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Completions of normed algebras of differentiable functions

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Abstract. We look at normed spaces of differentiable functions on compact plane sets, including the spaces of infinitely differentiable functions considered by Dales and Davie in [7]. For many compact plane sets the classical definitions give rise to incomplete spaces. We introduce an alternative definition of differentiability which allows us to describe the completions of these spaces. We also consider some associated problems of polynomial and rational approximation.

1. Introduction. In this paper we shall investigate problems concerning the normed spaces of differentiable functions on compact plane sets which were originally studied by Dales, Davie and McClure with particular reference to the Banach algebra case in [7] and [8]. These spaces are the $D(X, M)$ spaces where X is a perfect compact plane set and M is a sequence of positive real numbers. We shall give the definitions of these and some of the related spaces that we wish to study in Section 2.

There are many interesting problems concerning these spaces. One strand of their study concerns problems of approximation of functions by means of polynomials or rational functions. For example, in [8] Dales and McClure proved that, if X is the closed unit disk, then the polynomials are always dense in $D(X, M)$. In the one-dimensional case, some results on polynomial approximation were obtained by O'Farrell in [17] (see also Theorem 4.4.15 of [6]). It was shown under very mild conditions on M that the polynomials are dense in $D(I, M)$, where I is a closed interval. Further results on holomorphic and polynomial approximation, and related problems concerning extensions of functions in these spaces, were obtained in [10]. There are still many fascinating open problems in this area.

Another set of problems concerns the completeness of these spaces. Dales and Davie ([7]) gave some conditions on X which guaranteed the completeness of the $D(X, M)$ spaces. It was also noted in [1] that a union of any finite number of sets for which these spaces are complete gives another such

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set. Some further results on this are given in [13]. We shall discuss this problem further in Section 2. We prove that for any perfect, compact plane set which has infinitely many components all of these spaces are incomplete. We also give an example of an X which is a rectifiable Jordan arc and yet the normed algebra of once continuously differentiable functions on X (with the classical definition) is incomplete.

In the setting of normed or Banach algebras, work on these spaces includes the study of their endomorphisms. In [16], Kamowitz came close to classifying all the endomorphisms of $D(I,M)$, in terms of self-maps of the interval I. The second author and Kamowitz made further progress on the remaining problems, and investigated more general compact plane sets in [9]. We will not investigate these problems in this paper, but it is worth noting that some of the results of [9] depend on the completeness of the $D(X,M)$ spaces. (See also further joint papers of the second author and Kamowitz, and [3] for some work on homomorphisms between these spaces.)

In Section 3 we discuss various matters related to rectifiable arcs, including the Fundamental Theorem of Calculus for rectifiable paths, and the conditions of uniform regularity and pointwise regularity for compact plane sets. These latter conditions are sufficient to imply completeness of all the normed spaces defined in Section 2 ([7], [13]). In the case where the spaces are incomplete, it becomes important to investigate their completions. We do this in Section 4. There we determine the completions of the normed spaces above, at least for compact plane sets X such that the union of the images of all the injective, rectifiable arcs in X is a dense subset of X. In this setting we define a less restrictive notion of differentiation which ensures that the spaces we end up with are complete. The original versions of the spaces embed isometrically in our new versions, so the completions of the original spaces are simply their closure in the new spaces. Where the algebras considered in [9] were incomplete, the new versions are complete and all the arguments of Feinstein and Kamowitz remain valid in the new setting. This suggests that the new algebras may, in fact, be the correct place to study endomorphisms.

In Section 5 we investigate two related problems: For which compact plane sets are the new spaces constructed in Section 4 the same as the original spaces as defined in Section 2? For which compact plane sets are the original spaces dense in the new spaces? We also obtain some related polynomial and rational approximation results for these spaces. For some work on identifying the maximal ideal spaces in the original setting and on polynomial and rational/holomorphic approximation in some related spaces see, for example, [10], [12]–[15], [17] and [19].

We conclude, in Section 6, with some open problems.

2. Introductory concepts and results. We begin with some standard notation, definitions and results. Let X be a compact plane set. We denote the set of all continuous, complex-valued functions on X by $C(X)$. For $f \in C(X)$ we denote the uniform norm of f by $\|f\|_\infty$. More generally we denote the uniform norm of f on a closed subset E of X by $\|f\|_E$.

DEFINITION 2.1. Let X be a perfect, compact plane set. We say that a complex-valued function f defined on X is *complex-differentiable* at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\, z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the *complex derivative* of f at a. Using this concept of derivative, we define the terms *complex-differentiable on* X, *continuously complex-differentiable on* X, and *infinitely complex-differentiable on* X in the obvious way. We denote the nth complex derivative of f at a by $f^{(n)}(a)$, and we denote the set of infinitely complex-differentiable functions on X by $D^\infty(X)$. We denote the set of continuously complex-differentiable functions on X by $D^1(X)$. More generally, we define the corresponding algebras of n-times continuously differentiable functions, $D^n(X)$, again in the obvious way.

Let (M_n) be a sequence of positive real numbers. We define the space

$$D(X, M) = \left\{ f \in D^\infty(X) : \|f\| = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_\infty}{M_n} < \infty \right\}.$$

With pointwise addition, $D(X, M)$ is a normed space which is not necessarily complete.

If further the sequence M_n satisfies $M_0 = 1$ and, for all non-negative integers m, n, we have

$$\binom{m+n}{n} \le \frac{M_{m+n}}{M_m M_n}$$

then $D(X, M)$ is a normed algebra with pointwise multiplication.

In [7], Dales and Davie used this class of algebras to give an example of a commutative semisimple Banach algebra for which the peak points are of first category in the Shilov boundary, and an example of a commutative semisimple Banach algebra B and a discontinuous function F acting on B.

Clearly the restrictions of all (analytic) polynomials to X belong to all of the $D(X, M)$ spaces. It was further proved in [7] that the algebra $D(X, M)$ includes all of the rational functions with poles off X if and only if

$$\lim_{n \to \infty} \left(\frac{n!}{M_n} \right)^{1/n} = 0. \tag{1}$$

We say that (M_n) is a *non-analytic sequence* if (1) holds [7].

Each of the spaces $D^n(X)$ is a normed algebra, using the norm

$$\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}.$$

These spaces are often incomplete, even for fairly nice X: we give some examples of this below. However, for a given X, the completeness of $D^1(X)$ implies the completeness of all the others. This is a consequence of the following result.

THEOREM 2.2. *Let X be a perfect, compact plane set and let r be a positive integer. Suppose that $D^r(X)$ is complete. Then, for all integers $n \geq r$, $D^n(X)$ is complete and, for every sequence M of positive real numbers, $D(X, M)$ is complete.*

Proof. We give the proof for $D(X, M)$. The proof for $D^n(X)$ is similar but slightly easier. Let f_m be a Cauchy sequence in $D(X, M)$. It is clear that, for all non-negative integers k, the sequence $(f_m^{(k)})_{m=1}^\infty$ is Cauchy in $D^r(X)$ and so converges in $D^r(X)$ to a function g_k, say. By definition of the norm on $D^r(X)$, we see that $(f_m^{(k)})'$ converges uniformly to g_k' on X as $m \to \infty$. However, we also know that $(f_m^{(k)})' = f_m^{(k+1)}$ converges to g_{k+1} as $m \to \infty$ and so we have $g_k' = g_{k+1}$. The remainder of the proof is a standard functional analysis argument showing that $g_0 \in D(X, M)$ and that the sequence f_m converges in $D(X, M)$ to g_0; we omit the details. ∎

We now prove that if X has infinitely many components then $D(X, M)$ is incomplete, and hence all of the spaces $D^n(X)$ are incomplete. In the proof, and throughout the rest of this paper, we will frequently refer to sets which are both open and closed, and it will be convenient to call such sets *clopen* sets.

THEOREM 2.3. *Let $X \subseteq \mathbb{C}$ be a compact, perfect set which has infinitely many components, and let M be any sequence of positive real numbers. Then all of the spaces $D^n(X)$ and $D(X, M)$ are incomplete.*

Proof. By Theorem 2.2 it is sufficient to prove the result for $D(X, M)$. (The proof given below is, anyway, valid in all cases.) We are given that X has infinitely many connected components. Set $E_0 = X$. Then E_0 can be written as $E_0 = E_1 \cup F_1$ where E_1 and F_1 are non-empty, disjoint, clopen subsets of E_0 and E_1 has infinitely many components.

