then T_2 is also topologically mixing (resp. hypercyclic).*
(2) *Let $\{T(t)\}_{t \in \Delta(\alpha)}$ and $\{S(t)\}_{t \in \Delta(\alpha)}$ be strongly continuous semigroups of operators on X_1 and X_2 respectively with $\alpha \in [0, \pi/2] \cup \{\pi\}$ such that $S(t)\Phi = \Phi T(t)$ for all $t \in \Delta(\alpha)$. If $\{T(t)\}_{t \in \Delta(\alpha)}$ is topologically mixing (resp. hypercyclic), then $\{S(t)\}_{t \in \Delta(\alpha)}$ is also topologically mixing (resp. hypercyclic).*

The following result due to Ovsepian and Pełczyński [27] is needed to show the existence result.

THEOREM 2.3 ([27]). *Let X be a separable infinite-dimensional Banach space. Then there exist $(x_n)_{n=1}^{\infty} \subset X$ and $(f_m)_{m=1}^{\infty} \subset X^*$, the dual space of X, satisfying the following conditions:*

(1) $f_m(x_n) = \delta_{m,n}$, $m, n \in \mathbb{N}$.
(2) $\overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}} = X$.

(3) $f_m(x) = 0$ *for all* $m \in \mathbb{N} \Rightarrow x = 0$.
(4) $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$ *and* $\sup_{m\in\mathbb{N}} \|f_m\| = c < \infty$.

THEOREM 2.4. *Every separable infinite-dimensional Banach space X admits a topologically mixing holomorphic uniformly continuous semigroup of angle $\pi/2$.*

Proof. Let $(x_n)_{n=1}^{\infty} \subset X$ and $(f_m)_{m=1}^{\infty} \subset X^*$ be as in Theorem 2.3. Consider the bounded linear operator $S : X \to X$ defined by

$$Sx := \sum_{n=1}^{\infty} \frac{1}{2^n} f_{n+1}(x)x_n. \tag{2.2}$$

Our purpose is to show that $\{e^{tS^2}\}_{t\in\Delta(\pi/2)}$ is a topologically mixing holomorphic uniformly continuous semigroup of angle $\pi/2$. This semigroup is uniformly continuous by [18, Proposition I.3.5(i)]. Since S is the generator of a strongly continuous group, S^2 generates a holomorphic semigroup of angle $\pi/2$ by [3, Theorem 1.15] or [18, Corollary II.4.9].

Define $\Phi : \ell^1 \to X$ by $\Phi((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, which is a bounded linear operator with dense range.

CLAIM. *The semigroup $e^{t\tilde{S}^2} : \ell^1 \to \ell^1$ is topologically mixing, where*

$$\widetilde{S}((\alpha_j)_j) := (\alpha_2/2, \ldots, \alpha_{n+1}/2^n, \ldots).$$

From the definition it follows that $S\Phi = \Phi\widetilde{S}$ on ℓ^1 and therefore $e^{tS^2}\Phi = \Phi e^{t\widetilde{S}^2}$ for all $t \in \Delta(\pi/2)$. Applying Lemma 2.2(2) we deduce that $\{e^{tS^2}\}_{t\in\Delta(\pi/2)}$ is topologically mixing in X.

Proof of the Claim. We define $\beta_1 := 1$ and $\beta_i := 2^{1+\cdots+(i-1)}$ for $i > 1$, and $\Phi_\beta : \ell^1(\beta) \to \ell^1$ by $\Phi_\beta(\alpha_1, \alpha_2, \ldots) := (\beta_1\alpha_1, \beta_2\alpha_2, \ldots)$, which is a linear continuous surjective mapping.

According to Lemma 2.1, $\{e^{tB^2}\}_{t\in\Delta(\pi/2)}$ is topologically mixing on $\ell^1(\beta)$, since $\beta_i/\beta_{i+1} \le 1$ for all $i \in \mathbb{N}$. Therefore by Lemma 2.2(2), the semigroup $\{e^{t\widetilde{S}^2}\}_{\Delta(\pi/2)}$ is topologically mixing because $e^{t\widetilde{S}^2}\Phi_\beta = \Phi_\beta e^{tB^2}$ for every $t \in \Delta(\pi/2)$. ∎

As a consequence of the above theorem, all separable infinite-dimensional Banach spaces admit a topologically mixing holomorphic uniformly continuous semigroup. However, the countable product of lines $\omega = \mathbb{K}^{\mathbb{N}}$ endowed with the product topology is a (separable) Fréchet space which does not admit any hypercyclic uniformly continuous semigroup [12].

