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Polynomial functions on the classical projective spaces

by

Yu. I. Lyubich and O. A. Shatalova (Haifa)

Abstract. The polynomial functions on a projective space over a field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ come from the corresponding sphere via the Hopf fibration. The main theorem states that every polynomial function $\phi(x)$ of degree d is a linear combination of "elementary" functions $|\langle x, \cdot\rangle|^d$.

1. Spaces and operators. The classical projective spaces are $\mathbb{K}\mathrm{P}^{m-1}$ where $\mathbb{K}$ is one of three fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $m \geq 2$. The quaternion field $\mathbb{H}$ is noncommutative in contrast to $\mathbb{R}$ and $\mathbb{C}$. However, $\mathbb{R} \subset \mathbb{H}$ and each real number commutes with all quaternions. In general, $\mathbb{K}$ is an associative unital algebra over $\mathbb{R}$ of dimension $\delta = \delta(\mathbb{K}) = 1, 2, 4$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. The standard conjugation $\alpha \mapsto \overline{\alpha}$ is an involutive automorphism of $\mathbb{K}$ (or anti-automorphism if $\mathbb{K} = \mathbb{H}$ since $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$ in this case). For any $\alpha \in \mathbb{K}$ the real number $\operatorname{Re}\alpha = \frac{1}{2}(\alpha + \overline{\alpha})$ is the *real part* of α. (If $\mathbb{K} = \mathbb{R}$ we set $\overline{\alpha} = \alpha$.)

The space $\mathbb{K}\mathrm{P}^{m-1}$ can be constructed starting with $\mathbb{K}^m$ that consists of all m-tuples $x = (\xi_i)$, $\xi_i \in \mathbb{K}$. This is an m-dimensional right (for definiteness) linear space over $\mathbb{K}$; the addition in $\mathbb{K}^m$ is standard, the multiplication by a scalar $\alpha \in \mathbb{K}$ is $x\alpha = (\xi_i\alpha)$. Moreover, $\mathbb{K}^m$ is a Euclidean space provided with the inner product

$$\langle x, y\rangle = \sum \overline{\xi}_i \eta_i \quad (x = (\xi_i),\ y = (\eta_i)). \tag{1}$$

The properties of the latter are standard but with the fixed order of factors in the relations

$$\langle x\alpha, y\rangle = \overline{\alpha}\langle x, y\rangle, \quad \langle x, y\alpha\rangle = \langle x, y\rangle\alpha$$

if $\mathbb{K} = \mathbb{H}$. The corresponding Euclidean norm on $\mathbb{K}^m$ is

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum |\xi_i|^2}. \tag{2}$$

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We denote the unit sphere $\{x : \|x\| = 1\}$ by $\mathrm{S}(m, \mathbb{K})$. The multiplicative group

$$\mathrm{U}(\mathbb{K}) = \{\alpha \in \mathbb{K} : |\alpha| = 1\} \tag{3}$$

acts on $\mathrm{S}(m, \mathbb{K})$ by the rule $x \mapsto x\alpha$. The corresponding quotient space (the space of orbits) is $\mathbb{K}\mathrm{P}^{m-1}$ by definition. Let us emphasize that the mapping $x \mapsto x\alpha$ is not linear over $\mathbb{H}$ except for $\alpha \in \mathbb{R}$.

For any $m \geq 1$ a linear operator A in $\mathbb{K}^m$ can be described in terms of an $m \times m$ matrix (α_{ji}) over $\mathbb{K}$, so that if $x = (\xi_i)$ then $Ax = (\sum_i \alpha_{ji}\xi_i)$. As usual, a linear operator A is called *unitary* if $\langle Ax, Ax\rangle = \langle x, y\rangle$, or, equivalently, if $\|Ax\| = \|x\|$, which means that A is an isometry. These operators constitute the unitary group $\mathrm{U}_m(\mathbb{K})$. (This is the orthogonal group $\mathrm{O}(m)$ if $\mathbb{K} = \mathbb{R}$ and the symplectic group $\mathrm{Sp}(m)$ if $\mathbb{K} = \mathbb{H}$.) The group $\mathrm{U}_m(\mathbb{K})$ acts transitively on the unit sphere. Indeed, let $x, z \in \mathrm{S}(m, \mathbb{K})$. Then there are two orthonormal bases in $\mathbb{K}^m$, say $\{x_i\}_{i=1}^m$ and $\{z_i\}_{i=1}^m$, such that $x_1 = x$ and $z_1 = z$. The conditions $Ax_i = z_i$, $1 \leq i \leq m$, determine the required unitary operator.