Similarly we can write $E_1 = E_2 \cup F_2$ where E_2 and F_2 are non-empty disjoint clopen subsets of E_1 and E_2 has infinitely many components.

Clearly we can continue in this way to form sequences (E_n) and (F_n). For each $n \in \mathbb{N}$, choose a point $z_n \in F_n$. Then the sequence z_n has a convergent subsequence, z_{n_k}. Say $z_{n_k} \to z_0$ as $n \to \infty$. Now, we cannot have $z_0 \in F_{n_k}$ for any $k \in \mathbb{N}$ since the sets F_{n_k} are open and pairwise disjoint.

Define $f \in C(X)$ by

$$f(z) = \begin{cases} z_{n_k} & \text{for } z \in F_{n_k}, \\ z_0 & \text{for } z \in X \setminus \bigcup_{k=1}^{\infty} F_{n_k}. \end{cases}$$

Then f is constant on each of the clopen sets F_{n_k} and so has derivative 0 on their union. Thus if f were in $D^1(X)$ we would also have $f'(z_0) = 0$. However, for all k, we have

$$\frac{f(z_{n_k}) - f(z_0)}{z_{n_k} - z_0} = 1,$$

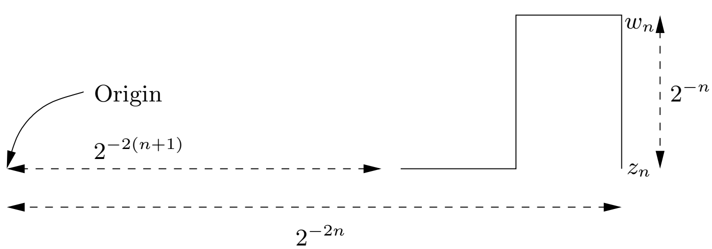
and so f is not in $D^1(X)$. Finally, note that there is an obvious sequence $(f_i) \subseteq D(X, M)$ such that $f_i \to f$ uniformly on X: for $i \in \mathbb{N}$, define $f_i \in D^1(X)$ by

$$f_i(z) = \begin{cases} z_{n_k} & \text{if } z \in F_{n_k} \text{ and } k \leq i, \\ z_0 & \text{for } z \in X \setminus \bigcup_{k=1}^{i} F_{n_k}. \end{cases}$$

It is easy to see that $f_i' = 0$ for all i and that (f_i) is a Cauchy sequence in $D(X, M)$. Since f is not even in $D^1(X)$, $D(X, M)$ is incomplete. ∎

The completeness of $D^1(X)$ is far from being a topological property of X: we conclude this section with an example where X is the image of a rectifiable Jordan arc in the plane and yet $D^1(X)$ is incomplete. (We will look at rectifiable curves in more detail later in this paper.)

EXAMPLE 2.4. Set $z_n = 2^{-2n}$ and $w_n = 2^{-2n} + 2^{-n}i$. We glue together the origin and the following paths (γ_n for $n \in \mathbb{N}$):



The resulting path γ can be parametrised by its arc-length. It is clear that γ is a rectifiable Jordan arc.

The exact position on the x-axis of the leftmost vertical line forming γ_n is irrelevant to the working of this example, so long as it lies (strictly) between $2^{-2(n+1)}$ and 2^{-2n}.

THEOREM 2.5. *Let X be the image of the path γ in the previous example. Then $D^1(X)$ is incomplete.*

Proof. Define $f \in C(X)$ by $f(0) = 0$ and for $x + yi$ in the image of γ_n,

$$f(x + yi) = \begin{cases} 3 \cdot 2^{n-1} y^3 - 9y^2/4 + 2^{-2n} & \text{if } x = 2^{-2n}, \\ 2^{-2(n+1)} & \text{otherwise,} \end{cases}$$

It is straightforward to check that the following conditions hold:

1. f is constant everywhere on Image(γ_n) except on the line joining the points z_n and w_n;
2. f is continuous on the whole of Image(γ), and is continuously differentiable on each Image(γ_n);
3. $f'(z_n) = f'(w_n) = 0$;
4. $f(z_n) = z_n$;
5. $f(w_n) = z_{n+1}$;
6. as $n \to \infty$, the supremum of $|f'|$ on the image of γ_n converges to zero.

Thus we have

$$\lim_{n\to\infty} \frac{f(z_n) - f(0)}{z_n - 0} = \lim_{n\to\infty} \frac{z_n - 0}{z_n - 0} = 1 \neq 0 = \lim_{n\to\infty} f'(z_n)$$

and so $f \notin D^1(X)$.

However, there is an obvious Cauchy sequence (f_n) of functions in $D^1(X)$ with $\|f - f_n\|_\infty \to 0$. We simply define $f_n(z)$ to be equal to $f(z)$ when the real part of z is larger than z_n, and constantly equal to $f(z_n)$ otherwise. Thus $D^1(X)$ is incomplete. ∎

3. Rectifiable paths and regularity conditions for compact plane sets. In this section we discuss families of rectifiable curves and some related conditions. We will assume that the reader is familiar with the elementary results and definitions concerning rectifiable paths including integration of continuous, complex-valued functions along rectifiable curves. For more details see, for example, Chapter 6 of [2].

Definition 3.1. A *path* is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where a and b are real numbers with $a < b$. We say that γ is a path *from* $\gamma(a)$ *to* $\gamma(b)$ with *endpoints* $\gamma^- = \gamma(a)$ and $\gamma^+ = \gamma(b)$.

Given $X \subseteq \mathbb{C}$, a *path in* X is a path whose image is a subset of X. (A Jordan arc in X is, of course, simply an injective path in X.)

The length of a rectifiable path γ will be denoted by $|\gamma|$.

We recall the following elementary connection between piecewise smooth paths, rectifiability and integration.

PROPOSITION 3.2 (see [5, pp. 58–62]). *Let X be a compact subset of $\mathbb{C}$ and $\gamma : [a,b] \to X$ be a piecewise smooth path in X. Then*:

1. γ *is rectifiable*;
2. $|\gamma| = \int_a^b |\gamma'(t)|\,dt$;
3. $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$ *for any* $f \in C(X)$.

The next result is an analogue of the Fundamental Theorem of Calculus. We have not been able to find a proof of it in the literature, although a similar theorem is given in [5] (Theorem 1.18, p. 65). However, the functions in that theorem are defined on open subsets of $\mathbb{C}$, whereas we need the same result for functions defined only on images of rectifiable paths. Elegant proofs of this general result using the method of repeated bisection have been shown to us by G. R. Allan, T. W. Körner and W. K. Hayman. The proof provided by Allan may be found in full in [4].

THEOREM 3.3 (Fundamental Theorem of Calculus for rectifiable paths). *Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path. Then for every $f \in D^1(\mathrm{Image}(\gamma))$ we have*

$$\int_\gamma f'(z)\,dz = f(\gamma^+) - f(\gamma^-).$$

We now discuss, in terms of rectifiable paths, some standard conditions a compact plane set X may satisfy which are sufficient to ensure the completeness of $D^1(X)$ (and hence of all the other spaces defined in Section 2). In [1] it was shown that the collection of sets X for which $D^1(X)$ is complete is closed under finite unions. (In fact the result is only stated there for the $D(X,M)$ spaces, but the proof for the other spaces is the same.)

DEFINITION 3.4. Let $X \subseteq \mathbb{C}$ be compact. We say X is *regular* at a point $z \in X$ if there is a constant $k_z > 0$ such that, for every $w \in X$, there is a path $\gamma : [a,b] \to X$ with $\gamma(a) = z$, $\gamma(b) = w$ and $|\gamma| \le k_z|z-w|$.

We say X is *pointwise regular* if X has more than one point and X is regular at every point $z \in X$. (In [13] such a set is simply said to be *regular*.) If, further, there is one constant $k > 0$ such that, for all z and w in X, there is a path $\gamma : [a,b] \to X$ with $\gamma(a) = z$, $\gamma(b) = w$ and $|\gamma| \le k|z-w|$ then we say that X is *uniformly regular*.