3. Topological mixing criteria. First of all, we state a criterion which is sufficient for an operator to be topologically mixing.

CRITERION 3.1 (Mixing Criterion for Operators). *Let X be a separable Banach space and $T \in L(X)$. If*

(a) *there exists a dense subset* $X_0 \subset X$ *such that* $\lim_{n\to\infty} T^n x = 0$ *for all* $x \in X_0$ *and*

(b) *there exist a dense subset* $Y_0 \subset X$ *and mappings* $S_n : Y_0 \to X$ *for each* $n \in \mathbb{N}$ *such that*

(b.1) $\lim_{n\to\infty} S_n y = 0$ *for all* $y \in Y_0$, *and*

(b.2) $\lim_{n\to\infty} T^n \circ S_n y = y$ *for all* $y \in Y_0$,

then T *is a topologically mixing operator.*

There is another mixing criterion, which was already given by Costakis and Sambarino [14]:

CRITERION 3.2 (Costakis–Sambarino). *Let* X *be a separable Banach space and* $T \in L(X)$. *If*

(a′) *there exist a dense subset* $X_0 \subset X$ *and an increasing sequence* (n_k) *of positive integers such that* $\sup_k (n_{k+1} - n_k) < \infty$ (*i.e.*, (n_k) *is syndetic*) *satisfying* $\lim_{k\to\infty} T^{n_k} x = 0$ *for all* $x \in X_0$ *and*

(b′) *there exist a dense subset* $Y_0 \subset X$ *and mappings* $S_{n_k} : Y_0 \to X$ *for each* $k \in \mathbb{N}$ *such that*

(b′.1) $\lim_{k\to\infty} S_{n_k} y = 0$ *for all* $y \in Y_0$, *and*

(b′.2) $\lim_{k\to\infty} T^{n_k} \circ S_{n_k} y = y$ *for all* $y \in Y_0$,

then T *is a topologically mixing operator.*

REMARK 3.3. Costakis–Sambarino's criterion might look less restrictive than ours, but it is actually equivalent: Let T satisfy (a′) and (b′) above. We define $S_n := T^{n_k - n} S_{n_k}$ if $n_{k-1} < n \leq n_k$, $k \in \mathbb{N}$, and $n_0 := 0$. We set $m := \sup_k (n_k - n_{k-1})$. Given $\varepsilon > 0$, $x \in X_0$, $y \in Y_0$, we pick $\delta > 0$ such that $\delta(1 + \|T\|^m) < \varepsilon$ and we find $k' \in \mathbb{N}$ satisfying

$$\|T^{n_k} x\| < \delta, \quad \|S_{n_k} y\| < \delta, \quad \|T^{n_k} \circ S_{n_k} y - y\| < \varepsilon, \quad \forall k \geq k'.$$

If $n > n_{k'}$, there are $j, j' \leq m$ and $k > k'$ such that $n = n_k - j = n_{k-1} + j'$, and then

$$\|T^n x\| \leq \|T\|^{j'} \|T^{n_{k-1}} x\| < \varepsilon, \quad \|S_n y\| \leq \|T\|^j \|S_{n_k} y\| < \varepsilon,$$
$$\|T^n \circ S_n y - y\| = \|T^{n_k} \circ S_{n_k} y - y\| < \varepsilon,$$

which implies that T satisfies our Mixing Criterion.

The Mixing Criterion can be adapted to semigroups with index set $[0, \infty)$ as follows.

CRITERION 3.4. *Let* $\{T(t)\}_{t\geq 0}$ *be a strongly continuous linear semigroup on a separable Banach space* X. *If*

(i) *there exists a dense subset* $X_0 \subset X$ *such that* $\lim_{t\to\infty} T(t)x = 0$ *for all* $x \in X_0$ *and*

(ii) *there exist a dense subset* $Y_0 \subset X$ *and a one-parameter family of mappings* $\{S(t)\}_{t\geq 0}$ *on* Y_0 *such that*

(ii.1) $\lim_{t\to\infty} S(t)y = 0$ *for all* $y \in Y_0$, *and*

(ii.2) $\lim_{t\to\infty} T(t)S(t)y = y$ *for all* $y \in Y_0$,

then $\{T(t)\}_{t\geq 0}$ *is a topologically mixing semigroup.*

A sufficient condition for a semigroup to be mixing can be stated in terms of syndetic sequences in $[0,\infty)$, by following the corresponding criterion for single operators of Costakis and Sambarino. As in the previous remark, it easily follows that this condition would be equivalent to the above criterion.