More generally, one can consider two systems $(x_i)_{i=1}^l$ and $(z_i)_{i=1}^l$ of vectors with the same Gram matrix and prove that they are unitarily equivalent, i.e. there exists a unitary operator A such that $Ax_i = z_i$, $1 \leq i \leq l$. Let us briefly show this for the case when the system $(x_i)_{i=1}^l$ is linearly independent. Then the conditions $Tx_i = z_i$, $1 \leq i \leq l$, determine a linear mapping $T : \mathrm{Span}\,(x_i) \to \mathrm{Span}\,(z_i)$. This T is an isometry since $\langle x_i, x_k\rangle = \langle z_i, z_k\rangle$, $1 \leq i, k \leq l$. It remains to set $A = T \oplus S$ where $S : \mathrm{Span}\,(x_i)^\perp \to \mathrm{Span}\,(z_i)^\perp$ is an arbitrary isometry. The latter does exist since the corresponding subspaces have the same dimension $m - l$.

Thus, the Euclidean geometry of the space $\mathbb{K}^m$ can be developed irrespective of $\mathbb{K}$. However, some special concepts and constructions relate to $\mathbb{K} = \mathbb{R}$ initially. In order to introduce them for any $\mathbb{K}$ we have to view $\mathbb{K}^m$ as a real linear space $\mathbb{R}^{\delta m}$, $\delta = \delta(\mathbb{K})$. Under this realification the inner product has to be replaced by

$$\langle x, y\rangle_{\mathbb{R}} = \mathrm{Re}\langle x, y\rangle, \tag{4}$$

but the norm remains the same since $\langle x, x\rangle$ is real. Accordingly, $\mathrm{S}(\delta m, \mathbb{R}) = \mathrm{S}(m, \mathbb{K})$, so $\mathrm{S}(m, \mathbb{K})$ is the standard real $(\delta m - 1)$-dimensional sphere.

2. Functions on $\mathbb{K}\mathrm{P}^{m-1}$. The functions we will focus on originate from the polynomials in the following sense.

Definition 1. A mapping $\psi : \mathbb{K}^m \to \mathbb{C}$ is called a *polynomial of degree* d if such is its realification $\mathbb{R}^{\delta m} \to \mathbb{C}$.

The set of all polynomials on $\mathbb{K}^m$ is an associative commutative unital algebra over $\mathbb{C}$. Moreover, if $\psi(x)$ is a polynomial then so is $\overline{\psi(x)}$. The simplest polynomials are constants, the coordinates ξ_i, their conjugates, etc. As usual, a polynomial ψ is called a *form* if it is homogeneous of some degree d, i.e.

$$\psi(x\lambda) = \lambda^d \psi(x), \quad \lambda \in \mathbb{R}. \tag{5}$$

Every (nonhomogeneous, in general) polynomial ψ of degree d is the sum of uniquely determined forms ψ_k of degrees $k = 0, 1, \ldots, d$, which are the *homogeneous components* of ψ. Hence,

$$\psi(x\lambda) = \sum_{k=0}^{d} \lambda^k \psi_k(x), \quad \lambda \in \mathbb{R}.$$

Therefore, ψ is a form of degree d as long as $\psi(x\lambda) = \lambda^d \psi(x)$ for $\lambda > 0$.