Clearly all pointwise and uniformly regular sets are perfect and path-connected. For points z and w in a set $X \subseteq \mathbb{C}$, we define

$$d(z,w) = \inf\{|\gamma| : \gamma \text{ is a rectifiable path from } z \text{ to } w \text{ in } X\}.$$

Dales and Davie showed that $D^1(X)$ is complete whenever X is a finite union of uniformly regular sets. However, as observed in [13], the proof given in [7] is equally valid for pointwise regular sets. Thus $D^1(X)$ is complete whenever X is a finite union of pointwise regular sets. We will give another proof of this fact in Section 5, after investigating the completions of these normed spaces in the next section.

We now note that for a compact plane set X to satisfy one of these two regularity conditions it is sufficient (though of course not necessary) for the boundary to satisfy the same condition.

THEOREM 3.5. *Let $X \subseteq \mathbb{C}$ be compact. If ∂X is uniformly regular then X is uniformly regular.*

Proof. We know that there is a constant $k > 0$ such that $d(z, w) \leq k|z - w|$ for all $z, w \in \partial X$. Let $z_1, z_2 \in X$. If the line segment connecting z_1 and z_2 is contained in X then $d(z_1, z_2) = |z_1 - z_2|$. Otherwise, the line segment must intersect ∂X at at least two points. In this case, let Z be the set of points of intersection, and (for $i = 1, 2$) let w_i be the closest point of Z to z_i.

We know that there is a path γ in ∂X between w_1 and w_2 such that $|\gamma| \leq k|w_1 - w_2|$. We have

$$\begin{aligned} d(z_1, z_2) &\leq |\gamma| + |z_2 - w_2| + |w_1 - z_1| \\ &\leq k|w_1 - w_2| + |z_1 - w_1| + |z_2 - w_2| \leq (k + 1)|z_1 - z_2|. \quad \blacksquare \end{aligned}$$

The proof of the same theorem for pointwise regularity requires a little more thought, but is essentially the same.

THEOREM 3.6. *Let $X \subseteq \mathbb{C}$ be compact. If ∂X is pointwise regular then X is pointwise regular.*

Proof. Let $z_1 \in X$. Choose a point $w_1 \in \partial X$ such that no other point in ∂X is closer to z_1 than w_1. We know that there is a constant $k_{w_1} > 0$ such that for any $w_2 \in \partial X$ we have $d(w_1, w_2) \leq k_{w_1}|w_1 - w_2|$. Set $c_{z_1} = 2 + 3k_{w_1}$. We show that, for all $z_2 \in X$, $d(z_1, z_2) \leq c_{z_1}|z_1 - z_2|$.

Take $z_2 \in X$. Again, choose a point $w_2 \in \partial X$ such that no other point in ∂X is closer to z_2 than w_2. If the line segment connecting z_1 to z_2 is contained in X then $d(z_1, z_2) = |z_1 - z_2| \leq c_{z_1}|z_1 - z_2|$. Otherwise we have

$$|z_1 - w_1| \leq |z_1 - z_2| \quad \text{and} \quad |z_2 - w_2| \leq |z_1 - z_2|.$$

Thus

$$\begin{aligned} d(z_1, z_2) &\leq k_{w_1}|w_1 - w_2| + |z_1 - w_1| + |z_2 - w_2| \\ &\leq k_{w_1}|w_1 - w_2| + 2|z_1 - z_2| \end{aligned}$$

Now

$$|w_1 - w_2| = |(w_1 - z_1) + (z_1 - z_2) + (z_2 - w_2)| \leq 3|z_1 - z_2|$$

and so $d(z_1, z_2) \leq c_{z_1}|z_1 - z_2|$ as required. $\blacksquare$

The following elementary lemma will be useful later.

LEMMA 3.7. *Each component of a finite union of pointwise regular sets is pointwise regular.*

Proof. We show that a union of two non-disjoint, pointwise regular compact plane sets is again pointwise regular. The result then follows easily from this.

Let X and Y be pointwise regular compact plane sets such that $X \cap Y \neq \emptyset$. Clearly there is a rectifiable path in $X \cup Y$ between each pair of points of $X \cup Y$. Take $z \in X \cup Y$. We show that $X \cup Y$ is regular at z. In view of the pointwise regularity of X and Y, this is clear if $z \in X \cap Y$. Otherwise, we may assume without loss of generality that $z \in X \setminus Y$. Then $\operatorname{dist}(z, Y) > 0$, where $\operatorname{dist}(z, Y)$ is the usual Euclidean point-set distance between z and Y. From this and the pointwise regularity of X it is now elementary to show that $X \cup Y$ is regular at z. ∎

Pointwise regularity is a local property in the following sense: say a set X is *locally pointwise regular* if each point in X has a pointwise regular compact neighbourhood in X. The following result is now an immediate consequence of compactness.

THEOREM 3.8. *Let $X \subseteq \mathbb{C}$ be a locally pointwise regular, compact plane set. Then X is a finite union of pointwise regular sets.*

We finish this section by noting that there are examples of rectifiable paths in the complex plane whose images are not pointwise regular. For example, the fact that the path γ in Example 2.4 has $D^1(\mathrm{Image}(\gamma))$ incomplete implies that γ cannot be pointwise regular.

We are now ready to introduce the new normed spaces that we wish to study.

4. The $\mathcal{F}$-differentiation spaces. In this section we investigate the completions of the normed spaces considered above by weakening the differentiability requirement on the functions.

One well known, related class of Banach spaces is obtained by looking at analytic functions on an open subset U of $\mathbb{C}$ with some specified number of the function's derivatives being bounded. This gives a set of Banach spaces corresponding to the spaces $D^n(X)$ above. Indeed, when X is the closure of U, the spaces $D^n(X)$ embed isometrically in these new, complete, spaces. A similar construction provides complete versions of the spaces $D(X, M)$. However, these constructions are only helpful for compact spaces X where the interior of X is dense in X. This is too restrictive for our purposes. Instead, we will mostly work with the larger class of compact plane sets X for which the union of the images of all rectifiable Jordan arcs in X is dense in X. We will then use appropriate sets of Jordan arcs to define our notion of derivative.

Definition 4.1. Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a set of paths in X. We say $\mathcal{F}$ is *useful* if the following conditions are satisfied.

1. Every path in $\mathcal{F}$ is a rectifiable Jordan arc.
2. If $\gamma \in \mathcal{F}$ is defined on $[a, b]$, then the restriction of γ to $[c, d]$ is in $\mathcal{F}$ whenever $[c, d] \subseteq [a, b]$ and $c < d$.

We write

$$\mathcal{F}(X) = \bigcup_{\gamma \in \mathcal{F}} \operatorname{Image}(\gamma).$$

Clearly the sets of rectifiable Jordan arcs and smooth, rectifiable Jordan arcs in X are both useful. Also, for any $L > 0$, the set of rectifiable Jordan arcs in X with length $\leq L$ is useful.

We are now ready to define the notion of differentiability associated with a set of rectifiable paths. Recall that the endpoints of a path γ are denoted by γ^- and γ^+.

Definition 4.2. Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a set of rectifiable paths in X. For $f \in C(X)$, we say $g \in C(X)$ is an $\mathcal{F}$-*derivative* of f if, for all $\gamma \in \mathcal{F}$, we have

$$\int_\gamma g(z)\, dz = f(\gamma^+) - f(\gamma^-).$$

Note that we assume neither that $\mathcal{F}(X) = X$, nor that there is a path in $\mathcal{F}$ between each pair of points of $\mathcal{F}(X)$: indeed $\mathcal{F}(X)$ may be disconnected.

We will mostly restrict attention to the case where $\mathcal{F}$ is a useful set of paths.

Definition 4.3. Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a set of rectifiable paths in X. Define

$$\mathcal{D}^1_{\mathcal{F}}(X) = \{f \in C(X) : f \text{ has an } \mathcal{F}\text{-derivative in } C(X)\}.$$

Clearly we would not expect $\mathcal{F}$-derivatives to be unique. We will see below, however, that their restriction to $\overline{\mathcal{F}(X)}$ is unique.

The following theorem is the $\mathcal{F}$-derivative analogue of a standard result of elementary real analysis.