Recently, Grivaux [22] showed that, if $K \in L(X)$ is a compact operator, then the set of vectors whose orbit under $T := I + K$ tends to zero cannot be dense in X. This in particular implies that no compact perturbation of the identity can satisfy the Mixing Criterion. If we consider in Lemma 2.1 an increasing sequence of weights (β_i) tending to infinity, then each operator in the semigroup is a compact perturbation of the identity. In consequence the corresponding semigroup is mixing but it does not satisfy the Mixing Criterion.

The following result relates, in a clear way, the mixing condition for semigroups to the mixing condition for the single operators corresponding to a fixed index of the semigroup.

THEOREM 3.5. *Let* $\{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup of operators defined on a Banach space* X. *The following are equivalent*:

(i) $\{T(t)\}_{t\geq 0}$ *is a topologically mixing semigroup.*

(ii) $T(t)$ *is topologically mixing as a single operator for all* $t > 0$.

(iii) $T(t_0)$ *is a topologically mixing operator for some* t_0.

Proof. (i)⇒(ii): Fix an arbitrary $t_1 > 0$; we prove that $T(t_1)$ is topologically mixing as a single operator. As $\{T(t)\}_{t\geq 0}$ is topologically mixing, given two non-void open sets U and V, there exists $s_0 > 0$ such that $T(t)(U) \cap V \neq \emptyset$ for all $t \geq s_0$. Consider $n_1 \in \mathbb{N}$ such that $n_1 t_1 \geq s_0$. Then $T(t_1)^n U \cap V \neq \emptyset$ for all $n \geq n_1$.

(ii)⇒(iii) is trivial.

(iii)⇒(i): Let $t_0 > 0$ be such that $T(t_0)$ is a mixing operator. Given non-empty open sets $U, V \subset X$, we find a 0-neighbourhood $\overline{W}$, and two non-empty open subsets $\overline{U} \subset U$, $\overline{V} \subset V$, satisfying

$$\overline{U} + \overline{W} \subset U, \qquad \overline{V} + \overline{W} \subset V.$$

By local equicontinuity, there is another 0-neighbourhood $W \subset \overline{W}$ such that $T(s)(W) \subset \overline{W}$ for each $s \leq t_0$. Since $T(t_0)$ is mixing, there is $m \in \mathbb{N}$ large enough so that

$$T(nt_0)(\overline{U}) \cap W \neq \emptyset, \qquad T(nt_0)(W) \cap \overline{V} \neq \emptyset, \qquad \forall n \geq m.$$

Now, given $t > mt_0$ there are $s, s' \leq t_0$ and $n \geq m$ satisfying $t = s + nt_0 = (n+1)t_0 - s'$. On the other hand, we find $u \in \overline{U}$ and $w \in W$ such that $T(nt_0)u \in W$ and $T((n+1)t_0)w \in \overline{V}$. Setting $\overline{w} := T(s')w \in \overline{W}$, we thus infer that

$$T(t)(u + \overline{w}) = T(s)(T(nt_0)u) + T((n+1)t_0 - s')(T(s')w)$$

belongs to $T(s)(W) + T((n+1)t_0)w \subset \overline{W} + \overline{V} \subset V$. From this we conclude that the semigroup is mixing. ∎

By the above theorem, if $\{T(t)\}_{t\geq 0}$ is a topologically mixing semigroup, then all $T(t)$ are topologically mixing operators. However, we do not know if the hypercyclic vectors are the same.

4. Topologically mixing translations. Let I be either $[0, \infty)$ or $\mathbb{R}$. By an *admissible weight function* on I we mean a measurable function $\varrho : I \to \mathbb{R}$ satisfying the conditions:

(1) $\varrho(x) > 0$ for all $x \in I$;
(2) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\varrho(x) \leq Me^{\omega t}\varrho(t+x)$ for all $x \in I$ and all $t > 0$.

We consider the following function spaces:

$$L^p_\varrho(I) = \{u : I \to \mathbb{C} \,:\, u \text{ measurable}, \int_I |u(\tau)|^p \varrho(\tau)\, d\tau < \infty\}$$

with

$$\|u\|_p = \Big(\int_I |u(\tau)|^p \varrho(\tau)\, d\tau\Big)^{1/p}, \quad p \geq 1,$$

and

$$C_{0,\varrho}(I) = \{u : I \to \mathbb{C} \,:\, u \text{ continuous}, \lim_{\tau\to\infty} \varrho(\tau)u(\tau) = 0\}$$

with

$$\|u\|_\infty = \sup_{\tau\in I} |u(\tau)|\varrho(\tau).$$

Similar spaces are considered for real-valued functions.