The forms of degree $d = 0$ are just constants. The forms of degree 1 are the sums of arbitrary linear functionals and their conjugates. In particular, $\langle x, y\rangle$ is a form of degree 1 with respect to x and y separately. Hence, the function

$$\psi_{2;y}(x) = |\langle x, y\rangle|^2 = \langle x, y\rangle\langle y, x\rangle$$

is a form of degree 2. Moreover, for every even d the function

$$\psi_{d;y}(x) = |\langle x, y\rangle|^d = (\psi_{2;y}(x))^{d/2} \tag{6}$$

is a form of degree d. We call it an *elementary form of degree d*. These forms play a central role in what follows.

Now let F be an arbitrary (nonempty) set. Any mapping $\phi : \mathbb{K}\mathrm{P}^{m-1} \to F$ can be lifted to $\pi^*\phi : \mathrm{S}(m, \mathbb{K}) \to F$ where $\pi^*\phi = \phi \circ \pi$ and $\pi : \mathrm{S}(m, \mathbb{K}) \to \mathbb{K}\mathrm{P}^{m-1}$ is the Hopf fibration (the natural factorization; cf. [1]). The mapping $\pi^*\phi : \mathrm{S}(m, \mathbb{K}) \to F$ is $\mathrm{U}(\mathbb{K})$-invariant in the sense that

$$(\pi^*\phi)(x\alpha) = (\pi^*\phi)(x), \quad |\alpha| = 1, \tag{7}$$

since $x\alpha$ and x belong to the same $\mathrm{U}(\mathbb{K})$-orbit.

Conversely, if $\theta : \mathrm{S}(m, \mathbb{K}) \to F$ is $\mathrm{U}(\mathbb{K})$-invariant then there exists a unique $\phi : \mathbb{K}\mathrm{P}^{m-1} \to F$ such that $\pi^*\phi = \theta$. This relation can be extended to the mappings $\psi : \mathbb{K}^m \to F$ by restriction $\widehat{\psi} = \psi|\mathrm{S}(m, \mathbb{K})$. If $\widehat{\psi} = \pi^*\phi$ then we say that ϕ is *generated* by ψ. The commutative diagram

$$\begin{tikzcd}
\mathbb{K}^m \arrow[r, "\psi"] & F \\
\mathrm{S}(m, \mathbb{K}) \arrow[u, "i"] \arrow[ur, "\widehat{\psi}"] \arrow[r, "\pi"'] & \mathbb{K}\mathrm{P}^{m-1} \arrow[u, "\phi"']
\end{tikzcd} \tag{8}$$

with the canonical embedding i summarizes the aforesaid. In this context the relation (7) turns into

$$\widehat{\psi}(x\alpha) = \widehat{\psi}(x), \quad |\alpha| = 1. \tag{9}$$

In particular, this general scheme is applicable to a polynomial ψ.

Definition 2. A mapping $\phi : \mathbb{KP}^{m-1} \to \mathbb{C}$ is called a *polynomial function* on $\mathbb{KP}^{m-1}$ if it is generated by a polynomial ψ.

This ψ is determined up to a summand $(1 - \langle x, x\rangle)\omega(x)$ where ω is an arbitrary polynomial. Indeed, we have

Lemma 1. *Any polynomial χ vanishing on the unit sphere is divisible by $1 - \langle x, x\rangle$.*

Proof. First, note that $1 - \langle x, x\rangle$ is irreducible, otherwise, it would be divisible by a polynomial f of degree 1. Then the affine hyperplane $\{x : f(x) = 0\}$ would lie on the unit sphere, which is impossible.

Now consider χ such that $\chi|\mathrm{S}(m, \mathbb{K}) = 0$ and assume for a while that $\chi(-x) = \chi(x)$. Then

$$\chi = \sum_k \chi_{2k}$$

where χ_{2k} is a form of degree $2k$, $0 \le k \le d/2$, $d = \deg \chi$. Hence,

$$\chi(x) = \sum_{k=0}^{d/2} \langle x, x\rangle^k \chi_{2k}(\widehat{x})$$

where $\widehat{x} = x/\|x\|$. By assumption,

$$\sum_{k=0}^{d/2} \chi_{2k}(\widehat{x}) = 0.$$

Therefore,

$$\chi(x) = \sum_{k=0}^{d/2} (1 - \langle x, x\rangle^k)\chi_{2k}(\widehat{x}) = (1 - \langle x, x\rangle) \sum_{k=0}^{d/2} \chi_{2k}(\widehat{x}) \sum_{j=0}^{k-1} \langle x, x\rangle^j$$

whence,

$$\langle x, x\rangle^{d/2}\chi(x) = (1 - \langle x, x\rangle) \sum_{k=0}^{d/2} \chi_{2k}(x) \sum_{j=0}^{k-1} \langle x, x\rangle^{d/2-k+j}.$$