Theorem 4.4. *Let X be a compact plane set and let $\mathcal{F}$ be a useful set of paths in X. Let f_n, g_n be uniformly convergent sequences in $C(X)$ with limits f, g respectively. Suppose that, for all n, g_n is an $\mathcal{F}$-derivative of f_n. Then g is an $\mathcal{F}$-derivative of f.*

Proof. This is essentially immediate from the definitions. ■

As we have already seen, the analogous statement for the original notion of differentiation is false: this is the reason why the spaces $D^1(X)$ are often incomplete.

Before going any further, we deal with the issue of "piecewise" curves. For every set $\mathcal{F}$ of paths in X, there is a corresponding set $\mathcal{F}_{\mathrm{PW}}$ of paths that are "piecewise-$\mathcal{F}$" paths. In other words, each path in $\mathcal{F}_{\mathrm{PW}}$ consists of finitely many paths in $\mathcal{F}$ that are joined together at their endpoints. The question is: do $\mathcal{F}$ and $\mathcal{F}_{\mathrm{PW}}$ lead to different theories of differentiation?

THEOREM 4.5. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Let $\mathcal{F}_{\mathrm{PW}}$ be the piecewise version of $\mathcal{F}$, as described above. Then $\mathcal{D}^1_{\mathcal{F}}(X) = \mathcal{D}^1_{\mathcal{F}_{\mathrm{PW}}}(X)$.*

Proof. Again this is an elementary consequence of the definitions. ∎

In view of this result we can now take $\mathcal{F}$ to be (for example) either the set of smooth Jordan arcs in X, or the set of piecewise smooth Jordan arcs in X; it does not make any difference to the resulting object $\mathcal{D}^1_{\mathcal{F}}(X)$. Also we may assume that the lengths of the paths in $\mathcal{F}$ are bounded: for example, for each $L > 0$, the same theory is obtained by using the set of all rectifiable Jordan arcs in X as is obtained by using the set of those rectifiable Jordan arcs in X whose length is at most L. (Every rectifiable curve is "piecewise short".)

Note that we always have $\overline{\mathcal{F}(X)} = \overline{\mathcal{F}_{\mathrm{PW}}(X)}$. Also, the set $\mathcal{F}_{\mathrm{PW}}$ is useful if $\mathcal{F}$ is useful (the converse is, however, not true).

We will prove that $\mathcal{D}^1_{\mathcal{F}}(X)$ is always a Banach algebra. As part of this we need to check that $\mathcal{F}$-derivatives behave in the way we expect with regard to sums, scalar multiples and products.

THEOREM 4.6. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Let $f_1, f_2 \in C(X)$ and $\lambda \in \mathbb{C}$. If $g_1, g_2 \in C(X)$ are $\mathcal{F}$-derivatives of f_1 and f_2 respectively then $g_1 + \lambda g_2$ is an $\mathcal{F}$-derivative of $f_1 + \lambda f_2$.*

Proof. Set $f = f_1 + \lambda f_2$ and $g = g_1 + \lambda g_2$. Clearly $g \in C(X)$. Now let $\gamma \in \mathcal{F}$. We have

$$\int_\gamma g(z)\,dz = \int_\gamma (g_1(z) + \lambda g_2(z))\,dz = \int_\gamma g_1(z)\,dz + \int_\gamma \lambda g_2(z)\,dz$$
$$= f_1(\gamma^+) - f_1(\gamma^-) + \lambda(f_2(\gamma^+) - f_2(\gamma^-)) = f(\gamma^+) - f(\gamma^-).$$

Thus g is an $\mathcal{F}$-derivative of f. ∎

COROLLARY 4.7. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Then $\mathcal{D}^1_{\mathcal{F}}(X)$ is a vector space.*

We now look at multiplication of functions in $\mathcal{D}^1_{\mathcal{F}}(X)$. First we note an elementary lemma concerning polynomials.

LEMMA 4.8. *Let $X \subseteq \mathbb{C}$ be compact and γ be an injective rectifiable path whose image is contained in X. Set $\mathcal{F} = \{\gamma\}$. Then for any polynomials p_1 and p_2 defined on X, the function $p_1'p_2 + p_1p_2'$ is an $\mathcal{F}$-derivative of p_1p_2.*

Proof. We know that p_1, p_2 and p_1p_2 are all complex-differentiable on any complex plane set, and $(p_1p_2)' = p_1'p_2 + p_1p_2'$. The result now follows from the Fundamental Theorem of Calculus for rectifiable paths (or indeed the special case of this theorem for polynomial functions). ■

THEOREM 4.9. *Let $X \subseteq \mathbb{C}$ be compact and γ be an injective rectifiable path whose image is contained in X. Let $\mathcal{F}$ be the set of all subpaths of γ (including γ itself). Then for any functions $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$ with $\mathcal{F}$-derivatives g_1 and g_2 respectively, the function $g_1f_2 + f_1g_2$ is an $\mathcal{F}$-derivative of f_1f_2.*

Proof. Set $Y = \mathrm{Image}(\gamma)$. Note that, since γ is injective, $\mathbb{C} \setminus Y$ must be connected. Hence by Mergelyan's (or Lavrentiev's) theorem we can choose two sequences of analytic polynomials p_n, q_n converging uniformly on Y to g_1, g_2 respectively. Now (anti-differentiating) choose analytic polynomials P_n, Q_n such that $P_n' = p_n$, $Q_n' = q_n$, $P_n(\gamma^-) = f_1(\gamma^-)$ and $Q_n(\gamma^-) = f_2(\gamma^-)$.

The Fundamental Theorem of Calculus for rectifiable paths tells us that p_n is an $\mathcal{F}$-derivative of P_n and similarly for q_n and Q_n. It now follows easily that P_n, Q_n converge uniformly on Y to f_1, f_2 respectively.

By the preceding lemma we know that $P_nq_n + p_nQ_n$ is an $\mathcal{F}$-derivative of P_nQ_n. Taking uniform limits, and applying 4.4, we see that $f_1g_2 + f_2g_1$ is an $\mathcal{F}$-derivative of f_1f_2, as required. ■

COROLLARY 4.10. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Then $\mathcal{D}^1_{\mathcal{F}}(X)$ is an algebra.*

Proof. Clear. ■

We now wish to establish the extent to which $\mathcal{F}$-derivatives are unique. We start with a simple lemma.

LEMMA 4.11. *Let $X \subseteq \mathbb{C}$ be compact and γ be a rectifiable path in X defined on $[a, b]$ and with $\gamma(a) \neq \gamma(b)$. Then there exist a constant $k > 0$ and a sequence (γ_n) of subpaths of γ defined on nested decreasing subintervals of $[a, b]$, with $|\gamma_n| = 2^{-(n-1)}|\gamma|$ and such that for each $n \in \mathbb{N}$, $|\gamma_n| < k|\gamma_n^+ - \gamma_n^-|$.*

Proof. Clearly $|\gamma| < k|\gamma(b) - \gamma(a)|$ for some $k > 0$. This will be our k. We set $\gamma_1 = \gamma$. Now suppose that $n > 1$ and that we have constructed the sequence of subpaths up to and including the path γ_{n-1}, defined on some interval $[a_{n-1}, b_{n-1}]$, with $|\gamma_{n-1}| = 2^{-(n-2)}|\gamma|$ and $|\gamma_{n-1}| < k|\gamma_{n-1}^+ - \gamma_{n-1}^-|$. Choose $c \in (a_{n-1}, b_{n-1})$ in the usual way to bisect the length of γ_{n-1}. Let γ_A and γ_B be the restrictions of γ_{n-1} to $[a_{n-1}, c]$ and $[c, b_{n-1}]$ respectively.

Suppose, for contradiction, that $|\gamma_A| \geq k|\gamma_A^+ - \gamma_A^-|$ and $|\gamma_B| \geq k|\gamma_B^+ - \gamma_B^-|$. Then

$$|\gamma_{n-1}| = |\gamma_A| + |\gamma_B| \geq k(|\gamma_A^+ - \gamma_A^-| + |\gamma_B^+ - \gamma_B^-|) \geq k|\gamma_{n-1}^+ - \gamma_{n-1}^-|,$$

which contradicts the choice of γ_{n-1}. Thus we must have $|\gamma_A| < k|\gamma_A^+ - \gamma_A^-|$

or $|\gamma_B| < k|\gamma_B^+ - \gamma_B^-|$. We now set γ_n to be either γ_A or γ_B accordingly, and the inductive construction may proceed. ∎

We are now ready to prove the uniqueness of $\mathcal{F}$-derivatives in the case where $\mathcal{F}(X)$ is dense in X. Note again that we do not assume that there is a path in $\mathcal{F}$ between each pair of points of $\mathcal{F}(X)$.