LEMMA 4.1 ([16, Lemma 4.2]). *Let I be either $[0,\infty)$ or $\mathbb{R}$ and let ϱ be an admissible weight function on I. For each $l > 0$ there are constants $0 < m \leq M$ (depending on ϱ and l only) such that for each $\sigma \in I$ and each $\tau \in [\sigma, \sigma + l]$,*

$$m\varrho(\sigma) \leq \varrho(\tau) \leq M\varrho(\sigma + l).$$

The *translation semigroup* $\{T(t)\}_{t\geq 0}$ with parameter $t \in [0, \infty)$ is defined as

$$[T(t)u](\tau) := u(\tau + t) \quad \text{for } u \in C_{0,\varrho}(I) \text{ or } L^p_\varrho(I).$$

Lemma 4.2 ([16, Theorems 4.7 and 4.8]). *Let X be one of the spaces $L^p_\varrho(I)$ or $C_{0,\varrho}(I)$ with an admissible weight function ϱ and let $\{T(t)\}_{t\geq 0}$ be the translation semigroup on X.*

(1) *If $I=[0,\infty)$, then $\{T(t)\}_{t\geq 0}$ is hypercyclic if and only if*

$$\liminf_{t\to\infty} \varrho(t)=0.$$

(2) *If $I=\mathbb{R}$, then $\{T(t)\}_{t\geq 0}$ is hypercyclic if and only if for each $\theta\in\mathbb{R}$,*

$$\liminf_{t\to\infty} \varrho(t+\theta)=\liminf_{t\to\infty} \varrho(-t+\theta)=0.$$

We give analogous results for topologically mixing translations.

Theorem 4.3. *Let X be one of the spaces $L^p_\varrho(I)$ or $C_{0,\varrho}(I)$ with an admissible weight function ϱ and let $\{T(t)\}_{t\geq 0}$ be the translation semigroup on X.*

(1) *If $I=[0,\infty)$, then $\{T(t)\}_{t\geq 0}$ is topologically mixing if and only if*

$$\lim_{t\to\infty} \varrho(t)=0.$$

(2) *If $I=\mathbb{R}$, then $\{T(t)\}_{t\geq 0}$ is topologically mixing if and only if*

$$\lim_{t\to\infty} \varrho(t)=\lim_{t\to\infty} \varrho(-t)=0.$$

Proof. We will prove the case $X=L^p_\varrho(I)$. The case $X=C_{0,\varrho}(I)$ is easier.

(1) First assume that $\{T(t)\}_{t\geq 0}$ is topologically mixing. Consider $U_n:=B(0,1/n)$, the ball of centre 0 and radius $1/n$, and $V_n:=B(u,1/n)$, where u is a fixed non-zero function with support in $[0,l]$ for some $l>0$. Since $\{T(t)\}_{t\geq 0}$ is topologically mixing, it follows that for each $n>0$, there exists $t(n)$ (defined as before) such that

$$T(t)U_n\cap V_n\neq\emptyset,\qquad t\geq t(n).$$

To derive a contradiction we will suppose that $\limsup_{t\to\infty}\varrho(t)\neq 0$. For every $n>0$, there exists $t_n>0$ such that $\lim_{n\to\infty} t_n=\infty$ and $\varrho(t_n)>\delta$ for some $\delta>0$. Hence there exists $u_n\in U_n$ such that $\|T(t_n)u_n-u\|<1/n$. Let $w_n:=u_{n|[t_n,t_n+l]}$. Then $\|T(t_n)w_n\|\to\|u\|\neq 0$.

On the other hand, using Lemma 4.1 we obtain

$$\begin{aligned}\|T(t_n)w_n\|^p&=\int_0^l |w_n(\tau+t_n)|^p\varrho(\tau)\,d\tau=\int_{t_n}^{t_n+l}|w_n(\tau)|^p\varrho(\tau-t_n)\,d\tau\\&\leq M\varrho(l)\int_{t_n}^{t_n+l}|w_n(\tau)|^p\,d\tau.\end{aligned}$$

Moreover,

$$\|u_n\|^p = \int_0^\infty |u_n(\tau)|^p \varrho(\tau)\,d\tau \geq \int_{t_n}^{t_n+l} |w_n(\tau)|^p \varrho(\tau)\,d\tau$$
$$\geq m\varrho(t_n) \int_{t_n}^{t_n+l} |w_n(\tau)|^p\,d\tau$$

for some $m, M > 0$. Hence by the two inequalities obtained before we have

$$\|T(t_n)w_n\|^p \leq \frac{M\varrho(l)}{m\varrho(t_n)}\,\|u_n\|^p \leq \frac{M\varrho(l)}{m\delta n^p} \leq C\,\frac{1}{n^p}.$$

Therefore, $\|T(t_n)w_n\|^p \to 0$ as $n \to 0$, which yields a contradiction. We then get $\lim_{t\to\infty} \varrho(t) = 0$.