The polynomial $\langle x, x\rangle^{d/2}$ is not divisible by $1 - \langle x, x\rangle$, hence $\chi(x)$ is divisible as required.

In the general case we can pass to $\chi(x)\chi(-x)$ and then conclude that the irreducible factor $1 - \langle x, x\rangle$ is contained in $\chi(x)$ or in $\chi(-x)$. However, this factor is even, hence it is contained in $\chi(x)$ in any case. ∎

If ϕ is a polynomial function on $\mathbb{KP}^{m-1}$ then the minimal degree of polynomials generating ϕ is called the *degree* of ϕ. Under this definition we have

LEMMA 2. *The degree of the elementary polynomial function $\phi_{d;y}$ generated by $\psi_{d;y}$ is equal to d.*

Proof. Obviously, $\deg \phi_{d;y} \le d$. Suppose that $\deg \phi_{d;y} < d$, so $\phi_{d;y}$ is generated by a polynomial ε with $\deg \varepsilon < d$. By Lemma 1,

$$|\langle x, y\rangle|^d = (1 - \langle x, x\rangle)\omega(x) + \varepsilon(x) \tag{10}$$

where ω is a polynomial. If $\eta(x)$ is the leading homogeneous component of $\omega(x)$ then (10) yields

$$|\langle x, y\rangle|^d = -\langle x, x\rangle\eta(x), \quad \deg \eta = d - 2. \tag{11}$$

This is a contradiction since on the left-hand side of (11) the irreducible factors are only $\langle x, y\rangle$ and $\langle y, x\rangle$ while the right-hand side contains the irreducible factor $\langle x, x\rangle$. ∎

Now we prove the following important

LEMMA 3. *For every polynomial function ϕ of degree d there exists a unique form ψ,* $\deg \psi = d$, *generating ϕ. This form is* $\mathrm{U}(\mathbb{K})$*-invariant*, *i.e.*

$$\psi(x\alpha) = \psi(x) \quad (x \in \mathbb{K}^m,\ |\alpha| = 1). \tag{12}$$

Proof. Let ψ_0 be a generating polynomial for ϕ such that $\deg \psi_0 = \deg \phi = d$. The even polynomial

$$\psi_1(x) = \frac{\psi_0(x) + \psi_0(-x)}{2}$$

also generates ϕ because of (9). Moreover, $\deg \psi_1 \le d$ but, in fact, $\deg \psi_1 = d$ since d is the minimal degree of polynomials generating ϕ. Hence, the number d is even and

$$\psi_1(x) = \sum_{k=0}^{d/2} \varkappa_{2k}(x)$$

where $\varkappa_{2k}$ is a form of degree $2k$. The form

$$\psi(x) = \sum_{k=0}^{d/2} \langle x, x\rangle^{d/2-k} \varkappa_{2k}(x)$$

is as required. Its uniqueness is obvious: two forms of the same degree coinciding on the unit sphere coincide everywhere. Finally, this form is $\mathrm{U}(\mathbb{K})$-invariant since for $|\alpha| = 1$ and $\widehat{x} = x/\|x\|$ we have

$$\psi(x\alpha) = \|x\|^d \widehat{\psi}(\widehat{x}\alpha) = \|x\|^d \widehat{\psi}(x) = \psi(x)$$

by (9) again. ∎

COROLLARY. *The degree of every polynomial function $\phi \ne 0$ is an even number.*

From now on d is a fixed even positive integer.