THEOREM 4.12. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X such that $\mathcal{F}(X)$ is dense in X. Then for $f \in C(X)$, any $\mathcal{F}$-derivative of f is unique.*

Proof. Let $f \in C(X)$. In view of Theorem 4.6, the difference of any two $\mathcal{F}$-derivatives of f is an $\mathcal{F}$-derivative of the constant function 0. Thus it is sufficient to prove that no non-zero $g \in C(X)$ can be an $\mathcal{F}$-derivative of 0.

Suppose, for contradiction, that $g \in C(X)$ is a non-zero $\mathcal{F}$-derivative of 0. Choose $z_0 \in X$ such that $g(z_0) \neq 0$. Then there exist $R > 0$ and $\delta > 0$ such that $|g(w)| \geq \delta$ for all $w \in X$ with $|w - z_0| < R$.

Choose a path $\gamma \in \mathcal{F}$ with $\mathrm{Image}(\gamma) \subseteq \{w \in X : |w - z_0| < R\}$. By Lemma 4.11, there exist a constant $k > 0$ and a sequence (γ_n) of subpaths of γ defined on nested decreasing subintervals of the domain of γ such that $|\gamma_n| \to 0$ and

$$|\gamma_n| < k|\gamma_n^+ - \gamma_n^-|$$

for each $n \in \mathbb{N}$. Now there must be a point $\alpha \in \bigcap_{n\in\mathbb{N}} \mathrm{Image}(\gamma_n)$, because these images are compact and nested. Clearly $|g(\alpha)| \geq \delta$. For $z \in X$, write

$$g(z) = g(\alpha) + r(z)$$

where $r \in C(X)$ and $r(z) \to 0$ as $z \to \alpha$ in X. Now choose $n \in \mathbb{N}$ such that $|r(z)| < \delta/2k$ for $z \in \mathrm{Image}(\gamma_n)$. We have

$$\int\limits_{\gamma_n} g(z)\,dz = \int\limits_{\gamma_n} g(\alpha)\,dz + \int\limits_{\gamma_n} r(z)\,dz = g(\alpha)(\gamma_n^+ - \gamma_n^-) + \int\limits_{\gamma_n} r(z)\,dz.$$

Now

$$|g(\alpha)(\gamma_n^+ - \gamma_n^-)| = |g(\alpha)|\;|\gamma_n^+ - \gamma_n^-| \geq \delta|\gamma_n^+ - \gamma_n^-|$$

and

$$\Big|\int\limits_{\gamma_n} r(z)\,dz\Big| \leq |\gamma_n| \sup\{|r(z)| : z \in \mathrm{Image}(\gamma_n)\} \leq \frac{\delta}{2}\,|\gamma_n^+ - \gamma_n^-|.$$

Thus we have $\int_{\gamma_n} g(z)\,dz \neq 0$. This contradicts the fact that g is an $\mathcal{F}$-derivative of 0, and the result follows. ∎

Note that, even if $\mathcal{F}(X)$ is not dense in X, it is clear that for any function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, any two $\mathcal{F}$-derivatives g_1 and g_2 of f must agree on $\overline{\mathcal{F}(X)}$. This point will be crucial in the following development of the analytic properties of $\mathcal{D}^1_{\mathcal{F}}(X)$.

Note that the converse to the previous theorem is clear: if $\mathcal{F}(X)$ is not dense in X then every f in $\mathcal{D}^1_{\mathcal{F}}(X)$ has infinitely many $\mathcal{F}$-derivatives. However, these $\mathcal{F}$-derivatives will all agree on $\overline{\mathcal{F}(X)}$.

We now define the norm we need to make $\mathcal{D}^1_{\mathcal{F}}(X)$ into a Banach algebra.

Definition 4.13. Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a non-empty, useful set of paths in X. For $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ we define

$$\|f\| = \|f\|_\infty + \|g\|_{\overline{\mathcal{F}(X)}}$$

where $g \in C(X)$ is any $\mathcal{F}$-derivative of f.

Note that $\|\cdot\|$ is well defined even when $\mathcal{F}(X)$ is not dense in X and $\mathcal{F}$-derivatives are non-unique, because we know that any two $\mathcal{F}$-derivatives of a function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ *do* agree on $\overline{\mathcal{F}(X)}$.

Theorem 4.14. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Then $\mathcal{D}^1_{\mathcal{F}}(X)$ is a normed space.*

Proof. Clearly we have $\|f\| \geq 0$ for all $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, and $\|f\| = 0$ if and only if $f = 0$. Choose $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ and $\lambda \in \mathbb{C}$. Let $g \in C(X)$ be an $\mathcal{F}$-derivative of f. We have already seen that λg is an $\mathcal{F}$-derivative of λf. We have

$$\|\lambda f\| = \|\lambda f\|_\infty + \|\lambda g\|_{\overline{\mathcal{F}(X)}} = |\lambda| \cdot \|f\|_\infty + |\lambda| \cdot \|g\|_{\overline{\mathcal{F}(X)}} = |\lambda| \cdot \|f\|.$$

Choose $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$, and $\mathcal{F}$-derivatives g_1 and g_2 respectively. We have already seen that $g_1 + g_2$ is an $\mathcal{F}$-derivative of $f_1 + f_2$. We have

$$\begin{aligned}\|f_1 + f_2\| &= \|f_1 + f_2\|_\infty + \|g_1 + g_2\|_{\overline{\mathcal{F}(X)}} \\ &\leq \|f_1\|_\infty + \|f_2\|_\infty + \|g_1\|_{\overline{\mathcal{F}(X)}} + \|g_2\|_{\overline{\mathcal{F}(X)}} = \|f_1\| + \|f_2\|. \ \blacksquare\end{aligned}$$

We now show that $\mathcal{D}^1_{\mathcal{F}}(X)$ is a Banach space.

Theorem 4.15. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Then $\mathcal{D}^1_{\mathcal{F}}(X)$ is complete.*

Proof. Set $Y = \overline{\mathcal{F}(X)}$. Let (f_n) be a Cauchy sequence in $\mathcal{D}^1_{\mathcal{F}}(X)$. For each $n \in \mathbb{N}$, choose an $\mathcal{F}$-derivative g_n of f_n. Then (f_n) is Cauchy in $C(X)$ and $(g_n|_Y)$ is Cauchy in $C(Y)$, so these sequences converge uniformly. Say $f_n \to f \in C(X)$ and $g_n|_Y \to g \in C(Y)$.

Extend g to $\tilde{g} \in C(X)$ by the Tietze extension theorem. It is now easy to check that $\tilde{g}$ is an $\mathcal{F}$-derivative of f, so $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ and hence $\mathcal{D}^1_{\mathcal{F}}(X)$ is complete. $\blacksquare$

The last thing we have to do to show that $\mathcal{D}^1_{\mathcal{F}}(X)$ is a Banach algebra is to show that $\|\cdot\|$ is an algebra norm. Fortunately this is not too difficult.