Conversely, suppose that $\lim_{t\to\infty} \varrho(t) = 0$. It is enough to prove that $\{T(t)\}_{t\geq 0}$ satisfies Criterion 3.4. Consider the set X_0 of all functions defined on $[0,\infty)$ with compact support. It is clear that X_0 is dense in X. In order to define $S(t)$, let $u \in X_0$ be a function with compact support on $[0,l]$ for some $l > 0$. Define

$$v_t(\tau) = \begin{cases} 0 & \text{if } \tau < t, \\ u(\tau - t) & \text{if } \tau \geq t. \end{cases}$$

Then

$$\|v_t\|^p = \int_t^\infty |v_t(\tau)|^p \varrho(\tau)\,d\tau = \int_t^{t+l} |u(\tau - t)|^p \varrho(\tau)\,d\tau$$
$$\leq M\varrho(t+l) \int_t^{t+l} |u(\tau - t)|^p\,d\tau = M\varrho(t+l) \int_0^l |u(\tau)|^p\,d\tau.$$

Moreover

$$\|u\|^p = \int_0^l |u(\tau)|^p \varrho(\tau)\,d\tau \geq m\varrho(0) \int_0^l |u(\tau)|^p\,d\tau.$$

Hence

$$\|v_t\|^p \leq \frac{M\varrho(t+l)}{m\varrho(0)}\,\|u\|^p.$$

Since $\lim_{t\to\infty} \varrho(t) = 0$, we see that $\|v_t\| \to 0$ as $t \to \infty$.

Define $S(t)u := v_t$, which converges to zero as $t \to \infty$ for all u with compact support. Moreover $T(t)S(t)u = u$ for all $u \in X_0$. The proof in the case $I = [0,\infty)$ is complete.

(2) Suppose that $\{T(t)\}_{t\geq 0}$ is a topologically mixing semigroup. If $\lim_{t\to\infty} \varrho(t) \neq 0$ then, by following the argument of (1), we reach a contradiction. If $\lim_{t\to\infty} \varrho(-t) \neq 0$ then, proceeding in a similar manner and

keeping the same notation for u, U_n and V_n, we find $t_n > 0$ such that $\lim_n t_n = \infty$, $\varrho(-t_n) > \delta$ for some $\delta > 0$, and

$$T(t_n)V_n \cap U_n \neq \emptyset \quad \forall n \in \mathbb{N}.$$

Hence there exists $v_n \in V_n$ such that $\|v_n - u\| < 1/n$ and $\|T(t_n)v_n\| < 1/n$. Let $w_n := v_{n|[0,l]}$. Then $\|w_n - u\| \to 0$ and $\|T(t_n)w_n\| \to 0$ as $n \to \infty$, since $\|w_n - u\| \le \|v_n - u\|$ and $\|T(t_n)w_n\| \le \|T(t_n)v_n\|$.

On the other hand, using Lemma 4.1, we obtain

$$\begin{aligned}\frac{1}{n^p} \ge \|T(t_n)w_n\|^p &= \int_{-t_n}^{l-t_n} |v_n(\tau + t_n)|^p \varrho(\tau)\, d\tau \\ &= \int_0^l |v_n(\tau)|^p \varrho(\tau - t_n)\, d\tau \ge m\varrho(-t_n) \int_0^l |v_n(\tau)|^p\, d\tau\end{aligned}$$

Moreover,

$$\|w_n\|^p = \int_0^l |v_n(\tau)|^p \varrho(\tau)\, d\tau \le M\varrho(l) \int_0^l |v_n(\tau)|^p\, d\tau$$

for some $m, M > 0$. Using these inequalities we get

$$\frac{1}{n^p} \ge \|T(t_n)w_n\|^p \ge \frac{m\varrho(-t_n)}{M\varrho(l)} \|w_n\|^p > \frac{ma}{M\varrho(l)} \|u\|^p \neq 0 \quad \forall n \in \mathbb{N},$$

yielding a contradiction.