Lemma 4. *The following properties are equivalent for a complex-valued function ψ on $\mathbb{K}^m$:*

(i) *ψ is a* $\mathrm{U}(\mathbb{K})$*-invariant form of degree d.*
(ii) *ψ is a polynomial such that*

$$\psi(x\lambda) = |\lambda|^d \psi(x) \qquad (x \in \mathbb{K}^m,\ \lambda \in \mathbb{K}). \tag{13}$$

Proof. (i)⇒(ii). For $\alpha = \lambda/|\lambda|$ we have $\psi(x\lambda) = |\lambda|^d\psi(x\alpha) = |\lambda|^d\psi(x)$.
(ii)⇒(i). For $\lambda > 0$ we have $\psi(x\lambda) = \lambda^d\psi(x)$, so the polynomial ψ is a form of degree d. Moreover, $\psi(x\alpha) = \psi(x)$ for $|\alpha| = 1$ by (13) again. ∎

The functions with the property (13) are called *absolutely homogeneous of degree d.*

The $\mathrm{U}(\mathbb{K})$-invariant forms (≡ absolutely homogeneous polynomials) of degree d constitute a complex linear space $\Phi_{\mathbb{K}}(m,d)$. All elementary forms $\psi_{d;y}$ belong to this space. Another example is $\langle x,x\rangle^{d/2}$. However, the corresponding polynomial function is the constant $\mathbf{1}$ irrespective of d.

Note that the space $\Phi_{\mathbb{K}}(m,d)$ is *symmetric* in the sense that

$$\psi \in \Phi_{\mathbb{K}}(m,d) \ \Rightarrow\ \overline{\psi} \in \Phi_{\mathbb{K}}(m,d). \tag{14}$$

Main Theorem. *The subspace* $\mathrm{Span}\{\psi_{d;y} : y \in \mathbb{K}^m\}$ *coincides with the whole space $\Phi_{\mathbb{K}}(m,d)$.*

Combining this result with Lemma 3 we obtain

Corollary. *In the space of all polynomial functions of degree $\le d$ there exists a basis $\{\psi_{d;y_i}\}_{i=1}^N$ where $\{y_i\}_{i=1}^N$ is a system of points on the unit sphere* $\mathrm{S}(m,\mathbb{K})$.

In the real case the Main Theorem is classical (see [6, pp. 29–32] for a proof and survey) but the complex and quaternionic cases are new. Our proof is equally valid for all three fields due to some general methods of functional analysis. In addition, we give a simple proof for the commutative fields $\mathbb{R}$ and $\mathbb{C}$ simultaneously.

3. The reproducing kernel. Let us provide the space $\Phi_{\mathbb{K}}(m,d)$ with the inner product

$$(\psi_1,\psi_2) = \int \overline{\psi}_1\psi_2\, d\sigma \tag{15}$$

where the integral is taken over the sphere $\mathrm{S}(m,\mathbb{K})$ against the normalized Lebesgue measure σ. For any $\psi \in \Phi_{\mathbb{K}}(m,d)$ and for any point $x \in \mathbb{K}^m$ the evaluation mapping $\psi \mapsto \psi(x)$ is a linear functional on $\Phi_{\mathbb{K}}(m,d)$. According to the Riesz Theorem,

$$\psi(x) = \int R(x,y)\psi(y)\, d\sigma(y) \tag{16}$$

where $R(x,\cdot) \in \Phi_{\mathbb{K}}(m,d)$, i.e. $R(x,y)$ is a $\mathrm{U}(\mathbb{K})$-invariant form of degree d with respect to the variable y while x is a parameter. The function $R(x,y)$ of two vector variables x,y is uniquely determined; it is called the *reproducing kernel.*

Now let A be a unitary operator in $\mathbb{K}^m$. Then A is also unitary (orthogonal) in $\mathbb{R}^{\delta m}$ (see (4)). It follows from (16) that

$$(17)\qquad (\widetilde{A}\psi)(x) \equiv \psi(Ax) = \int R(Ax,y)\psi(y)\,d\sigma(y) = \int R(Ax,Ay)(\widetilde{A}\psi)(y)\,d\sigma(y)$$

since the measure σ is orthogonally invariant. When ψ runs over $\Phi_{\mathbb{K}}(m,d)$, the function $\widetilde{A}\psi$ runs over the same space. Comparing (17) to (16) we obtain

$$(18)\qquad R(Ax,Ay) = R(x,y), \qquad A \in \mathrm{U}_m(\mathbb{K}),$$

i.e. the kernel R is unitarily invariant.