THEOREM 4.16. *Let $X \subseteq \mathbb{C}$ be compact and $\mathcal{F}$ be a useful set of paths in X. Then the norm $\|f\| = \|f\|_\infty + \|g\|_{\overline{\mathcal{F}(X)}}$ (where g is any $\mathcal{F}$-derivative of f) is an algebra norm on $\mathcal{D}^1_{\mathcal{F}}(X)$.*

Proof. Choose $f_1, f_2 \in \mathcal{D}^1_{\mathcal{F}}(X)$, and $\mathcal{F}$-derivatives g_1 and g_2 respectively. We have already seen that $g_1 f_2 + f_1 g_2$ is an $\mathcal{F}$-derivative of $f_1 f_2$. We have

$$\begin{aligned}
\|f_1 f_2\| &= \|f_1 f_2\|_\infty + \|g_1 f_2 + f_1 g_2\|_{\overline{\mathcal{F}(X)}} \\
&\le \|f_1\|_\infty \|f_2\|_\infty + \|g_1 f_2\|_{\overline{\mathcal{F}(X)}} + \|f_1 g_2\|_{\overline{\mathcal{F}(X)}} \\
&\le \|f_1\|_\infty \|f_2\|_\infty + \|g_1\|_{\overline{\mathcal{F}(X)}} \|f_2\|_\infty + \|f_1\|_\infty \|g_2\|_{\overline{\mathcal{F}(X)}} \\
&\le \|f_1\|_\infty \|f_2\|_\infty + \|g_1\|_{\overline{\mathcal{F}(X)}} \|f_2\|_\infty \\
&\quad + \|f_1\|_\infty \|g_2\|_{\overline{\mathcal{F}(X)}} + \|g_1\|_{\overline{\mathcal{F}(X)}} \|g_2\|_{\overline{\mathcal{F}(X)}} \\
&= (\|f_1\|_\infty + \|g_1\|_{\overline{\mathcal{F}(X)}})(\|f_2\|_\infty + \|g_2\|_{\overline{\mathcal{F}(X)}}) = \|f_1\| \cdot \|f_2\|. \ \blacksquare
\end{aligned}$$

To avoid any complications arising from non-uniqueness of $\mathcal{F}$-derivatives, when we come to higher derivatives we will restrict attention to the case where $\mathcal{F}(X)$ is dense in X.

We next show that $D^1(X) \subseteq \mathcal{D}^1_{\mathcal{F}}(X)$, and note conditions under which the inclusion is isometric. We also see the connection between the two kinds of derivative.

THEOREM 4.17. *Let $X \subseteq \mathbb{C}$ be compact and perfect and $\mathcal{F}$ be a useful set of paths in X. Then $D^1(X) \subseteq \mathcal{D}^1_{\mathcal{F}}(X)$. Indeed, for each $f \in D^1(X)$ the derivative (in the old sense) f' is also an $\mathcal{F}$-derivative of f. If $\overline{\mathcal{F}(X)} = X$ then the inclusion above is isometric.*

Proof. Choose $f \in D^1(X)$. Then f' (in the old sense) exists and is in $C(X)$. The Fundamental Theorem of Calculus for rectifiable paths gives us

$$\int_\gamma f'(z)\, dz = f(\gamma^+) - f(\gamma^-)$$

for all $\gamma \in \mathcal{F}$ and so f' is an $\mathcal{F}$-derivative of f. The rest is clear. $\blacksquare$

Note that when $\overline{\mathcal{F}(X)} = X$ the completion of $D^1(X)$ is simply its closure in $\mathcal{D}^1_{\mathcal{F}}(X)$.

We now introduce the new versions of the higher derivatives. As mentioned above, we will simplify matters by restricting attention to the case where $\overline{\mathcal{F}(X)} = X$. In view of the equality (in this setting) of the two kinds of derivatives when both are defined, we may safely use the notation f' for the derivative of f in either sense.

Given such X and $\mathcal{F}$, it is clear how to define (inductively) the notion of *n-times $\mathcal{F}$-differentiable* and the nth $\mathcal{F}$-derivative of a function f. An easy induction using the above theorem shows that if f is in $D^n(X)$ then f is n-times $\mathcal{F}$-differentiable and the old nth derivative $f^{(n)}$ is also the nth

$\mathcal{F}$-derivative of f. Thus we may use the notation $f^{(n)}$ for the new notion of derivative also. Moreover, in view of our earlier results, there is no problem (in this setting) in checking that the standard Leibniz formula is still valid for the new notion of nth derivative.

We can now define spaces corresponding to the $D^n(X)$ spaces and the $D(X,M)$ spaces. We denote these new spaces by $\mathcal{D}^n_{\mathcal{F}}(X)$ and $\mathcal{D}_{\mathcal{F}}(X,M)$. For $f \in \mathcal{D}^n_{\mathcal{F}}(X)$, we define

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_{\overline{\mathcal{F}(X)}}}{k!}$$

(with the usual convention that $f^{(0)} = f$). Similarly we define the norm on $\mathcal{D}_{\mathcal{F}}(X,M)$ corresponding to that on $D(X,M)$.

Because $\mathcal{D}^1_{\mathcal{F}}(X)$ is complete, the new spaces are all Banach spaces and the old spaces are contained isometrically in the new spaces (the argument for this is the same as the proof that whenever $D^1(X)$ is complete then so are the $D^n(X)$ and $D(X,M)$ spaces). The spaces $\mathcal{D}^n_{\mathcal{F}}(X)$ are always Banach algebras. When M is an algebra sequence, $\mathcal{D}_{\mathcal{F}}(X,M)$ is also a Banach algebra. The completions of the old spaces are simply their closures in the new spaces.

In the next section we will investigate questions concerning the density or otherwise of $D^1(X)$ in $\mathcal{D}^1_{\mathcal{F}}(X)$, along with some related questions of polynomial, rational and holomorphic approximation in these spaces and the higher derivative spaces.

5. Approximation results. We will show that in many cases, $\mathcal{D}^1_{\mathcal{F}}(X)$ is itself the completion of $D^1(X)$. We begin with some cases where the two spaces are equal. In this first result, part of the conclusion (the fact that $D^1(X)$ is complete) was previously observed in [13].

THEOREM 5.1. *Let $X \subseteq \mathbb{C}$ be compact, perfect and the union of finitely many pointwise regular sets. Let $L > 0$, and let $\mathcal{F}$ be a useful set of paths in X which includes all injective rectifiable paths with length $\le L$ in X. Then $D^1(X) = \mathcal{D}^1_{\mathcal{F}}(X)$ (and hence $D^1(X)$ is complete).*

Proof. As we observed before, every rectifiable path is "piecewise of length at most L" and so we may assume that $\mathcal{F}$ is, in fact, the set of *all* injective rectifiable paths in X. By Lemma 3.7, each component of X is pointwise regular and so we have $\overline{\mathcal{F}(X)} = X$. Thus $\mathcal{F}$-derivatives are unique and $D^1(X) \subseteq \mathcal{D}^1_{\mathcal{F}}(X)$, the inclusion being isometric.

Choose $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ and let $g \in C(X)$ be the $\mathcal{F}$-derivative of f. Choose $a \in X$ and $(z_n) \subseteq X$ such that $z_n \ne a$ and $z_n \to a$. Then z_n is eventually in the same component as a (call the component U). So without loss of generality we can assume that $z_n \in U$ for every $n \in \mathbb{N}$.

Since U is pointwise regular, there is a $k(a) > 0$ such that, for each $n \in \mathbb{N}$, there is an injective, rectifiable path γ_n from a to z_n in U, such that $|\gamma_n| \le k_a|a - z_n|$.

Set

$$I_n = \frac{f(z_n) - f(a)}{z_n - a} - g(a) = \left(\frac{1}{z_n - a}\int_{\gamma_n} g(z)\,dz\right) - g(a).$$

We have

$$g(a) = \frac{1}{z_n - a}\int_{\gamma_n} g(a)\,dz$$

for each $n \in \mathbb{N}$, so

$$\begin{aligned} |I_n| &= \left|\frac{1}{z_n - a}\int_{\gamma_n} (g(z) - g(a))\,dz\right| \\ &\le \frac{|\gamma_n|}{|z_n - a|}\cdot \sup\{|g(z) - g(a)| : z \in \mathrm{Image}(\gamma_n)\} \\ &\le k_a \cdot \sup\{|g(z) - g(a)| : z \in \mathrm{Image}(\gamma_n)\} \to 0 \end{aligned}$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{f(z_n) - f(a)}{z_n - a} = g(a)$$

as required. ∎

Note that it is *not* enough just to have $\mathcal{F}$ being a useful set of paths in X with $\overline{\mathcal{F}(X)} = X$, as the following example shows.

EXAMPLE 5.2. Let X be the unit square, $[0,1] \times [0,1] \subseteq \mathbb{C}$. Let $\mathcal{F}$ be the set of paths of the form $\gamma : [a,b] \to X$, $\gamma(t) = k+ti$ for some set $[a,b] \subseteq [0,1]$. It is clear that $\mathcal{F}$ is useful and $\overline{\mathcal{F}(X)} = X$, in fact $\mathcal{F}(X) = X$.