Conversely, suppose that $\lim_{t\to\infty} \varrho(t) = \lim_{t\to\infty} \varrho(-t) = 0$. It is enough to prove that $\{T(t)\}_{t\ge0}$ satisfies Criterion 3.4. Consider the set X_0 of all functions defined on $\mathbb{R}$ with compact support, which is a dense subspace of X. Using the same technique as in (1) we find that $T(t)u$ converges to zero as $t \to \infty$ for all $u \in X_0$. In order to define $S(t)$, let $u \in X_0$ be a function with compact support on $[-l, l]$ for some $l > 0$. If we define

$$(S(t)u)(\tau) = \begin{cases} 0 & \text{if } \tau - t < -l, \\ u(\tau - t) & \text{if } \tau - t \ge -l, \end{cases}$$

we easily prove that $\lim_{t\to\infty} S(t)u = 0$. Moreover $T(t)S(t)u = u$, and the proof is complete. ∎

A bounded linear operator S on a Banach space X is a *generalized backward shift* if the kernel of S, denoted by $\ker S$, is a one-dimensional subspace and $\bigcup\{\ker S^n : n = 0, 1, 2, \dots\}$ is dense in X [20]. Inspired by the arguments of [26, Section 5], we can establish the following two results.

THEOREM 4.4. *If S is a generalized backward shift on a separable Banach space X, then $\{e^{tS}\}_{t\ge0}$ is a topologically mixing uniformly continuous semigroup.*

Proof. Since S is a generalized backward shift, there exists a sequence $\{e_j\}_{j=1}^\infty$ of non-zero vectors such that $Se_1 = 0$, $Se_{j+1} = e_j$ for $j \in \mathbb{N}$ and $\overline{\operatorname{span}\{e_j : j \in \mathbb{N}\}} = X$ (see [20]).

Define $w := (w_j)_{j=1}^\infty$, where $w_j := \|e_j\|$, and

$$\Psi : \ell^1(w) \to X, \qquad (x_1, x_2, \dots) \mapsto \Psi(x_1, x_2, \dots) = \sum_{j=1}^\infty x_j e_j.$$

Since $w_j = \|e_j\| = \|Se_{j+1}\| \le \|S\|\,\|e_{j+1}\| = \|S\| w_{j+1}$, we have

$$w_j / w_{j+1} \le \|S\| \quad \text{for every } j \in \mathbb{N}.$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\ell^1(w) \arrow[r, "e^{tB}"] \arrow[d, "\Psi"'] & \ell^1(w) \arrow[d, "\Psi"] \\
X \arrow[r, "e^{tS}"] & X
\end{tikzcd}$$

where B is the backward shift defined on $\ell^1(w)$.

On the other hand, e^{tB} is topologically mixing (see the proof of [16, Theorem 5.2]). By Lemma 2.2(2) we conclude that $\{e^{tS}\}_{t\ge0}$ is topologically mixing. ∎

In particular, if S is a generalized backward shift on a Banach space, then e^S is a topologically mixing operator. This is related to the following question of [14]: Characterize those unilateral backward shifts T for which $I + T$ is topologically mixing.

THEOREM 4.5. *If S is a generalized backward shift on a separable Banach space, then $I + S$ is a topologically mixing operator.*

Proof. Using the same argument of the last theorem, we find that the following diagrams commute:

$$\begin{tikzcd}
\ell^1 \arrow[r, "I+T"] \arrow[d, "\Phi"'] & \ell^1 \arrow[d, "\Phi"] \\
\ell^1(w) \arrow[r, "I+B"] \arrow[d, "\Psi"'] & \ell^1(w) \arrow[d, "\Psi"] \\
X \arrow[r, "I+S"] & X
\end{tikzcd}$$

where the weights $\omega = (\omega_n)$ and the mapping Ψ are given in the proof of the previous theorem, T is a weighted backward shift operator defined on ℓ^1 by

$$T(x_1, x_2, \dots) = \left(\frac{\omega_1}{\omega_2} x_2, \frac{\omega_2}{\omega_3} x_3, \dots \right)$$

and

$$\Phi : \ell^1 \to \ell^1(\omega), \qquad (x_1, x_2, \dots) \mapsto \Phi(x_1, x_2, \dots) = \left(\frac{x_1}{\omega_1}, \frac{x_2}{\omega_2}, \cdots\right).$$

Hence, $(I + B)\Phi = \Phi(I + T)$ and $(I + S)\Psi = \Psi(I + B)$. By [22, Lemma 2.3] the operator $I + T$ is topologically mixing, so Lemma 2.2(1) shows that $I + B$ is topologically mixing. Using Lemma 2.2(1) again, we get the desired result. ∎

5. Chaotic semigroups and infinitesimal generators. In [15] deLaubenfels and Emamirad made the following

CONJECTURE 5.1. *Suppose $\mathcal{A}$ is a commutative algebra of operators that contains a chaotic operator. Let $T \in \mathcal{A}$ which is not a multiple of the identity. Then there exists a constant $c > 0$ such that cT is chaotic.*

Our purpose in this section is to provide an example that disproves the above conjecture.