Let $x,y \in \mathrm{S}(m,\mathbb{K})$. Since any pair $z,w \in \mathrm{S}(m,\mathbb{K})$ with $\langle z,w\rangle = \langle x,y\rangle$ can be represented as $z = Ax$, $w = Ay$, $A \in \mathrm{U}_m(\mathbb{K})$, we conclude that $R(x,y)$ depends only on $\langle x,y\rangle$. Moreover, $R(x,y\alpha) = R(x,y)$ for all α, $|\alpha| = 1$, hence, $R(x,y)$ depends only on $|\langle x,y\rangle|$ or, equivalently,

$$(19)\qquad R(x,y) = P(|\langle x,y\rangle|^2) \qquad (x,y \in \mathrm{S}(m,\mathbb{K}))$$

where $P = P(s)$, $0 \le s \le 1$. Thus, (16) takes the form

$$(20)\qquad \psi(x) = \int P(|\langle x,y\rangle|^2)\psi(y)\,d\sigma(y) \qquad (x,y \in \mathrm{S}(m,\mathbb{K})).$$

LEMMA 5. *$P(s)$ is a polynomial of degree $\le d$.*

Proof. Let $\{\chi_i\}_{i=1}^m$ be an orthonormal basis in $\Phi_{\mathbb{K}}(m,d)$. Then for any fixed $x \in \mathrm{S}(m,\mathbb{K})$ we have

$$P(|\langle x,y\rangle|^2) = \sum_{i=1}^n \varphi_i(x)\chi_i(y)$$

where

$$\varphi_i(x) = \int P(|\langle x,y\rangle|^2)\overline{\chi_i(y)}\,d\sigma(y) = \overline{\chi_i(x)}$$

by (14) and (16). Therefore,

$$(21)\qquad P(|\langle x,y\rangle|^2) = \sum_{i=1}^n \overline{\chi_i(x)}\chi_i(y).$$

Now we fix $y = y_0 \in \mathrm{S}(m,\mathbb{K})$ and let $x = x(\vartheta) = y_0\cos\vartheta + x_0\sin\vartheta$ where $x_0 \in \mathrm{S}(m,\mathbb{K})$, $\langle x_0,y_0\rangle = 1$. Then (21) yields

$$P(\cos^2\vartheta) = \sum_{i=1}^n \overline{\chi_i(y_0 + x_0\tan\vartheta)}\,\chi_i(y_0)\cos^d\vartheta = \sum_{j=0}^d p_j \sin^{d-j}\vartheta\cos^j\vartheta$$

where p_j are some complex coefficients. Actually, $p_j = 0$ for odd j since $P(\cos^2\vartheta)$ is an even function. Hence,

$$P(\cos^2\vartheta) = \sum_{k=0}^{d/2} p_{2k}\sin^{d-2k}\vartheta\cos^{2k}\vartheta,$$

and, by the substitution $\cos^2\vartheta = s$, we obtain

$$P(s) = \sum_{k=0}^{d/2} p_{2k}(1-s)^{d/2-k}s^k, \quad 0 \le s \le 1. \ \blacksquare$$

Remark. Lemma 5 is closely related to the addition formula for spherical functions on projective spaces (cf. [2], [3]).

4. Proof of the Main Theorem. We start with a formula which shows that the elementary polynomials constitute a Jordan chain for the Laplacian Δ. Let us emphasize that when applying Δ we mean that all functions on $\mathbb{K}^m$ are viewed as depending on the real coordinates in $\mathbb{R}^{\delta m}$.