Define $f \in C(X)$ by $f(x + iy) = x$ for $x + iy \in X$. It is clear from the Cauchy–Riemann equations that $f \notin D^1(X)$. However $f \in \mathcal{D}^1_{\mathcal{F}}(X)$, with (unique) $\mathcal{F}$-derivative $g \in C(X)$ given by $g(x + iy) = 0$ for all $x + iy \in X$.

In this example, $D^1(X)$ is a proper closed subalgebra of $\mathcal{D}^1_{\mathcal{F}}(X)$.

In view of this example, it is worth investigating conditions on $\mathcal{F}$ which ensure that functions in $\mathcal{D}^1_{\mathcal{F}}(X)$ are analytic on the interior of X. The following lemma and its immediate corollary give one class of useful sets of paths with this property.

LEMMA 5.3. *Let $X \subseteq \mathbb{C}$ be compact and let $\mathcal{F}$ be the set of all injective rectifiable paths in X. Then every $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ is analytic on the interior of X.*

Proof. Choose a point $z \in \mathrm{int}(X)$. Then $B(z,r) \subseteq \mathrm{int}(X)$ for some $r > 0$. Set $Y = \overline{B(z,r)}$. Let $\mathcal{G}$ be the set of paths in $\mathcal{F}$ whose images are contained in Y. Clearly $\mathcal{G}$ is in fact the set of injective rectifiable paths in Y.

Choose $f \in \mathcal{D}^1_{\mathcal{F}}(X)$. Then $f|_Y \in \mathcal{D}^1_{\mathcal{G}}(Y)$. Since Y is pointwise regular, by Theorem 5.1 we have $\mathcal{D}^1_{\mathcal{G}}(Y) = D^1(Y)$. Thus $f|_Y \in D^1(Y)$ and hence f is analytic at z. ■

The following corollary is now immediate.

COROLLARY 5.4. *Let $X \subseteq \mathbb{C}$ be compact, let $L > 0$ and let $\mathcal{F}$ be a useful set of paths in X which includes all those injective rectifiable paths in X which have length at most L. Then every $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ is analytic on the interior of X.*

In order to show that the original spaces are often dense in the new spaces, we will look at some related questions concerning polynomial and rational approximation. We first extend a definition from the original paper of Dales and Davie. (See [7] and [12] for some work on these spaces in the original setting.)

DEFINITION 5.5. Let X be a perfect, compact plane set, and let D be any of the normed algebras of functions on X discussed in this paper such that D includes all rational functions with poles off X. We define D_{R} to be the closure in D of the rational functions with poles off X and D_{P} to be the closure in D of the polynomial functions.

Curiously, it appears to be an open question whether or not D_{R} is always equal to D, even if we restrict attention to the case where D is $D^1(X)$. It is, however, easy to see that the continuous character space of D_{R} is always equal to X (recall that D may be incomplete): the proof of Theorem 1.8 of [7] goes through without need for any modifications. It is also elementary to see that whenever $\mathbb{C} \setminus X$ is connected then $D_{\mathrm{P}} = D_{\mathrm{R}}$: for example, this follows from the fact that the spectrum of the coordinate functional Z must be the same in the completions of D_{R} and D_{P}.

THEOREM 5.6. *Let X be a compact plane set and suppose that $\mathcal{F}$ is a useful set of paths in X with $\overline{\mathcal{F}(X)} = X$. Let $D = \mathcal{D}^1_{\mathcal{F}}(X)$. Consider the following subsets of D: B, the set of all $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ such that the $\mathcal{F}$-derivative of f is the zero function; A, the linear span of the idempotents in D; and C, the closure of A in D. Then C is equal to the set of all functions in $C(X)$ which are constant on every component of X. Moreover we have $A \subseteq D^1(X)$, $C \subseteq B$ and $C \subseteq D_{\mathrm{R}}$.*

Proof. It is clear that all of the subsets of D mentioned are in fact subalgebras of D and that $A \subseteq B$ and $A \subseteq D^1(X)$. It is also clear that all of the idempotents in $C(X)$ are in A, and that B is a closed subalgebra of D. Since the derivatives of all elements involved are 0, C is equal to the uniformly closed linear span in $C(X)$ of the idempotents in $C(X)$, and this is easily seen to be equal to the set of all functions in $C(X)$ which are constant on every component of X.

Finally, we turn to the rational approximation result. For this we need only prove that all of the idempotents are in D_{R}. This is, of course, immediate from the Shilov idempotent theorem, but can also be seen directly by extending any idempotent to be a function analytic on a neighbourhood of X and applying Runge's theorem. ∎

Note that Example 5.2 shows that C need not coincide with B unless further conditions are placed on $\mathcal{F}$.

We now prove some closely related approximation results.

THEOREM 5.7. *Let X be a perfect, compact plane set such that $\mathbb{C}\setminus X$ is connected and X has empty interior. Suppose that $\mathcal{F}$ is a useful set of paths in X with the following properties*: $\overline{\mathcal{F}(X)} = X$, *the set of lengths of paths in $\mathcal{F}$ is bounded above*, *and every pair of distinct points of X which are in the same component of X can be joined by a path in $\mathcal{F}$. Set $D = \mathcal{D}^1_{\mathcal{F}}(X)$. Then $D = D_{\mathrm{R}} = D_{\mathrm{P}}$ and $\mathcal{D}^1_{\mathcal{F}}(X)$ is the completion of $D^1(X)$.*

Proof. It is clear that the second part of the conclusion follows from the first, and we have already mentioned that $D_{\mathrm{P}} = D_{\mathrm{R}}$ in this setting. We prove the rational approximation result. Set $L = \sup\{|\gamma| : \gamma \in \mathcal{F}\}$.

Choose $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ with $\mathcal{F}$-derivative $g \in C(X)$. By Mergelyan's theorem we can find a polynomial $p \in C(X)$ such that

$$\|p - g\|_X < \min\left\{\frac{\varepsilon}{3}, \frac{\varepsilon}{3L}\right\}.$$

We have, for any path $\gamma_0 \in \mathcal{F}$,

$$\Big|\int_{\gamma_0} (p(z) - g(z))\,dz\Big| \le |\gamma_0| \cdot \|p - g\|_X < \frac{\varepsilon|\gamma_0|}{3L} \le \frac{\varepsilon}{3}.$$

Choose an analytic polynomial F whose derivative is p. Certainly we have $F \in D_{\mathrm{R}}$.

Since $F - f$ is uniformly continuous on X, we may choose $\delta > 0$ such that whenever $z, w \in X$ with $|z - w| < \delta$ then $|(F - f)(z) - (F - f)(w)| < \varepsilon/3$.

Noting that every component of X is the intersection of the clopen sets containing it, by compactness we may choose components $K_1, \ldots, K_n$ of X and pairwise disjoint clopen subsets $U_1, \ldots, U_n$ of X such that for each i we have

$$U_i \subseteq \{z \in \mathbb{C} : \mathrm{dist}(z, K_i) < \delta\} \quad \text{and} \quad X = \bigcup_{i=1}^{n} U_i.$$

For each i, choose a point $z_i \in K_i$. Define h on X as follows: $h(z) = F(z) - F(z_i) + f(z_i)$ for $z \in U_i$. Then h is F plus a linear combination of idempotents, so $h \in D_{\mathrm{R}}$. We now look at $h - f$. First note that $h' = p$, so

$\|h' - g\|_X < \varepsilon/3$. We now wish to estimate $\|h - f\|_X$. Although we have not assumed that $K_i \subseteq U_i$, our choice of δ ensures that if $|F(z) - F(z_i) + f(z_i) - f(z)|$ is small on K_i then $|h - f|$ is small on U_i. Let $z \in K_i$. Choose any path $\gamma \in \mathcal{F}$ from z_i to z. Then

$$F(z) - F(z_i) + f(z_i) - f(z) = \int_\gamma (p(w) - g(w))\, dw,$$

and so $|F(z) - F(z_i) + f(z_i) - f(z)| < \varepsilon/3$ for $z \in K_i$. It follows from our choice of δ that $|h(z) - f(z)| < 2\varepsilon/3$ for $z \in U_i$ and so $|h - f|_X < 2\varepsilon/3$. Thus $\|h - f\| < \varepsilon$. ∎

A similar theorem is valid when the interior is non-empty, provided that we ensure that the functions in $\mathcal{D}^1_{\mathcal{F}}(X)$ are analytic on the interior of X. (This is of course also necessary for rational approximation.) We saw some conditions which were sufficient for this above in Lemma 5.3 and Corollary 5.4. Here is one fairly general version of the result for X with interior.