REMARK 5.2. There is a chaotic operator R such that no multiple of the semigroup generated by R is hypercyclic.

Let $T = I + \varepsilon B$ on $\ell^1(\mathbb{N})$, where B is the backward shift on $\ell^1(\mathbb{N})$, and let $\varepsilon > 0$. By [20, Sect. 6], T is chaotic. The operator T^2 inherits the periodic points of T, and it is also hypercyclic [2, Theorem 1], therefore chaotic. Hence by [8, Proposition 2.14 and Theorem 2.3], $T^2 \oplus T^2$ is hypercyclic and equals $T^2 \oplus (-T)^2 = (T \oplus -T)^2$, so $T \oplus -T$ is hypercyclic. On the other hand, the set

$$\{x \oplus y \mid x, y \text{ are periodic points for } T\}$$

is dense in $X \oplus X$ and it consists of periodic points for $T \oplus -T$, hence $R := T \oplus -T$ is also chaotic.

If ε is sufficiently small then, for each multiple of e^R, at least one of the two components of its spectrum does not intersect the unit circle, therefore it cannot be hypercyclic. This gives a negative answer to Conjecture 5.1.

REMARK 5.3. There exists a chaotic uniformly continuous semigroup such that no multiple of its generator is hypercyclic.

Let $T(t) = e^{tS}$ on $\ell^1(\mathbb{N})$, where $S := 2B$ and B is the backward shift operator on $\ell^1(\mathbb{N})$. Then $\sigma(S) = \overline{B}(0, 2)$ and S is chaotic by [20, Sect. 6]. Moreover, $T(t)$ (as a single operator!) is chaotic by [16, Theorem 5.3].

This implies that $T(t) \oplus e^{5ti}T(t)$ is chaotic, and the spectrum of its infinitesimal generator $S \oplus (5i + S)$ is $\overline{B}(0, 2) \cup \overline{B}(5i, 2)$.

Thus, for any multiple of the infinitesimal generator, there is a connected component of its spectrum which does not intersect the unit circle and, hence, it cannot be hypercyclic.

6. Final remarks. A Banach space X is called *hereditarily indecomposable* if whenever Y and Z are closed infinite-dimensional subspaces of X satisfying $Y \cap Z = \{0\}$, then $Y + Z$ is non-closed.

Gowers and Maurey provided the first example of a hereditarily indecomposable Banach space [21]. In particular, they proved that every bounded linear operator T on X can be written as $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is *strictly singular* (i.e. it has no bounded inverse on any infinite-dimensional subspace).

Notice that the dual of a hereditarily indecomposable space may be far from being hereditarily indecomposable [4]. However, Ferenczi proved that the space defined by Gowers and Maurey in [21], X_{GM}, which is hereditarily indecomposable, has the property that X_{GM}^* is also hereditarily indecomposable [19, Corollary 22].

From the proof of the main theorem of [9], [6, Theorem 3.3], and [25, Theorem 2.8] we can deduce the following result:

THEOREM 6.1. *Let X be a hereditarily indecomposable Banach space such that X^* is also hereditarily indecomposable. Then all hypercyclic operators on X are of the form*

$$T = \lambda I + S$$

with S strictly singular, $|\lambda| = 1$ and $\sigma(S) = 0$. Moreover, all hypercyclic semigroups on X are of the form

$$e^{(\theta i I + S)t},$$

where S is strictly singular and $\sigma(S) = 0$ and θ is a real number.

On the one hand, if S is a generalized backward shift in a Banach space, then $I + S$ and e^S are hypercyclic operators by [26, Theorem 5.2] and Theorem 4.4, respectively.

On the other hand, if we consider the operator $S = 2iI + B$ on $\ell^1(\mathbb{N})$, where B is the backward shift, then $\sigma(S) \subset B(2i, 1)$ and

(1) $I + S$ cannot be hypercyclic.
(2) e^S is hypercyclic.