Lemma 6. *For $x \in \mathrm{S}(m,\mathbb{K})$, $y \in \mathbb{K}^m$, and $\delta = \delta(\mathbb{K})$ we have the formula*

$$\Delta_y(|\langle x,y\rangle|^d) = d(d+\delta-2)|\langle x,y\rangle|^{d-2}. \tag{22}$$

Proof. The linear functional $\xi(y) = \langle x,y\rangle$ is a coordinate of y for an orthonormal basis which includes x. In its turn,

$$\xi(y) = \sum_{l=0}^{\delta-1}\eta_l(y)\varepsilon_l$$

where $\varepsilon_0 = 1$ and the remaining ε_l are the basic imaginary units of $\mathbb{K}$ if $\mathbb{K} \ne \mathbb{R}$. The coefficients $\eta_l(y)$ are the coordinates of y for the corresponding orthonormal basis in $\mathbb{R}^{\delta m}$. We have

$$|\langle x,y\rangle|^d = \Big(\sum_{l=0}^{\delta-1}\eta_l^2\Big)^{d/2},$$

and (22) follows immediately. $\blacksquare$

In order to prove the Main Theorem, it suffices to derive $\theta = 0$ when $\theta \in \Phi_{\mathbb{K}}(m,d)$ satisfies

$$\int\overline{\theta}(x)|\langle x,y\rangle|^d\,d\sigma(x) = 0 \tag{23}$$

for all $y \in \mathbb{K}^m$. To this end we apply the Laplacian to (23) using (22) and then iterate this procedure. This yields

$$\int\overline{\theta}(x)|\langle x,y\rangle|^{2k}\,d\sigma(x) = 0$$

for $0 \le k \le d/2$. By Lemma 5,

$$\int \overline{\theta}(x) P(|\langle x, y\rangle|^2)\, d\sigma(x) = 0.$$

It follows from (20) that $\overline{\theta}(y) = 0$.

5. An application (cf. [5] and the references therein). We consider the problem of existence of isometric embeddings $\ell_2^m \to \ell_p^n$ or, equivalently, of existence of m-dimensional Euclidean subspaces in ℓ_p^n over $\mathbb{K}$. The latter is $\mathbb{K}^n$ endowed with the norm

$$\|x\| = \Big(\sum_{k=1}^{n} |\xi_k|^p\Big)^{1/p}, \qquad p \ge 1, \tag{24}$$

so ℓ_2^m is what we considered before (cf. (2)). Such an isometric embedding with $m \ge 2$ cannot exist except for the case when p is an even integer. This necessary condition is sufficient if n is large enough, $n \ge N_{\mathbb{K}}(m,p)$. An explicit form of the function $N_{\mathbb{K}}(m,p)$ is unknown; however, there are some lower and upper bounds for it. In this theory a crucial role is played by an equivalence between isometric embeddings $f : \ell_2^m \to \ell_p^n$ and cubature formulas

$$\int \psi\, d\sigma = \sum_{k=1}^{n} \psi(x_k)\varrho_k, \quad \psi \in \varPhi_{\mathbb{K}}(m,p), \tag{25}$$

where $\{x_k\}_{k=1}^n \subset \mathrm{S}(m,\mathbb{K})$ and ϱ_k are some positive numbers.

The way from (25) to an isometric embedding f is direct. Indeed, (25) is applicable to every elementary polynomial $\psi_{d;y}$, so that

$$\int |\langle x, y\rangle|^p\, d\sigma(x) = \sum_{k=1}^{n} |\langle x_k, y\rangle|^p \varrho_k, \qquad y \in \mathbb{K}^m. \tag{26}$$

As a function of y the integral in (26) is a unitary invariant form of degree p. All such forms are constant on $\mathrm{S}(m,\mathbb{K})$, so they are all proportional to $\langle y, y\rangle^{p/2}$ on the whole space $\mathbb{K}^m$. Therefore, (26) can be written as the identity

$$\sum_{k=1}^{n} |\langle z_k, y\rangle|^p = \langle y, y\rangle^{p/2}, \qquad y \in \mathbb{K}^m. \tag{27}$$

Geometrically, (27) means that the $\mathbb{K}$-linear mapping

$$f(y) = (\langle z_k, y\rangle)_{k=1}^n \tag{28}$$

is an isometric embedding $\ell_2^m \to \ell_p^n$.