Theorem 5.8. *Let X be a perfect, compact plane set such that $\mathbb{C} \setminus X$ is connected and let $r > 0$. Suppose that $\mathcal{F}$ is a useful set of paths in X with the following properties: $\overline{\mathcal{F}(X)} = X$, the set of lengths of paths in $\mathcal{F}$ is bounded above, every pair of distinct points of X which are in the same component of X can be joined by a path in $\mathcal{F}$, and $\mathcal{F}$ includes all injective rectifiable paths in X of length $\le r$. Set $D = \mathcal{D}^1_{\mathcal{F}}(X)$. Then $D = D_{\mathrm{R}} = D_{\mathrm{P}}$ and $\mathcal{D}^1_{\mathcal{F}}(X)$ is the completion of $D^1(X)$.*

The proof is the same as that of 5.7 in view of the fact that, since f is analytic on the interior of X, so is the $\mathcal{F}$-derivative g of f. As g is continuous on X we may still apply Mergelyan's theorem.

When $\mathbb{C} \setminus X$ is not connected then the polynomials cannot be dense. If we attempt to imitate the above proofs using rational functions we hit the obstacle that it may not be possible to anti-differentiate these. However, if a rational function is uniformly close to an $\mathcal{F}$-derivative, we may obtain good estimates on the residues at the poles and this may allow us to modify the rational function slightly to obtain one which may be anti-differentiated. Here is one rational approximation result valid for finitely connected X.

Theorem 5.9. *Let X be a perfect, compact plane set such that $\mathbb{C} \setminus X$ has only finitely many bounded components, say $U_1, \ldots, U_n$. Choose one point a_j in each of the bounded components U_j $(1 \le j \le n)$. Suppose that $\mathcal{F}$ is a useful set of paths in X satisfying the conditions of Theorem 5.8 and, in addition, for each j with $1 \le j \le n$ there is a closed curve γ_j in $\mathcal{F}$ with non-zero winding number about a_j. Set $D = \mathcal{D}^1_{\mathcal{F}}(X)$. Then $D = D_{\mathrm{R}}$ and $\mathcal{D}^1_{\mathcal{F}}(X)$ is the completion of $D^1(X)$.*

Proof. The proof is again similar to that of Theorem 5.7. Let $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ and let g be the $\mathcal{F}$-derivative of f. Then g is continuous on X and analytic on the interior of X, so it is standard (see for example [11]) that g may be uniformly approximated on X by a sequence of rational functions, say r_k. By Runge's theorem we may further assume that the poles of the rational functions r_k all lie in $\{a_1, \ldots, a_n\}$. As $\int_{\gamma_j} g(z)\,dz = 0$ for each j, the residue at each a_j of r_k tends to 0 as $k \to \infty$. Thus we may modify the sequence r_k (subtracting rational functions with simple poles in $\{a_1, \ldots, a_n\}$ if necessary) to show that g may be uniformly approximated on X by anti-differentiable rational functions. The remainder of the proof is identical to that of Theorem 5.7, using such a rational function r in place of the polynomial p used there and taking F to be a rational anti-derivative of r. ∎

These theorems cover many cases where completeness has previously been an issue, for example the simple sets and the radially self-absorbing sets considered in [10] (we define these below), and the combs and stars considered in [4].

Recall the following definition from [10].

Definition 5.10. Let $X \subseteq \mathbb{C}$ be non-empty and compact. Then X is *radially self-absorbing* if, for every $r > 1$, we have $X \subseteq \operatorname{int}(rX)$.

We conclude this section by transferring to our new spaces a result about holomorphic approximation for radially self-absorbing sets, originally proved for the $D(X, M)$ spaces in [10] (Lemma 3.1). The bulk of the proof is identical, but we need to use Lemma 5.3.

Theorem 5.11. *Let $X \subseteq \mathbb{C}$ be compact and radially self-absorbing. Let $\mathcal{F}$ be the set of injective rectifiable paths in X. Let M be a sequence of positive numbers. Set*

$$S = \{f \in D(X, M) : f \text{ extends to be analytic on a neighbourhood of } X\}.$$

Then S is dense in $\mathcal{D}_{\mathcal{F}}(X, M)$.

Proof. Note that $\overline{\mathcal{F}(X)} = X$ and so $D(X, M)$ embeds isometrically in $\mathcal{D}_{\mathcal{F}}(X, M)$.

Choose $f \in \mathcal{D}_{\mathcal{F}}(X, M)$. Then by Lemma 5.3, f is analytic on $\operatorname{int}(X)$. For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, set $g_n(z) = nz/(n+1)$ and set $F_n = f \circ g_n$. Then F_n is analytic on $\frac{n+1}{n}\operatorname{int}(X)$, which is a neighbourhood of X, and so $F_n|_X \in D^\infty(X)$.

Set $f_n = F_n|_X$. We have, for all $n \in \mathbb{N}$ and all $k \geq 0$, $\|f_n^{(k)}\|_\infty \leq \|f^{(k)}\|_\infty$. Thus $f_n \in D(X, M)$ for each $n \in \mathbb{N}$. Now $\|f_n^{(k)} - f^{(k)}\|_\infty \to 0$ as $n \to \infty$. Hence, by dominated convergence for series, $\|f_n - f\| \to 0$ as $n \to \infty$. ∎

Remarks. This holomorphic approximation result also shows that the new space is the completion of the old. The same result is, of course, valid for the $\mathcal{D}^n_{\mathcal{F}}(X)$ spaces (with a slightly easier proof).

In the algebra setting, the completeness of our $\mathcal{D}_{\mathcal{F}}(X, M)$ algebras allows us to apply the holomorphic functional calculus to the coordinate functional Z, as in Corollary 3.2 of [10], to see that the polynomials are dense in our spaces: the same proof goes through without the need for any changes. Of course, this result from [10] follows from our result, without need for the completeness assumption made there. This eliminates the need to appeal to the Runge argument given in Section 5 of that paper to cover the possibility that the normed algebras concerned might be incomplete.

6. Open problems. We conclude with some open problems.

1. Do there exist a compact plane set X and a non-analytic sequence M such that the rational functions with poles off X are not dense in $D(X, M)$?

We have mentioned a small number of positive results on polynomial and rational approximation, but in general this problem is wide open. In particular the answer for $D(X, M)$ is apparently not known for the "square annulus" obtained by deleting an open square from the middle of a compact square. (Note that in view of these open problems, some authors have worked directly with the closures in these spaces of the set of rational functions instead.)

2. If X is a radially self-absorbing set, is $D^1(X)$ already complete? More generally, suppose that X is the closure of a bounded, connected open subset of $\mathbb{C}$. Is $D^1(X)$ already complete?

3. Let X be a compact plane set and let $\mathcal{F}$ be the set of all injective, rectifiable paths in X. Suppose that $\overline{\mathcal{F}(X)} = X$. Is it always true that $\mathcal{D}^1_{\mathcal{F}}(X)$ is the completion of $D^1(X)$? Is it always true that the rational functions are dense in $\mathcal{D}^1_{\mathcal{F}}(X)$?

Note that here the answer for the square annulus is easily seen to be yes, using Theorem 5.9. More generally, any function $f \in \mathcal{D}^1_{\mathcal{F}}(X)$ which may be extended to have continuous first-order partial derivatives on a neighbourhood of X and whose $\overline{\partial}$ derivative vanishes on X may be approximated in $\mathcal{D}^1_{\mathcal{F}}(X)$ by rational functions (see [19]). However, even functions in $D^1(X)$ need not in general have such extensions. Our question on rational approximation is equivalent to the question of whether a dense set of functions may be so extended.

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