QUESTION. Let S be a continuous linear operator on a Banach space X, whose spectrum is connected and contains zero. If $I + S$ is hypercyclic, is e^S necessarily hypercyclic? Is the converse true?

References

[1] S. I. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal. 148 (1997), 384–390.

[2] —, *Hypercyclic and cyclic vectors*, ibid. 128 (1995), 374–383.

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck, *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Math. 1184, Springer, Berlin, 1986.

[4] S. A. Argyros and V. Felouzis, *Interpolating hereditarily indecomposable Banach spaces*, J. Amer. Math. Soc. 13 (2000), 243–294.

[5] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly 99 (1992), 332–334.

[6] T. Bermúdez, A. Bonilla, and A. Martinón, *On the existence of chaotic and hypercyclic semigroups in Banach spaces*, Proc. Amer. Math. Soc. 131 (2003), 2435–2441.

[7] L. Bernal-González, *On hypercyclic operators on Banach spaces*, ibid. 127 (1999), 1003–1010.

[8] J. P. Bès and A. Peris, *Hereditarily hypercyclic operators*, J. Funct. Anal. 167 (1999), 94–112.

[9] J. Bonet, F. Martínez-Giménez, and A. Peris, *A Banach space which admits no chaotic operator*, Bull. London Math. Soc. 33 (2001), 196–198.

[10] —, —, —, *Linear chaos on Fréchet spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (2003), 1649–1655.

[11] J. Bonet and A. Peris, *Hypercyclic operators on non-normable Fréchet spaces*, J. Funct. Anal. 159 (1998), 587–595.

[12] J. A. Conejero, *Operadores y semigrupos de operadores en espacios de Fréchet y espacios localmente convexos*, Ph.D. thesis, Univ. Politècnica de València, 2004.

[13] G. Costakis and A. Peris, *Hypercyclic semigroups and somewhere dense orbits*, C. R. Acad. Sci. Paris Sér. I Math. 335 (2002), 895–898.

[14] G. Costakis and M. Sambarino, *Topologically mixing hypercyclic operators*, Proc. Amer. Math. Soc. 132 (2004), 385–389.

[15] R. deLaubenfels and H. Emamirad, *Chaos for functions of discrete and continuous weighted shift operators*, Ergodic Theory Dynam. Systems 21 (2001), 1411–1427.

[16] W. Desch, W. Schappacher, and G. F. Webb, *Hyperbolic and chaotic semigroups of linear operators*, ibid. 17 (1997), 1–27.

[17] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, Reading, MA, 1989.

[18] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer, New York, 2000.

[19] V. Ferenczi, *Quotient hereditarily indecomposable Banach spaces*, Canad. J. Math. 51 (1999), 566–584.

[20] G. Godefroy and J. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. 98 (1991), 229–269.

[21] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.

[22] S. Grivaux, *Hypercyclic operators, mixing operators and the bounded steps problem*, J. Operator Theory, to appear.

[23] K. G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), 345–381.

[24] —, *Recent developments in hypercyclicity*, Rev. R. Acad. Cien. Ser. A Mat. 97 (2003), 273–286.

[25] C. Kitai, *Invariant closed sets for linear operators*, Ph.D. thesis, Univ. of Toronto, 1982.

[26] F. Martínez-Giménez and A. Peris, *Chaos for backward shift operators*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 12 (2002), 1703–1715.

[27] R. I. Ovsepian and A. Pełczyński, *On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in* l^2, Studia Math. 54 (1975), 149–159.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, New York, 1992.

[29] V. Protopopescu and Y. Azmy, *Topological chaos for a class of linear models*, Math. Models Methods Appl. Sci. 2 (1992), 79–90.

[30] S. Rolewicz, *On orbits of elements*, Studia Math. 32 (1969), 17–22.

[31] K. Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

Teresa Bermúdez and Antonio Bonilla
Departamento de Análisis Matemático
Universidad de La Laguna
E-38271 La Laguna (Tenerife), Spain
E-mail: tbermude@ull.es
abonilla@ull.es

José A. Conejero
Facultat d'Informàtica
Departament de Matemàtica Aplicada
and IMPA
Universitat Politècnica de València
E-46022 València, Spain
E-mail: aconejero@mat.upv.es

Alfredo Peris
E.T.S. Arquitectura
Departament de Matemàtica Aplicada
and IMPA
Universitat Politècnica de València
E-46022 València, Spain
E-mail: aperis@mat.upv.es

Received June 29, 2004
Revised version January 13, 2005 (5444)