The converse way is short due to the Main Theorem. Indeed, (28) is a general form of a linear mapping $f : \ell_2^m \to \mathbb{K}^n$. If f is isometric then

(27) holds, so we can return to (26). Thus, (25) is valid for all elementary polynomials. By the Main Theorem it is valid for all $\phi \in \Phi_{\mathbb{K}}(m,p)$.

Concluding this section we note that the cubature formula (25) can be transferred to $\mathbb{K}\mathrm{P}^{m-1}$. Indeed, by the substitution $\widehat{\psi} = \pi^*\phi$ we obtain

$$\int \phi \, d\sigma_* = \sum_{k=1}^{n} \phi(\pi x_k)\varrho_k \tag{29}$$

where σ_* is the π-induced measure on $\mathbb{K}\mathrm{P}^{m-1}$. Formula (29) is valid for all polynomial functions of degree $\leq p$ on the projective space $\mathbb{K}\mathrm{P}^{m-1}$.

6. The commutative case (cf. [4]). If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ then there is a monomial basis of the space $\Phi_{\mathbb{K}}(m,d)$, namely,

$$\Big\{\xi_1^{i_1} \cdots \xi_m^{i_m} : \sum_{l=1}^{m} i_l = d\Big\}$$

in the real case and

$$\Big\{\xi_1^{i_1} \cdots \xi_m^{i_m} \overline{\xi}_1^{j_1} \cdots \overline{\xi}_m^{j_m} : \sum_{l=1}^{m} i_l = \sum_{l=1}^{m} j_l = d/2\Big\}$$

in the complex case. Further we use the abridged notation

$$I = [i_l]_{l=1}^m, \qquad |I| = \sum_{l=1}^{m} i_l, \qquad I! = \prod_{l=1}^{m} i_l!$$

and

$$x^I = \prod_{l=1}^{m} \xi_l^{i_l}, \qquad \overline{x}^I = \prod_{l=1}^{m} \overline{\xi}_l^{i_l}.$$

Then

$$|\langle x,y\rangle|^d = \langle x,y\rangle^d = \sum_{I\,:\,|I|=d} \frac{d!}{I!}\, x^I y^I \qquad (\mathbb{K} = \mathbb{R})$$

and

$$|\langle x,y\rangle|^d = \langle x,y\rangle^{d/2}\langle y,x\rangle^{d/2} = \sum_{I,J\,:\,|I|=|J|=d/2} \frac{(d/2)!}{I!J!}\, x^J\overline{x}^I y^I \overline{y}^J \qquad (\mathbb{K} = \mathbb{C}).$$

By substitution in (23) we obtain

$$\sum_{I\,:\,|I|=d} \frac{1}{I!}\, y^I(\theta, x^I) = 0 \quad (\mathbb{K} = \mathbb{R})$$

and

$$\sum_{I,J\,:\,|I|=|J|=d/2} \frac{1}{I!J!}\, y^I\overline{y}^J(\theta, x^J\overline{x}^I) = 0 \quad (\mathbb{K} = \mathbb{C}).$$

Hence, θ is orthogonal to all basis monomials in both cases, therefore, $\theta = 0$.

References

[1] E. L. Grinberg, *Spherical harmonics and integral geometry on projective spaces*, Trans. Amer. Math. Soc. 279 (1983), 187–203.
[2] S. G. Hoggar, *Zonal functions and the symplectic group*, preprint, Mat. Inst. Aarhus Univ., 1977/1978.
[3] T. Koornwinder, *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. 25 (1973), 236–246.
[4] Yu. I. Lyubich, *Almost Euclidean subspaces of real ℓ_p^n with p an even integer*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1850, Springer, 2004, 179–192.
[5] Yu. I. Lyubich and O. A. Shatalova, *Isometric embeddings of finite dimensional ℓ_p-spaces over the quaternions*, St. Petersburg Math. J. 16 (2004), 15–32.
[6] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. 96 (1992), no. 463.

Technion – Israel Institute of Technology
Haifa 32000, Israel
E-mail: lyubich@tx.technion.ac.il
oksanashatalov@msn.com

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